FOGLIO 3 : basi, lemma scambio, dimensione

↳ il n° di vettori di una base di V è minore/uguale al n° di vettori di un insieme di generatori di V

* B è base di uno spazio vettoriale V se è contemporaneamente
  → un insieme massimale di vettori linearmente indipendenti
  → un insieme minimale di generatori per V

* Ogni base di uno spazio vettoriale ha la medesima cardinalità e coincide con la dimensione dello spazio vettoriale

FOGLIO 3

ES 1 → d) $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} 2x + y + z = 0 \\ y - z = 0 \end{cases} \right\}$

$$\begin{cases} y = z \\ x = (-z - y)\frac{1}{2} \end{cases} \qquad \begin{cases} y = z \\ x = -z \end{cases} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad B_U = \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \qquad \Rightarrow \dim U = 1$$

ES 2 → b) $U = \left\{ \begin{pmatrix} a+c \\ b+c \\ a+c \\ -c \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; a, b, c \in \mathbb{R} \right\}$

$$\begin{pmatrix} a+c \\ b+c \\ a+c \\ -c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

verifichiamo che i tre vettori che generano U siano l.i.:

$$\begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{matrix} \xrightarrow{III - I} \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{matrix} \xrightarrow{III - II} \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} \quad ✓$$

$$B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\} \quad \Rightarrow \dim U = 3$$

ricaviamo le eq. cartesiane di U: (4 inc - 3 par = 1 eq)

$$\begin{cases} x = a + c \\ y = b + c \\ z = a + c \\ t = -c \end{cases} \rightarrow c = -t \qquad \begin{cases} x = a - t \\ y = b - t \\ z = a - t \end{cases} \begin{matrix} \\ \rightarrow b = y + t \\ \rightarrow a = z + t \end{matrix} \qquad \{ x = z + t - t \quad \Rightarrow \quad U : \{ x = z$$

d) $U = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{cases} 2x + y + t = 0 \\ y - z + 2t = 0 \end{cases} \right\}$

$$y = z - 2t$$

$$x = \frac{1}{2}(-y - t) = \frac{1}{2}(-z + 2t - t) = -\frac{1}{2}z + \frac{1}{2}t$$

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z + \frac{1}{2}t \\ z - 2t \\ z \\ t \end{pmatrix} = z \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ -2 \\ 0 \\ 1 \end{pmatrix} \Rightarrow B_U = \left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -1 \\ 2 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$\Rightarrow \dim U = 2$$

ESERCIZIO 2 cerchiamo un insieme di generatori di U e verifichiamo che siano l.i.

a) $U = \{(a, b, a+b, -b) \in \mathbb{R}^4 \mid a, b \in \mathbb{R}\}$ trovare una base e scrivere U in forma cartesiana

$$\begin{pmatrix} a \\ b \\ a+b \\ -b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\rangle$$

non sono multipli l'uno dell'altro

sono dei generatori di U ma sono anche lin. indip.

$$\Rightarrow B_U = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right\} \Rightarrow \dim U = 2 = \# B_U$$

NB. 4 incognite
2 parametri
$\Rightarrow 4-2 = 2$ eq

$$\begin{cases} x = a \\ y = b \\ z = a+b \\ t = -b \end{cases} \rightarrow \begin{matrix} a = x \\ b = y \end{matrix} \quad \begin{cases} z = x+y \\ t = -y \end{cases} \rightarrow \begin{cases} x = z - y \\ y = -t \end{cases}$$

oppure considero la generica eq. lineare omogenea nelle variabili x, y, z, t

$\alpha \cdot x + \beta \cdot y + \gamma \cdot z + \delta \cdot t = 0$ con $\alpha, \beta, \gamma, \delta \in \mathbb{R}$

impogniamo che questa eq. sia soddisfatta dai vettori che generano U

$\Rightarrow$ sarà soddisfatta da tutti gli $u \in U$!

