Proposition: Let $f : \mathbb{R} \to \mathbb{R}$ monotone nondecreasing function
(Then $f \in L^1_{loc}(\mathbb{R})$ since it has a countable number of jumps).

$[x > y \Rightarrow f(x) \geq f(y)]$

Let $T_f(\phi) = \int_{\mathbb{R}} f \phi \, dx$. Then $(T_f)'$ is a positive distribution (that is: the derivative in the sense of distributions of a monotone non decreasing function is positive).

In particular $(T_f)'$ has order $0 \Rightarrow \exists \mu \in \mathcal{M}(\mathbb{R})$ $(T_f)' = T_\mu$.

Proof

Let $\phi \in C^\infty_c(\mathbb{R})$

BASIC OBSERV

$$\lim_{h \to 0^+} \frac{\phi(x+h) - \varphi(x)}{h} = \varphi'(x) \quad \text{UNIFORMLY}$$

$$\varphi(x+h) = \varphi(x) + \phi'(x) \cdot h + \phi''(\xi_{x,x+h}) \frac{h^2}{2}$$

$$\left| \frac{\varphi(x+h) - \varphi(x)}{h} - \varphi'(x) \right| \leq \frac{h}{2} \|\phi''\|_\infty$$

$$\lim_{h \to 0^+} \left\| \frac{\phi(x+h) - \phi(x)}{h} - \phi'(x) \right\|_\infty \leq \lim_{h \to 0} \frac{h}{2} \|\varphi''\|_\infty = 0$$

Let $\phi \in C_c^\infty(\mathbb{R})$ $\phi \geq 0$.

We want to prove that $(T_f)'(\phi) \geq 0$.

$$(T_f)'(\phi) = -\int_{\mathbb{R}} f(x)\,\phi'(x)\,dx = -\int_{\mathbb{R}} \lim_{h\to 0} \frac{\phi(x+h)-\phi(x)}{h} f(x)\,dx$$

| by uniform convergence / Lebesgue dominated convergence

$$= -\lim_{h\to 0^+} \int_{\mathbb{R}} f(x) \frac{\phi(x+h)-\phi(x)}{h} = \lim_{h\to 0^+} \frac{1}{h} \int_{\mathbb{R}} \underbrace{f(x)\phi(x) - f(x)\phi(x+h)}\,dx$$

change of variable $x \to x-h$

$$= \lim_{h\to 0^+} \frac{1}{h}\left[\int_{\mathbb{R}} f(x)\phi(x)\,dx - \int_{\mathbb{R}} f(x-h)\phi(x)\,dx\right] =$$

$$= \lim_{h\to 0^+} \underbrace{\frac{1}{h}}_{\geq 0} \int_{\mathbb{R}} \underbrace{[f(x)-f(x-h)]}_{\geq 0} \underbrace{\phi(x)}_{\geq 0}\,dx \geq 0 \qquad \square.$$

since f is MONOTONE NON DECREASING $x \geq x-h$

Let $C = \bigcap C_n$ Cantor set

$C_0 = [0\,1]$ $\quad C_1 = \frac{1}{3}C_0 \cup \left(\frac{1}{3}C_0 + \frac{2}{3}\right)$ $\qquad C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{1}{3}C_n + \frac{2}{3}\right)$.

$f : [0\,1] \to [0,1]$ Cantor function (DEVIL'S STAIRCASE)

$f(0) = 0$ $\quad f(1) = 1$ $\qquad$ (so f can be extended to $\mathbb{R}$
$f(x) = 0 \;\; x \le 0 \quad f(x) = 1 \;\; x \ge 1$)

f CONTINUOUS, MONOTONE NON DECREASING, CONSTANT ON EVERY INTERVAL NOT INTERSECTING C.

$f : C \to [0\,1]$ $\qquad$ write $x \in C$ in base 3 $\to$ it can be written as $0, x_1 x_2 \ldots$ $\quad x_i \in \{0, 2\}$

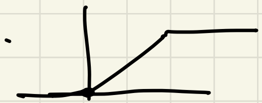
$x \in C \longrightarrow 0, \frac{x_1}{2} \frac{x_2}{2} \ldots \in [0,1]$ (number in $[0,1]$ written in base 2.)

$f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right) = \frac{1}{2}$ $\quad f\left(\frac{1}{9}\right) = f\left(\frac{2}{9}\right) = \frac{1}{4}$ $\quad f\left(\frac{7}{9}\right) = f\left(\frac{8}{9}\right) = \frac{3}{4}$ $\ldots$

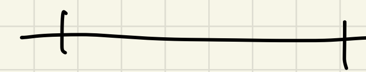
Another way to construct it

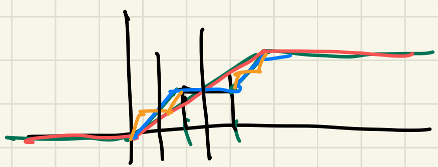
$f_0 : \mathbb{R} \to \mathbb{R}$ $\qquad f_0(x) \equiv 0 \quad x \le 0 \qquad f_0(x) \equiv 1 \quad x \ge 1$

and f_0 linear in C_0.



