Def SUPPORT of DISTRIBUTION: $\operatorname{supp} T = U \setminus \omega_0$ (closed in U)

where $\omega_0 = \bigcup \{ A \subseteq U \text{ open s.t. } \forall \phi \in C_c^\infty(U) \ \operatorname{supp} \phi \subseteq A \ \ T\varphi = 0 \}$.

So if $\phi \equiv 0$ on some A open set with $\operatorname{supp} T \subseteq A \Rightarrow T\varphi = 0$

If $\phi = 0$ on $\operatorname{supp} T \not\Rightarrow T\phi = 0$ $\left(\begin{array}{l} \text{ex: } T(\varphi) = \varphi'(0) \text{ in } \mathcal{D}'(\mathbb{R}) \ \operatorname{supp} T = \{0\} \\ \varphi(0) = 0 \not\Rightarrow \varphi'(0) = 0 \, ! \end{array} \right)$

Prop If $\operatorname{supp} T$ is COMPACT then T has finite order

proof $K = \operatorname{supp} T$ (compact)

Let $K \subset\subset U_1 \subset\subset U_2 \subset\subset U$ $\quad U_1, U_2$ bdd open in U

Take $\psi \in C_c^\infty(U)$ such that $\psi \equiv 1$ on $\overline{U_1}$ $\operatorname{supp} \psi \subseteq \overline{U_2}$

$\Rightarrow \forall \phi \in C_c^\infty(U) \quad \varphi \cdot \psi \in C_c^\infty(U) \ \operatorname{supp} \phi\psi \subseteq \overline{U_2}$

$\varphi(1-\psi) = 0$ in $U_1 \Rightarrow T(\varphi(1-\psi)) = 0 \Rightarrow T\varphi = T(\varphi\psi)$

$$|T\varphi| = |T(\varphi\psi)| \leq C_{K_2} \sup_{|\alpha| \leq p_{K_2}} \|D^\alpha \varphi\psi\|_\infty \leq \tilde{C}_{K_2}(\psi) \sup_{|\alpha| \leq p_{K_2}} \|D^\alpha \varphi\|_\infty \to \text{ORDER} \leq p_{K_2}$$

DERIVATIVES IN THE SENSE OF DISTRIBUTIONS

$\frac{\partial}{\partial x} T(\phi) = (-1) T(\frac{\partial}{\partial x_i} \phi)$

Def Let $f \in L^1_{loc}(U)$ and $T_f(\phi) = \int_U \phi f \, dx \quad \forall \phi \in C^\infty_c(U)$

Then f admits WEAK DERIVATIVE in the e_i-direct.

if $\exists v_i \in L^1_{loc}(U)$ s.that $\frac{\partial}{\partial x_i} T_f = T_{v_i}$

that is $\forall \phi \in C^\infty_c(U) \quad (-1) \int_U f(x) \frac{\partial}{\partial x_i} \phi \, dx = \int_U v_i \phi \, dx$.

WEAK GRADIENT $Df = (v_1, \dots, v_n)$

Prop If a weak derivative exists it is unique.

Assume not true v_i, w_i weak derivatives of f in the e_i-direction

$\Downarrow$

$$\int_U v_i \phi \, dx = -\int_U f \frac{\partial \phi}{\partial x_i} dx = \int_U w_i \phi \, dx \Rightarrow \int_U (v_i - w_i) \phi \, dx = 0 \quad \forall \phi \in C^\infty_c(U)$$

$\Rightarrow$ LEMMA DI DEBOIS-REYMOND $v_i - w_i = 0$ a.e $\Rightarrow v_i = w_i$ $\square$

Note that $\forall f \in L^1_{loc}(U)$, f admits a derivative in the sense of distribution $\left(\frac{\partial}{\partial x_i} T_f\right)$

but it is NOT ALWAYS TRUE that f admits a WEAK DERIVATIVE

Example 1: a function which admits a weak derivative
(a.e derivative coincides with weak derivative)

ex Let $f(x) = \frac{1}{|x|^r}$ in $\mathbb{R}^n$ with $r < n$ (so $f \in L^1_{loc}(\mathbb{R}^n)$)

If $r < n-1$ then f admits weak derivatives which are $\frac{\partial f}{\partial x_i} = -r \frac{x_i}{|x|^{r+2}}$

NB $|v_i| = +r \frac{|x_i|}{|x|^{r+2}} \leq r \frac{1}{|x|^{r+1}} \in L^1_{loc}(\mathbb{R}^n) \iff r+1 < n$

$\boxed{r < n-1}$

If $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\phi \subseteq \mathbb{R}^n \setminus \{0\}$

$$\int_{\mathbb{R}^n} \frac{1}{|x|^r} \frac{\partial \phi}{\partial x_i}\, dx = \int_U \frac{1}{|x|^r} \frac{\partial \phi}{\partial x_i}\, dx = -\int_U \left(-r \frac{x_i}{|x|^{r+2}}\right)\phi(x)\, dx$$

[$U \subseteq \mathbb{R}^n$ open set, class C^1, $0 \notin U$.
$\operatorname{supp}\phi \subset U$.