$$\begin{cases} \alpha + \gamma = 0 \\ \beta + \gamma - \delta = 0 \end{cases} \rightarrow \begin{cases} \alpha = -\gamma \\ \beta = -\gamma + \delta \end{cases} \quad (\gamma, \delta) \begin{matrix} \rightarrow (1, 0) \rightarrow \\ \rightarrow (0, 1) \rightarrow \end{matrix} \begin{cases} -x - y + z = 0 \\ y + t = 0 \end{cases}$$

(γ, δ) var. libere

ESERCIZIO 3

e) base W_1, W_2, $W_1 \cap W_2$, $W_1 + W_2$ e dimensioni

$$W_1 = \{(x,y,z,t) \in \mathbb{R}^4 \mid x + 2y + z - t = 0\}$$

$$W_2 = \langle (1,0,0,0), (1,0,1,2), (1,0,1,0) \rangle$$

W_1: $\rightarrow x = -2y - z + t$

$$\begin{pmatrix} -2y - z + t \\ y \\ z \\ t \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathcal{B}_{W_1} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \dim W_1 = 3$$

W_2: per trovare una base di W_2 estraggo dai generatori dati un sottoinsieme linearmente indipendente massimale

$$\Rightarrow \mathcal{B}_{W_2} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{sono LI}$$

$\dim W_2 = 3$

$W_1 \cap W_2$:

metodo con eq. cartesiane :

W_2 in forma cartesiana $\rightarrow \begin{cases} x = a+b+c \\ y = 0 \\ z = b+c \\ t = 2b \end{cases} \Rightarrow y = 0$

4 incognite
3 parametri $\Rightarrow$ 1 eq.

$$\Rightarrow W_1 \cap W_2 : \begin{cases} x + 2y + z - t = 0 \\ y = 0 \end{cases} \rightarrow \begin{cases} x = -z + t \\ y = 0 \end{cases}$$

$$\begin{pmatrix} -z+t \\ 0 \\ z \\ t \end{pmatrix} = z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow B_{W_2 \cap W_1} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\dim W_1 \cap W_2 = 2$

metodo con le basi : $v \in W_1 \cap W_2 \Rightarrow v \in W_1$ e $v \in W_2$

$$\alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \delta \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \sigma \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

risolvo il sistema :

$$\left(\begin{array}{cccccc|c} -2 & -1 & 1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{c} \begin{array}{cccccc} \alpha & \beta & \gamma & \delta & \sigma & \varepsilon \end{array} \\ \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right) \end{array}$$

$$\begin{cases} \alpha = 0 \\ \beta - \sigma - \varepsilon = 0 \\ \gamma - 2\sigma = 0 \\ \delta + 2\varepsilon = 0 \end{cases} \qquad \begin{cases} \alpha = 0 \\ \beta = \sigma + \varepsilon \\ \gamma = 2\sigma \\ \delta = 2\varepsilon \end{cases}$$

$$\Rightarrow 0 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \sigma + \varepsilon \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 2\sigma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -\varepsilon + \sigma \\ 0 \\ \varepsilon + \sigma \\ 2\sigma \end{pmatrix} = \sigma \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + \varepsilon \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

altro metodo:

conosco eq. cartesiana di W_1: $x + 2y + z - t = 0$

scrivo W_2 in forma parametrica $W_2: \begin{cases} x = a+b+c \\ y = 0 \\ z = b+c \\ t = 2b \end{cases}$

verifico per quali a, b, c $\begin{pmatrix} a+b+c \\ 0 \\ b+c \\ 2b \end{pmatrix}$ soddisfa l'eq. cartesiana di W_1:

$$\begin{cases} (a+\cancel{b}+c) + 2(0) + (\cancel{b}+c) - (\cancel{2b}) = 0 \\ a + 2c = 0 \\ a = -2c \end{cases} \Rightarrow \begin{pmatrix} a+b+c \\ 0 \\ b+c \\ 2b \end{pmatrix} = \begin{pmatrix} -c+b \\ 0 \\ b+c \\ 2b \end{pmatrix} = b\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} + c\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$B_{W_1 \cap W_2} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

dalla formula di Grassmann so che:

$$\dim(W_1 + W_2) = \underset{\substack{\parallel \\ 3}}{\dim W_1} + \underset{\substack{\parallel \\ 3}}{\dim W_2} - \underset{\substack{\parallel \\ 2}}{\dim W_1 \cap W_2}$$