$f_1 : \mathbb{R} \to \mathbb{R} \qquad f_1 = f_0$ on $\mathbb{R} \setminus C_0$





$f_1 = \frac{1}{2}$ in $C_0 \setminus C_1$

f_1 linear and continuous

$f_2 = f_1$ on $\mathbb{R} \setminus C_1$ + affine

$$f_2(x) = \begin{cases} \left(\frac{1}{2}\right)^2 & x \in (C_1 \setminus C_2) \cap [0, \frac{1}{2}] \\ \frac{1}{2} + \left(\frac{1}{2}\right)^2 & x \in (C_1 \setminus C_2) \cap [\frac{1}{2}, 1] \end{cases}$$

and so on $\qquad f_n(x) = f_{n-1}(x)$ on $\mathbb{R} \setminus C_{n-1}$

$$f_n(x) = \begin{cases} (1/2)^n & x \in C_{n-1} \setminus C_n \cap [0, (\frac{1}{2})^n] \\ (\frac{1}{2})^{n-1} + (\frac{1}{2})^n & x \in C_{n-1} \setminus C_n \cap [(\frac{1}{2})^n, (\frac{1}{2})^{n-1}] \\ \cdots & \cdots \end{cases} \quad + \text{ affine}$$

$\forall\, m > k \qquad f_m(x) = f_k(x) \quad \forall x \in \mathbb{R} \setminus C_k$

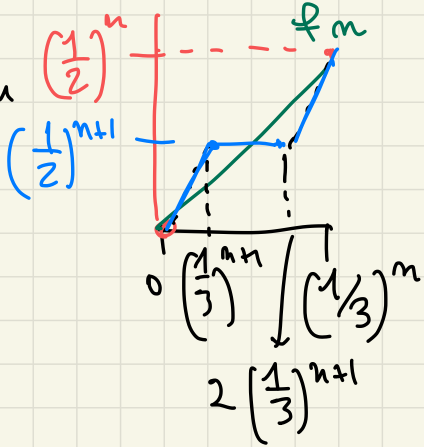
(and they are constant)

$$\|f_{n+1} - f_n\|_\infty \le \frac{1}{3}\left(\frac{1}{2}\right)^{n+1}$$

$\downarrow$

$f_{n+1} = f_n$ on $\mathbb{R} \setminus C_n$ by definition

on C_n



just look to $\left[0, \left(\frac{1}{3}\right)^n\right] \subseteq C_n$

(for all the other intervals is the same)

$$f_n\left(\frac{1}{3}\right)^{n+1} - f_{n+1}\left(\frac{1}{3}\right)^{n+1} = \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{3}\right)$$

f_n is a Cauchy sequence in $\mathcal{C}[0,1], \|\cdot\|_\infty$.

$f_n \to f$ Cantor function. ($f \in \mathcal{C}[0,1]$, monotone non decreasing, constant on intervals not intersecting C)

$\mathbb{R} \setminus C$ is union of intervals and $|C| = 0$

$f'(x) = 0$ a.e.

Nonetheless $(T_f)' \neq 0$ since if $(T_f)' = 0 \Rightarrow$

$0 = (T_f)'(\phi) = -\int_{\mathbb{R}} f \phi' dx \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}) \to$ Corollary of DB-R $f =$ constant a.e. (NOT TRUE!).

So $f' = 0$ a.e but $(T_f)' \neq 0$ $(T_f)' = T_\mu$ $\exists \mu$ since $(T_f)'$ is positive and so has order 0.

$$\forall \phi \in \mathcal{C}_c^\infty(U) \qquad \int_{\mathbb{R}} \phi'(x) f(x) dx = -\int_{\mathbb{R}} \phi(x) d\mu(x)$$

μ is supported on C (T_μ is supported on C).

Let $A \subseteq \mathbb{R}$ open set such that $A \cap C = \phi \Rightarrow$ take $\phi \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp} \phi \subseteq A$. $\int_A \phi' f \, dx = -\int_A \phi f' \, dx = 0$ since f is constant on every connected component of A

$\operatorname{supp} \mu = C$

$\mathbb{R} = \mathbb{R} \setminus C \cup C \qquad |C| = 0 \qquad \mu(\mathbb{R} \setminus C) = 0$

$\rightarrow \mu \perp \mathcal{L}$ $\quad$ μ is singular but it is not a sum of Dirac deltas.

μ is the derivative in the sense of distributions of the Cantor function.

$\downarrow$

Cantorian measure (measure supported on a set of measure 0, which is the derivative in the sense of distribution of a function f continuous with $f' = 0$ a.e)

More generally it is possible to prove the following!

