$\mathcal{D}'(U)$ = space of distributions =

$= \{ T : C_c^\infty(U) \to \mathbb{R}$ linear and continuous $\}$

(w.r.t. convergence in $C_c^\infty(U)$).

as a dual space it inherits a notion of convergence

$$T_n \to T \quad \text{if} \quad \forall \phi \in C_c^\infty(U) \quad T_n(\varphi) \to T(\varphi)$$

With the topology associated to this convergence $\mathcal{D}'(U)$ is a locally convex Hausdorff space, σ compact (NOT METRIZABLE)

- $f_n, f \in L^p_{loc}(U)$ $\quad T_{f_n}(\phi) := \int_U f_n \phi \, dx \qquad T_f(\varphi) = \int_U f \phi \, dx$

$f_n \rightharpoonup f$ weakly in $L^p(K)$ for all $K \subset\subset U$ then $T_{f_n} \to T_f$ in the sense of distribution

weakly* in $L^\infty(K)$ for all $K \subset\subset U$ then $T_{f_n} \to T_f$

$f_n \to f$ a.e $\not\Rightarrow$ $T_{f_n} \to T_f$

$$T_{\mu_n} \phi = \int_U \phi \, d\mu_n \qquad T_\mu \phi = \int_U \phi \, d\mu$$

$\mu_n \rightharpoonup \mu$ in $\mathcal{M}(U)$ $\Rightarrow$ $T_{\mu_n} \to T_\mu$

ORDER of the distribution T: minimum $p \in \mathbb{N}$ (if it exists) s.t. $\forall K \subset\subset U$ $\exists c_K$ for which $\forall \varphi \in C_c^\infty(U)$ supp $\phi \subseteq K$

$$|T(\varphi)| \leq c_K \sup_{|\alpha| \leq p} \|D^\alpha \phi\|_\infty$$

<u>Observation</u>: A distribution of order $p \geq 0$ can be extended to a continuous linear functional

$$T : \underbrace{C_c^p(U)}_{\text{we say } f_n \to f \text{ in } C_c^p(U)} \longrightarrow \mathbb{R}$$

we say $f_n \to f$ in $C_c^p(U)$ if $\exists K \subset\subset U$ such that supp f_n, supp $f \subseteq K$ $\|D^\alpha f_n - D^\alpha f\|_\infty \to 0$ $\forall |\alpha| \leq p$.

therefore: Distributions of order 0 on U are exactly signed Radon measure on U.

T distribution of order 0 $\longleftrightarrow$ $\exists \mu^+, \mu^- \in \mathcal{M}(U)$
$T\varphi = \int_U \phi \, d\mu^+ - \int \phi \, d\mu^-$

Prop If T is positive ($T\varphi \geq 0 \ \forall \varphi \geq 0$) then T has order 0

proof $\phi \in C_c^\infty(U)$ fix $K \supset \operatorname{supp} \phi$

Fix $\psi \in C_c^\infty(U)$ $\psi \equiv 1$ on K $0 \leq \psi \leq 1$

$-\|\varphi\|_\infty \psi \leq \phi \leq \|\phi\|_\infty \psi \implies |T\varphi| \leq \|\varphi\|_\infty T(\psi)$. $C_K = T\psi$ $\square$.

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Ex $T \in D'(\mathbb{R}^n)$ $T(\varphi) = \varphi'(0)$ has order 1

$|T\varphi| \leq \|\varphi'\|_\infty \ \forall \varphi \in C_c^\infty(\mathbb{R}) \implies$ order of $T \leq 1$.

Let $\psi \in C_c^\infty(\mathbb{R})$ $\psi'(0) = 1$. Let $\psi_n(x) = \psi(nx)$ $\psi_n'(0) = n$

$T\psi_n = \psi_n'(0) = n$

Let $K = [-R, R]$ such that $\operatorname{supp} \psi \subseteq [-R,R] \implies \operatorname{supp} \psi_n \subseteq [-R,R]$

If T is of order 0 $|T\psi_n| = \psi_n'(0) = n \leq C_K \|\psi_n\|_\infty = C_K\|\psi\|_\infty$ ABSURD

Another example of a distribution of order 1.

in $\mathcal{D}'(\mathbb{R})$. PRINCIPAL VALUE $\mathrm{p.v}\,\frac{1}{x}$

$$T(\varphi) := \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx \qquad \forall\, \phi \in \mathcal{C}^\infty_c(\mathbb{R})$$

well defined fix $\phi \in \mathcal{C}^\infty_c(\mathbb{R})$ $\exists\, a>0$ such that $\operatorname{supp}\phi \subset (-a,a)$

$$\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,dx = \int_{-a}^{-\varepsilon} \frac{\phi(x)}{x}\,dx + \int_{\varepsilon}^{a} \frac{\phi(x)}{x}\,dx = \int_{-a}^{-\varepsilon} \frac{\phi(x)-\phi(0)}{x} + \underbrace{\cancel{\int_{-a}^{-\varepsilon} \frac{\phi(0)}{x}\,dx}}$$

$$+ \int_{\varepsilon}^{a} \frac{\phi(x)-\phi(0)}{x}\,dx + \underbrace{\cancel{\int_{\varepsilon}^{a} \frac{\phi(0)}{x}\,dx}}$$