$\Rightarrow \dim W_1 + W_2 = 4 = \dim \mathbb{R}^4$

$\Rightarrow$ una base di $W_1 + W_2$ é la base canonica di $\mathbb{R}^4$

$$\Rightarrow B_{W_1 + W_2} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Es 5 $\rightarrow$ c) $\quad S = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \right\}$

per verificare se i vettori sono l.i. li metto in riga e riduco in forma a scala la matrice ottenuta (il sottospazio generato dalle righe non viene alterato dalle operazioni elementari sulle righe). Le righe non nulle sono l.i.

$$\begin{matrix}1&0&0&0\\0&1&1&1\\1&1&1&0\\1&1&1&1\end{matrix} \xrightarrow{\substack{III-I\\IV-I}} \begin{matrix}1&0&0&0\\0&1&1&1\\0&1&1&0\\0&1&1&1\end{matrix} \xrightarrow{\substack{III-II\\IV-II}} \begin{matrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&0\end{matrix}$$

$\left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$ sono l.i $\Rightarrow B_S = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\} \Rightarrow \dim S = 3$

essendo che $\dim \mathbb{R}^4 = 4 \Rightarrow S$ non è un insieme di generatori di $\mathbb{R}^4$

per completare S ad una base di $\mathbb{R}^4$ "aggiungo il pivot mancante":

$$\begin{matrix}\underline{1}&0&0&0\\0&\underline{1}&1&1\\0&0&0&\underline{-1}\\0&0&0&0\end{matrix} \rightarrow$$

dalla matrice manca un pivot nella terza colonna

$\Rightarrow$ scelgo $\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$$\Rightarrow B_{\mathbb{R}^4} = B_S + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \right\}$$

ES 6 $\rightarrow$ a) $V = \left\{ \begin{pmatrix} b \\ b \\ a+b \end{pmatrix} \in \mathbb{R}^3 \mid a, b \in \mathbb{R} \right\}$

$$\begin{pmatrix} b \\ b \\ a+b \end{pmatrix} = b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + a \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow B_V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \Rightarrow \dim V = 2$$

W t.c. $V \oplus W = \mathbb{R}^3 \Rightarrow \underbrace{\dim V \oplus W}_{3} = \underbrace{\dim V}_{2} + \dim W - \underbrace{\dim V \cap W}_{0}$

$\dim W = 1 \Rightarrow$ cerco il vettore che completa V a una base di $\mathbb{R}^3$ (ovvero il pivot mancante)

$$\left.\begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}\right\} V \qquad V \oplus W = \mathbb{R}^3 \qquad W = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$$
$$\left.\begin{matrix} 0 & 1 & 0 \end{matrix}\right\} W$$

b) $V = \left\{ \begin{pmatrix} 2a \\ a \\ -3a \end{pmatrix} \in \mathbb{R}^3 \mid a \in \mathbb{R}^3 \right\}$ $\begin{pmatrix} 2a \\ a \\ -3a \end{pmatrix} = a \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ $B_V = \left\{ \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \right\}$

W é generato dai vettori che completano V a una base di $\mathbb{R}^3$

$$\left.\begin{matrix} 2 & 1 & 3 \end{matrix}\right\} V$$
$$\left.\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\} W \Rightarrow W = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

ES 7 $\Rightarrow$ d) $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{matrix} 2x + t = 0 \\ x - z + 2t = 0 \end{matrix} \right\}$

cerco dei generatori di V: $\left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 2 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{II - 2I} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 2 & -3 & 0 \end{array}\right)$

$$\begin{cases} 2z - 3t = 0 \\ x - z + 2t = 0 \end{cases} \qquad \begin{cases} z = \frac{3}{2} t \\ x = -2t + z \end{cases} \qquad \begin{cases} z = \frac{3}{2} t \\ x = -\frac{1}{2} t \end{cases}$$

$$\begin{pmatrix} -\frac{1}{2} t \\ y \\ \frac{3}{2} t \\ t \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{t}{2} \begin{pmatrix} -1 \\ 0 \\ 3 \\ 2 \end{pmatrix} \Rightarrow B_V = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 3 \\ 2 \end{pmatrix} \right\} \quad \dim V = 2$$

