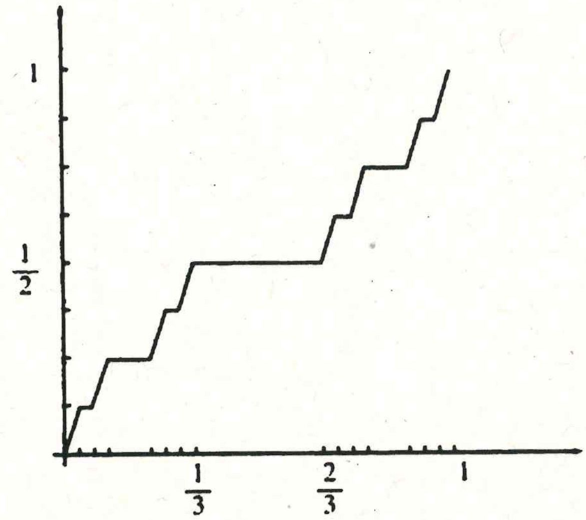

FIG. 10. The graph of the function f_3 on $[0, 1]$.

Example 3.34 Let us define by induction a sequence of increasing, onto functions $f_h : \mathbf{R} \to [0, 1]$ setting $f_0(t) = 0 \vee t \wedge 1$ and

$$f_{h+1}(t) = \frac{1}{2} \cdot \begin{cases} f_h \circ \psi_1^{-1}(t) & \text{if } t \in (-\infty, 1/3] \\ 1 & \text{if } t \in [1/3, 2/3] \\ 1 + f_h \circ \psi_2^{-1}(t) & \text{if } t \in [2/3, \infty) \end{cases} \qquad \forall h \geq 0.$$

It can be easily checked by induction that $f_h(t) = 0$ for $t \leq 0$, $f_h(t) = 1$ for $t \geq 1$ and

(a) $\|f_{h+1} - f_h\|_\infty \leq 2^{-h-1}/3$;

(b) $f_n = f_k$ is constant in any interval of $\mathbf{R} \setminus C_k$ if $n \geq k \geq 0$.

By (a) (f_h) is a Cauchy sequence in $C([0, 1])$, hence uniformly converging in $[0, 1]$ (and then in $\mathbf{R}$) to some continuous function f. This construction of f and of the f_h is equivalent to the one of Example 1.67, but we do not need that in the following.

The function f is still increasing and maps $[0, 1]$ onto $[0, 1]$; in particular $f \in BV(0, 1)$ and Df is a probability measure in $(0, 1)$. On the other hand, from (b) we infer that f is constant in any connected component of $(0, 1) \setminus C$. These properties allow us to conclude that f is a Cantor function, because Df has no atoms (f is continuous) and $D^a f = 0$ ($f' = 0$ in the complement of C, a set with full measure in $(0, 1)$).

The distributional derivative of f_{h+1} is given by $\psi_{1\#}(Df_h)/2$ on $(-\infty, 1/3)$ and by $\psi_{2\#}(Df_h)/2$ on $(2/3, \infty)$, hence

$$Df_{h+1} = \tfrac{1}{2}\left[\psi_{1\#}(Df_h) + \psi_{2\#}(Df_h)\right].$$

Since, by Proposition 3.13, (Df_h) weakly* converges to (Df) in $\mathbf{R}$, from Remark 1.71 we infer that the measures $\psi_{i\#}(Df_h)$ weakly* converge in $\mathbf{R}$ to $\psi_{i\#}(Df)$. Passing to the limit as $h \to \infty$ we obtain that $\nu = Df$ satisfies (3.28), hence $Df = c^{-1}\mathcal{H}^\gamma \llcorner C$ and, by integration, $f(t) = c^{-1}\mathcal{H}^\gamma([0, t] \cap C)$ for any $t \geq 0$.