$\frac{\phi(0)}{x}$ is ODD !

$$\frac{\phi(x)-\varphi(0)}{x} = \frac{\int_0^x \phi'(y)\,dy}{x} \underset{y = xt}{=} \int_0^1 \frac{\phi'(tx)\,\cancel{x}\,dt}{\cancel{x}} = \int_0^1 \phi'(tx)\,dt$$

$$\lim_{\varepsilon\to 0^+} \int_{-a}^{-\varepsilon}\int_0^1 \phi'(tx)\,dt\,dx + \int_{\varepsilon}^{a}\int_0^1 \phi'(tx)\,dt\,dx =$$

$$= \int_{-a}^{a}\int_0^1 \phi'(tx)\,dt\,dx = \text{p.v.}\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\phi(x)}{x}\,dx$$

Let us compute the order of p.v. $\frac{1}{x}$.

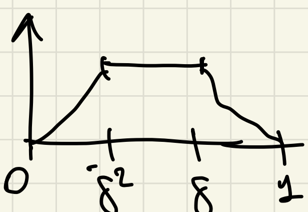
$$\left|\text{p.v}\frac{1}{x}(\varphi)\right| = \left|\int_{-a}^{a}\int_0^1 \phi'(tx)\,dt\,dx\right| \le \|\phi'\|_\infty\,(\operatorname{diam} K)$$

if $\operatorname{supp}\phi \subseteq K$.

ORDER ≤ 1

Note that it cannot have order 0

take ϕ_δ



$\delta > 0$ $\quad \phi_\delta \equiv 1$ on $[\delta^2, \delta]$ $\quad \varphi \ge 0\ \forall x$

$\phi_\delta \equiv 0 \quad x \le 0 \quad x \ge 1$

$\phi \in C_c^\infty(\mathbb{R})$

$$\text{p.v.}\frac{1}{x}(\phi_\delta) = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\phi_\delta(x)}{x} = \lim_{\varepsilon\to 0^+}\int_\varepsilon^{\delta^2} \frac{\phi_\delta(x)}{x} + \int_{\delta^2}^{\delta}\frac{\phi_\delta(x)}{x} + \int_\delta^1 \frac{\phi_\delta(x)}{x}\,dx$$

$$\ge \int_{\delta^2}^{\delta}\frac{1}{x}\,dx = \log\delta - \log\delta^2 = -\log\delta$$

p.v. $\frac{1}{x}$ is of order 1 (it cannot be of order 0 since $\forall \delta \in (0,1)$ $\exists \phi_\delta \in \mathcal{C}_c^\infty(\mathbb{R})$ supp $\phi_\delta \subseteq [0,1]$ p.v. $\frac{1}{x}\, \phi_\delta \geq -\log \delta$.)

Since p.v. $\frac{1}{x}$ has order 1 $\rightarrow$ IT IS NOT ASSOCIATED to a Radon measure, (or to a L^1_{loc} function).

Ex DISTRIBUTION of ∞ ORDER

$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R})$ $\quad T\phi = \sum_{n=0}^{+\infty} \phi^{(n)}(n)$ $\quad$ $\phi^{(n)}(x)$ = derivative of order n of ϕ

$\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R})$ let $k \in \mathbb{N}$ such that supp $\phi \subseteq \underbrace{[-k,k]}_{= K}$

$$\Rightarrow T\varphi = \sum_{n=0}^{k} \phi^{(n)}(n) \leq \underbrace{k}_{= C_K} \sup_{n \leq \underbrace{k}_{\rightarrow p_K}} \|\phi^{(n)}\|_\infty$$

T is a distribution, but it has order $+\infty$.

$\boxed{\text{Differential operators on distributions}}$

Let $T \in \mathcal{D}'(U)$ $\quad \alpha = (\alpha_1, \dots, \alpha_n) \quad \alpha_i \in \mathbb{N}$

$D^\alpha T$ is the distribution defined as

$$D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \phi) \qquad \forall \phi \in C^\infty_c(U).$$

Note that since $\phi \in C^\infty_c(U) \rightarrow \partial^{\alpha_i}_{x_i} \partial^{\alpha_j}_{x_j} \phi = \partial^{\alpha_j}_{x_j} \partial^{\alpha_i}_{x_i} \phi$

The definition is a generalization of Gauss-Green / integration by parts

If T is associated to $f \in C^1(U) \cap L^1_{loc}(U)$

$$T = T_f \rightarrow \left(\frac{\partial}{\partial x_i} T_f\right)(\phi) = (-1)^1 T_f\left(\frac{\partial}{\partial x_i}\phi\right) = -\int_U f(x) \frac{\partial}{\partial x_i}\phi(x)\,dx$$

$$\alpha = (0 \dots \underset{i}{1} \dots 0)$$

GAUSS GREEN.

$$= \int_U \frac{\partial}{\partial x_i} f \, \phi(x)\,dx$$

$(\phi \equiv 0 \ \partial U)$

$$\frac{\partial T_f}{\partial x_i} = -T_{\frac{\partial f}{\partial x_i}}$$

$\boxed{\text{Note that } T_n \rightarrow T \Rightarrow D^\alpha T_n \rightarrow D^\alpha T \quad \forall \alpha}$