G.G (integration by parts)

If $\{0\} \in \operatorname{supp}\phi$ Let U open bdd set, class C^1, $0 \in U$
$\operatorname{supp}\phi \subseteq U$.

$$\int_{\mathbb{R}^n} \frac{1}{|x|^r} \frac{\partial \phi}{\partial x_i}\, dx = \lim_{\epsilon \to 0^+} \int_{U \setminus B(0,\epsilon)} \frac{\partial \phi}{\partial x_i} \frac{1}{|x|^r}\, dx =$$

(G.G / integration by parts)

$$= \lim_{\epsilon \to 0^+} \left[-\int_{U \setminus B(0,\epsilon)} \phi \cdot \left(-r \frac{x_i}{|x|^{r+2}}\right) dx + \cancel{\int_{\partial U} \phi \cdot \frac{1}{|x|^r}\, dS} + \int_{\partial B(0,\epsilon)} \phi(x) \cdot \frac{1}{|x|^r} \left(-\frac{x_i}{|x|}\right) d\mathcal{H}^{n-1} \right.$$

i-comp. of exterior normal to $U \setminus B(0,\epsilon)$
at $x \in \partial B(0,\epsilon)$

$$= \lim_{\varepsilon \to 0^+} -\int_{U \setminus B(0,\varepsilon)} \underbrace{\phi(x)}_{\in C_c^\infty(U)} \cdot \underbrace{\left[-r\frac{x_i}{|x|^{r+2}}\right]}_{\in L^1(U)} dx + \int_{\partial B(0,\varepsilon)} \phi(x) \cdot \frac{1}{\varepsilon^r} \cdot \left(-\frac{x_i}{|x|}\right) d\mathcal{H}^{n-1}$$

$$\left| \int_{\partial B(0,\varepsilon)} \phi(x) \frac{1}{\varepsilon^r} \left(-\frac{x_i}{|x|}\right) d\mathcal{H}^{n-1} \right| \le \int_{\partial B(0,\varepsilon)} |\phi(x)| \frac{1}{\varepsilon^r} d\mathcal{H}^{n-1} \le$$

$$\le \|\phi\|_\infty \frac{1}{\varepsilon^r} \cdot \mathcal{H}^{n-1}(\partial B(0,\varepsilon)) = \|\phi\|_\infty \frac{1}{\varepsilon^r} \omega_n n \varepsilon^{n-1} = \underbrace{\|\phi\|_\infty \omega_n n \varepsilon^{n-1-r}}_{\to 0 \text{ as } \varepsilon \to 0}$$

$n-1-r > 0$ by assumption.

$$= -\int_U \phi(x) \cdot \left[-r\frac{x_i}{|x|^{r+2}}\right] dx.$$

Obs In this case derivative a.e = weak derivative.

Not always true.

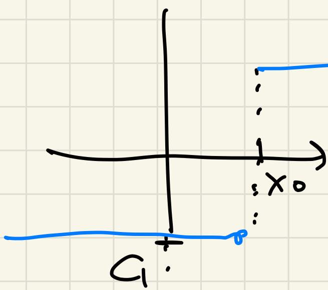
If f has weak derivative & f is differentiable a.e in U with $\frac{\partial f}{\partial x_i} \in L^1_{loc}(U)$ $\Longrightarrow$ WEAK DERIVATIVE = DERIVATIVE a.e.

Not always true that if a function in $L^1_{loc}(U)$ is differentiable a.e. $\Rightarrow$ its derivative a.e. is the weak derivative

ex 1 FUNCTIONS WITH JUMPS $\rightarrow$ derivative in the sense of distribution is a Dirac-delta

Let $f(x) = \begin{cases} c_1 & x < x_0 \\ c_2 & x > x_0 \end{cases}$ $\quad f \in L^1_{loc}(\mathbb{R})$

$f' = 0$ a.e. (derivative a.e.)



$$T_f(\phi) = \int_{\mathbb{R}} \phi f \, dx \qquad T \in \mathcal{D}'(\mathbb{R})$$

$$= \int_{-\infty}^{x_0} \phi c_1 + \int_{x_0}^{+\infty} \phi c_2 \, dx$$

$$(T_f)'(\varphi) = (-1)\int_{\mathbb{R}} \phi' f \, dx = (-1)\int_{-\infty}^{x_0} c_1 \phi' \, dx + (-1)\int_{x_0}^{+\infty} c_2 \phi' \, dx =$$

$$= (-1)\cdot c_1 \, \phi(x_0) + (-1) c_2 [-\phi(x_0)] = (c_2 - c_1)\,\phi(x_0) = (c_2 - c_1)\,\delta_{x_0}(\phi)$$