Let f be a monotone non decreasing function in $\mathbb{R}$.

then the derivative in the sense of distributions of f is a Radon measure μ such that

$$\mu = \mu_{ac} + \mu_j + \mu_c$$

$\mu_{ac} << \mathcal{L}$ and the density of μ_{ac} is f' (derivative a.e of f) $\rightarrow f_{ac}(x) = \int_0^x f'(t)\,dt + f_{ac}(0)$

$\mu_j + \mu_c$ = singular part

$\mu_j = \sum_k \left[f(x_k^+) - f(x_k^-)\right] \delta_{x_k}$ where $\{x_k\}$ are the jumps of f (COUNTABLY MANY)

$f_j(x) = \sum_{k,\, x_k \le x} \left(f(x_k^+) - f(x_k^-)\right)$

μ_c = continuous part = derivative in the sense of distribution of f_c, continuous non decreasing function $f_c' = 0$ a.e.

$f - f_j$ is a continuous function, still non decreasing

$= f_{ac} + f_c$. f_c continuous

PRODUCT of CONVOLUTION

$T \in \mathcal{D}'(U)$, $\phi \in C_c^\infty(\mathbb{R}^n) \to \forall x \in \mathbb{R}^n \quad \phi^x(y) := \phi(x-y)$

$V_\phi =$ open set in $\mathbb{R}^n$ $= \{x \in \mathbb{R}^n \quad x-y \in U \quad \forall y \in \operatorname{supp}\phi\}$

$x \in V_\phi \iff x - \operatorname{supp}\phi \subseteq U$ (V_ϕ can also be empty!)

If $V_\phi \neq \emptyset$, if $x \in V_\phi \Rightarrow \operatorname{supp}\phi^x \subseteq U$

$x \in V_\phi$

$y \in \operatorname{supp}\phi^x \iff x-y \in \operatorname{supp}\phi \iff y \in x - \operatorname{supp}\phi \iff y \in U$

for the definition of V_ϕ

If $x \in V_\phi \Rightarrow \operatorname{supp}\phi^x \subseteq U \quad \phi^x \in C_c^\infty(U)$

We define $\underbrace{T * \phi}(x) := T(\phi^x) \quad \forall x \in V_\phi$.

$T * \phi : V_\phi \longrightarrow \mathbb{R}$ it is a function!
$x \longmapsto T(\phi^x)$

[if $U = \mathbb{R}^n \Rightarrow V_\phi = \mathbb{R}^n$
$\forall \phi \in C_c^\infty(\mathbb{R}^n)$]

Obs. $f \in L^1_{loc}(U)$ $\quad T = T_f$

$$T * \phi(x) = T(\phi^x) = T_f(\phi^x) = \int_U f(y)\, \phi^x(y)\, dy =$$

$$= \int_U f(y)\, \phi(x-y)\, dy = f * \phi(x)$$

$T * \phi = f * \phi$ if $T = T_f$

Proposition

1) $T * \phi \in C(V_\phi)$ $\qquad x_n \longrightarrow x$ in $V_\phi \Rightarrow \phi^{x_n} \to \phi^x$ in $C_c^\infty(U)$

$\Rightarrow T(\phi^{x_n}) \to T(\phi^x) \Rightarrow T * \phi(x_n) \to T * \phi(x)$

2) δ_0 is the unit of convolution

$$\delta_0 * \phi(x) := \delta_0(\phi^x) = \delta_0(\phi(x - \cdot)) = \phi(x-0) = \phi(x)$$

3) $\forall \alpha \in \mathbb{N}^n$ $\qquad D^\alpha(T * \phi)(x) = (D^\alpha T) * \phi(x) = [T * (D^\alpha \phi)](x)$

$\Longrightarrow T * \phi \in C^\infty(V_\phi)$.

proof $\alpha = e_1$ $\qquad x \in V_\phi \Rightarrow x + he_1 \in V_\phi$ $\quad$ h small

$$\frac{T(\phi^{x+he_1}) - T(\phi^x)}{h} = T\Bigg(\underbrace{\frac{\phi^{x+he_1} - \phi^x}{h}}\Bigg) \xrightarrow[h \to 0]{} T\left(\frac{\partial}{\partial x_1}\phi^x\right)$$

$$\forall y \quad \frac{\phi^{x+he_1}(y) - \phi^x(y)}{t} = \frac{\phi(x+he_1-y) - \phi(x-y)}{t} \to \frac{\partial \phi}{\partial x_1}(x-y)$$

$$\Downarrow$$

$$\lim_{t \to 0} \frac{T*\phi(x+he_1) - T*\phi(x)}{t} = T\left(\frac{\partial}{\partial x_1}\phi^x\right) =$$

$$= \left(T * \frac{\partial}{\partial x_1}\phi\right)(x)$$

So for every order α. $\qquad D^\alpha(T*\phi)(x) = (T * D^\alpha \phi)(x)$

$$\left(\frac{\partial}{\partial x_1}T\right) * \phi(x) = \frac{\partial}{\partial x_1} T(\phi^x) = -T\left(\frac{\partial}{\partial y_1}\phi^x\right) = -T\left((-1)\frac{\partial}{\partial x_1}\phi(x-\cdot)\right)$$

$$= T\left(\frac{\partial}{\partial x_1}\phi(x-\cdot)\right) = T\left(\frac{\partial}{\partial x_1}\phi^x\right) = \left[T * \frac{\partial}{\partial x_1}\phi\right](x).$$