$V \oplus W = \mathbb{R}^4 \Rightarrow \dim W = 2$

$$\begin{matrix} -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \end{matrix} \Big\} V$$

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \Big\} W \Rightarrow B_W = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Foglio 5 es 9 a)

$$V_\lambda = \left\langle \begin{pmatrix}1\\\lambda\\0\end{pmatrix} \begin{pmatrix}-\lambda\\-3\\2\end{pmatrix} \begin{pmatrix}2\\1\\\lambda\end{pmatrix} \begin{pmatrix}0\\-\lambda\\1\end{pmatrix} \right\rangle \quad \forall \lambda \in \mathbb{R} \quad \dim V\,?$$

La dimensione di un sottospazio è uguale alla cardinalità della sua base

$\Rightarrow$ cerchiamo una base di V_λ $\forall \lambda \in \mathbb{R}$.

$$\begin{matrix}1 & \lambda & 0\\ 0 & -\lambda & 1\\ 2 & 1 & \lambda\\ -\lambda & -3 & 2\end{matrix} \xrightarrow[IV+\lambda I]{III-2I} \begin{matrix}1 & \lambda & 0\\ 0 & -\lambda & 1\\ 0 & 1-2\lambda & \lambda\\ 0 & -3+\lambda^2 & 2\end{matrix} \xrightarrow[IV + II\,\frac{\lambda^2-3}{\lambda} \iff \lambda \neq 0]{III + II \cdot \frac{1-2\lambda}{\lambda} \iff \lambda \neq 0}$$

valuto prima i casi per $\lambda = 0$

$$\begin{matrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 0 & -3 & 2\end{matrix} \longrightarrow \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -3 & 2\end{matrix} \xrightarrow{IV+3II-2III} \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} = A_{V_{\lambda=0}}$$

$$\boxed{B_{V_{\lambda=0}} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \quad \dim(V_{\lambda=0}) = 3 = rk(A_{V_{\lambda=0}})}$$

ora posso considerare i casi $\lambda \neq 0$ e quindi dividere per λ:

$$\begin{matrix}1 & \lambda & 0\\ 0 & -\lambda & 1\\ 0 & 0 & \frac{\lambda^2-2\lambda+1}{\lambda}\\ 0 & 0 & \frac{\lambda^2+2\lambda-3}{\lambda}\end{matrix} \xrightarrow{IV - III \cdot \frac{\lambda^2+2\lambda-3}{\lambda^2-2\lambda+1}} \iff \begin{aligned} \lambda^2-2\lambda+1 &\neq 0\\ (\lambda-1)^2 &\neq 0\\ \lambda &\neq 1\end{aligned}$$

valuto prima il caso $\lambda = 1$:

$$\begin{pmatrix}1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = A_{V_{\lambda=1}}$$

$$\boxed{B_{V_{\lambda=1}} = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix} \begin{pmatrix}0\\-1\\1\end{pmatrix} \right\} \quad \dim V_{\lambda=1} = 2 = rk\, A_{V_\lambda}}$$

ora considero i casi $\lambda \neq 1$ (posso dividere per $(\lambda-1)^2$)

$$\begin{pmatrix}1 & \lambda & 0\\ 0 & -\lambda & 1\\ 0 & 0 & \frac{(\lambda-1)^2}{\lambda}\\ 0 & 0 & 0\end{pmatrix} = A_{V_{\lambda\neq 0,1}}$$

$$\boxed{B_{V_{\lambda\neq 0,1}} = \left\{ \begin{pmatrix}1\\\lambda\\0\end{pmatrix} \begin{pmatrix}0\\-\lambda\\1\end{pmatrix} \begin{pmatrix}0\\0\\\frac{(\lambda-1)^2}{\lambda}\end{pmatrix} \right\} \quad \dim V_\lambda = 3 = rk\, A_{V_\lambda} \text{ per } \lambda \neq 0,1}$$