$(\phi \in C^\infty_c(\mathbb{R})\,!)$

the derivative of f in the sense of distributions is a measure $\quad \mu = \underbrace{(c_2 - c_1)}_{\text{height of the jump.}} \delta_{x_0}$ ← place where f jumps

$f(x_0^+) - f(x_0^-)$

$f(x_0^+) = \lim_{x \to x_0^+} f(x) \qquad f(x_0^-) = \lim_{x \to x_0^-} f(x)$

$\mu \perp \mathcal{L}$ so $\nexists\, v \in L^1_{loc}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v(x)\phi(x)\,dx = (c_2 - c_1)\,\varphi(x_0) \qquad \forall \varphi \in C_c^\infty(\mathbb{R})$$

$\Rightarrow$ f has NOT WEAK DERIVATIVE.

Exercise: take $f : \mathbb{R} \to \mathbb{R}$ $\quad f \in C^1(-\infty, x_1)$, $f \in C^1(x_1, x_2)$, $f \in C^1(x_2, +\infty)$ $\quad f(x_1^+) \neq f(x_1^-)$ finite $\quad f(x_2^+) \neq f(x_2^-)$ finite

then $(T_f)'(\varphi) = \int_U \phi f'(x)\,dx + (f(x_1^+) - f(x_1^-))\,\phi(x_1) +$
$+ f(x_2^+) - f(x_2^-))\,\phi(x_2)$.

$\mu_{ac} \ll \mathcal{L}$ density f'

$(T_f)' = T_\mu \quad \mu = \mu_{ac} + \mu_s \ldots \quad \mu_s \perp \mathcal{L} \quad \mu_s = (f(x_1^+) - f(x_1^-))\delta_{x_1} + (f(x_2^+) - f(x_2^-))\delta_{x_2}$

$f' = \frac{1}{x}$ a.e NOT THE WEAK DERIVATIVE BUT IS THE DERIVATIVE IN THE SENSE OF DISTRIBUTIONS

Ex 2 $\quad f(x) = \lg|x|$ in $\mathbb{R}$ $\quad f(x) \in L^1_{loc}(\mathbb{R})$

the derivative in the sense of distributions of $f(x)$ is p.v. $\frac{1}{x}$!

(DERIVATIVE IN THE SENSE OF DISTRIBUTION of f is A DISTR. OF ORDER 1 $\rightarrow$ NO FUNCTION...

proof $\quad T_f(\phi) = \int_{\mathbb{R}} \phi(x) \lg|x|\,dx$

| DERIVATIVE A.E = DERIVATIVE IN THE SENSE OF DISTRIB. f HAS NO WEAK DERIVATIVE

$$(T_f)'(\phi) = -\int_{\mathbb{R}} \phi'(x) \lg|x|\,dx = \lim_{\varepsilon \to 0^+} -\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \phi'(x)\lg|x|\,dx =$$

INT. BY PARTS

$$= \lim_{\varepsilon \to 0^+} \left[\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \phi(x)\frac{1}{x}\,dx + \lg(\varepsilon)\phi(\varepsilon) - \lg|-\varepsilon|\,\phi(-\varepsilon) \right] =$$

$$= \lim_{\varepsilon \to 0^+} \left[\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x} + \underbrace{(\log(\varepsilon)\varepsilon)}_{\downarrow\, 0} \underbrace{\left(\frac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{\varepsilon}\right)}_{\text{bounded}} \right] =$$

$$\left(\lim_{\varepsilon \to 0^+} \frac{\varphi(\varepsilon)-\varphi(-\varepsilon)}{\varepsilon} = 2\varphi'(0) \right)$$

$$= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x}$$

$$= \text{p.v}\,\frac{1}{x}(\varphi).$$

~ . ~

In conclusion: if $f \in L^1_{loc}(U)$. T_f is a distn. of order 0

$\frac{\partial T_f}{\partial x_i}$ is a distribution of order at most 1.

If $\frac{\partial}{\partial x_i} T_f$ is a distribution of order 0 $\quad \frac{\partial}{\partial x_i} T_f = T_\mu \quad \exists \mu$ signed Radon measure

$\mu \ll \mathcal{L} \Rightarrow \mu$ has a density $\frac{\partial f}{\partial x_i}$ = WEAK DERIVATIVE of f

$\mu = \mu_{ac} + \mu_s \quad \mu_{ac} \ll \mathcal{L} \quad \mu_s \perp \mathcal{L} \quad \mu_s \neq 0 \Rightarrow f$ has NOT WEAK DERIVATIVE