Risposta: $\dim V_\lambda = 2$ per $\lambda = 1$; $\dim V_\lambda = 3$ per $\lambda \neq 1$

FOGLIO 5 - es. 9

b) $V_\lambda = \left\langle \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ \lambda^2/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ \lambda-2 \\ \lambda \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ -\lambda \end{pmatrix} \right\rangle$ $\quad$ $\dim V = ?$ $\quad \forall \lambda \in \mathbb{R}$

$$A_\lambda = \begin{pmatrix} 1/2 & 0 & 1 & \lambda^2/2 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & \lambda-2 & \lambda \\ 1 & -1 & 2 & -\lambda \end{pmatrix} \xrightarrow{\mathrm{I}\cdot 2} \begin{pmatrix} 1 & 0 & 2 & \lambda^2 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & \lambda-2 & \lambda \\ 1 & -1 & 2 & -\lambda \end{pmatrix} \xrightarrow[-(\mathrm{IV}-\mathrm{I})]{\mathrm{III}+\mathrm{I}} \begin{pmatrix} 1 & 0 & 2 & \lambda^2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & \lambda & \lambda+\lambda^2 \\ 0 & +1 & 0 & +\lambda+\lambda^2 \end{pmatrix}$$

$$\xrightarrow[\mathrm{IV}-\mathrm{II}]{\mathrm{III}-2\mathrm{II}} \begin{pmatrix} 1 & 0 & 2 & \lambda^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & \lambda+\lambda^2 \\ 0 & 0 & 0 & \lambda+\lambda^2 \end{pmatrix}$$

caso $\lambda = 0 \rightsquigarrow A_0 = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad$ $\dim V_0 = 2$

caso $\left.\begin{array}{l} \lambda + \lambda^2 = 0 \\ \lambda(\lambda+1) = 0 \end{array}\right| \Rightarrow \lambda = -1 \rightsquigarrow {}_{-1} = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad$ $\dim V_{-1} = 3$

caso $\lambda \neq 0, -1 \Rightarrow \dim V_\lambda = 4$

$$B_{V_0} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$B_{V_{-1}} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$B_{V_\lambda} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ \lambda^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \lambda \\ \lambda+\lambda^2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lambda+\lambda^2 \end{pmatrix} \right\}$$ per $\lambda \neq 0, -1$

ESERCIZIO → In $\mathbb{R}^5$ trovare una base del sottospazio W

$$W = \left\langle \begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix} \begin{pmatrix}2\\-1\\1\\1\\-1\end{pmatrix} \begin{pmatrix}0\\-1\\-1\\1\\-3\end{pmatrix} \begin{pmatrix}0\\1\\-1\\2\\3\end{pmatrix} \begin{pmatrix}2\\-3\\3\\-3\\-7\end{pmatrix} \right\rangle$$

trovare un sottospazio W' t.c. $W \oplus W' = \mathbb{R}^5$

operando sui generatori di W operazioni elementari si trova una base di W

$$\begin{matrix}1&0&1&0&1\\2&-1&1&1&-1\\0&-1&-1&1&-3\\0&1&-1&2&3\\2&-3&3&-3&-7\end{matrix} \longrightarrow \begin{matrix}1&0&1&0&1\\0&1&-1&2&3\\0&-1&-1&1&-3\\0&-1&-1&1&-3\\0&-2&2&-2&-6\end{matrix} \longrightarrow \begin{matrix}1&0&1&0&1\\0&1&-1&2&3\\0&0&-2&3&0\\0&0&0&0&0\\0&0&0&0&0\end{matrix}$$

$$\mathcal{B}_W = \left\{ \begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix} \begin{pmatrix}0\\1\\-1\\2\\3\end{pmatrix} \begin{pmatrix}0\\0\\-2\\3\\0\end{pmatrix} \right\} \qquad \dim W = 3$$

$$W' \text{ t.c. } \underbrace{W}_{\dim 3} \oplus W' = \underbrace{\mathbb{R}^5}_{\dim 5} \Rightarrow W' \dim 2$$

$$\mathcal{B}_{W'} = \left\{ \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\}$$

ESERCIZIO → trovare delle eq. cartesiane per il sottospazio

$V = \langle (3,-1,1,4)(1,1,1,2)(0,2,1,1) \rangle$

$$\left(\begin{array}{ccc|c}3&1&0&x\\-1&1&2&y\\1&1&1&z\\4&2&1&t\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1&1&1&z\\0&2&3&z+y\\0&-2&-3&x-3z\\0&-2&-3&t-4z\end{array}\right)$$

$$\left(\begin{array}{ccc|c}1&1&1&z\\0&2&3&z+y\\0&0&0&x-3z+z+y\\0&0&0&t-4z+z+y\end{array}\right) \qquad \begin{cases} x+y-2z=0 \\ y-3z+t=0 \end{cases}$$

FOGLIO 4 - ES3

$V = < \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -4 \\ -3 \end{pmatrix} >$ eq cartesiane? dim V?

dim V $\rightarrow$ $\begin{matrix} 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ -1 & 1 & -4 & -3 \end{matrix} \xrightarrow[\text{III}+\text{I}]{\text{II}-\text{I}} \begin{matrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & -3 \end{matrix} \xrightarrow{\text{III}-\text{II}} \begin{matrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{matrix}$ ⊛

$\rightarrow$ dim V = 2

eq cartesiane $V = \left\{ a \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; a, b \in \mathbb{R} \right\}$

ho 4 incognite - 2 parametri $\Rightarrow$ avrò 2 equazioni

$\begin{cases} x = a + b \\ y = 2a + b \\ z = a + 2b \\ t = b \end{cases} \quad \begin{cases} x = a + t \\ y = 2a + t \\ z = a + 2t \\ b = t \end{cases} \quad \begin{cases} a = x - t \\ y = 2x - t \\ z = x + t \\ b = t \end{cases} \Rightarrow \begin{cases} t + y - 2x = 0 \\ z - x - t = 0 \end{cases}$

V' di $\mathbb{R}^4$ t.c. $V \subseteq V'$ e $\dim V' = \dim V + 1$

$V' : \{ -2x + y + t = 0 \rightarrow V' = < \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} >$

$\dim V' = 2 + 1 = 3$

oppure potevo prendere $V' = < \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} >$

è l.i. rispetto agli altri due vettori perché "completa" i pivot mancanti in ⊛ (andava bene anche $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$)

$\begin{cases} x = a + b \\ y = 2a + b \\ z = a + 2b \\ t = b + c \end{cases} \quad \left. \begin{matrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{matrix} \right| \begin{matrix} x \\ y \\ z \\ t \end{matrix} \rightarrow \left. \begin{matrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{matrix} \right| \begin{matrix} x \\ y - 2x \\ z - x \\ t \end{matrix}$

$\left. \begin{matrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \right| \begin{matrix} x \\ y - 2x \\ z - x + y - 2x \\ t + y - 2x \end{matrix} \rightarrow \{ -3x + z + y = 0$

$$\begin{cases} x = a+b \\ y = 2a+b \\ z = a+2b \\ t = b+c \end{cases} \rightarrow a = \cancel{x} - b \quad \begin{cases} y = 2(x-b)+b = 2x-2b+b = 2x-b \\ z = x-b+2b = x+b \\ t = b+c \end{cases}$$

$$\begin{cases} y = 2x-b \\ z = x+b \\ t = b+c \end{cases} \rightarrow b = 2x-y \quad \begin{cases} z = x+2x-y \\ t = 2x-y+c \end{cases} \quad \begin{cases} z = 3x-y \end{cases}$$

$$\rightarrow c = t-2x+y$$

$$V' : \begin{cases} 3x - y + z = 0 \end{cases}$$