Automata, Languages and Computation

Chapter 5 : Context-Free Grammars and Languages

Master Degree in Computer Engineering
University of Padua
Lecturer : Giorgio Satta

Lecture based on material originally developed by :
Gösta Grahne, Concordia University

Derivation trees

1. Context-free grammars : we consider devices defining structures more complex than regular languages

2. Parse trees : tree representation of a derivation

3. CFGs and ambiguity : some strings might have more than one parse tree

4. Relation with regular languages : CFGs can simulate FAs or regular expressions

Informal example of CFL

Let $L_{pal} = \{w \mid w \in \Sigma^*, \ w = w^R\}$, also called the language of all **palindrome** strings

Example : (ignore case, spaces, and punctuation characters)
`"Madam I'm Adam"` is a palindrome;
`"A man, a plan, a canal, Panama!"` is a palindrome

Informal example of CFL

Let $\Sigma = \{0, 1\}$ and assume L_{pal} is a regular language

Let n be the constant from the pumping lemma. We pick $w = 0^n 1 0^n \in L_{pal}$, $w \geqslant n$

Let $w = xyz$ be such that $y \neq \epsilon$ and $|xy| \leqslant n$

If $k = 0$, $xz \notin L_{pal}$: the number 0's to the left of 1 is smaller than the number of 0's to its right

Informal example of CFL

We **inductively** define L_{pal}

Base ϵ, 0, and 1 are palindrome strings

Induction
If w is a palindrome strings, then $0w0$ and $1w1$ are also palindrome strings

Nothing else is a palindrome string

CFG example

CFGs are a formalism for **recursively** defining languages such as L_{pal}, using **rewriting rules**

1. $P \rightarrow \epsilon$
2. $P \rightarrow 0$
3. $P \rightarrow 1$
4. $P \rightarrow 0P0$
5. $P \rightarrow 1P1$

P is a **variable** representing strings of a language. In this grammar P is also the initial symbol

Compare variables with recursive functions in programming languages

Definition

A **context-free grammar** (CFG for short) is a tuple

$$G = (V, T, P, S)$$

where

- V is a finite set of **variables** (also called **nonterminals**)
- T is a finite set of **terminal symbols**, representing the language alphabet
- P is a finite set of **productions** having the form $A \to \alpha$, where A (head, or left-hand side) is a variable and α (body or right-hand side) is a string in $(V \cup T)^*$
- S is a variable called **initial symbol**

Example

A CFG for palindrome strings is

$$G_{pal} = (\{P\}, \{0, 1\}, A, P)$$

with

$$A = \{P \rightarrow \epsilon, P \rightarrow 0, P \rightarrow 1, P \rightarrow 0P0, P \rightarrow 1P1\}$$

Example

The language of all regular expressions over the alphabet $\{0, 1\}$ can be defined by the CFG

$$G_{regEx} = (\{E\}, T, P, E)$$

where T is defined as (ϵ **overloaded** !)

$$\{\emptyset,\ \epsilon,\ \mathbf{0},\ \mathbf{1},\ +,\ .,\ {}^*,\ (,\ )\}$$

and P is defined as

$$\begin{aligned}\{&E \rightarrow \emptyset,\ E \rightarrow \epsilon,\ E \rightarrow \mathbf{0},\ E \rightarrow \mathbf{1},\\ &E \rightarrow E.E,\ E \rightarrow E + E,\ E \rightarrow E^*, E \rightarrow (E)\}\end{aligned}$$

Don't get confused: this defines the syntax of regular expressions, not the generated language

Example

Consider a simplified form of the **arithmetic expressions** as used in most common programming languages

$+$ and $*$ are arithmetic operators; operands are **identifiers** generated by the regular expression

$$(\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} + \boldsymbol{b} + \mathbf{0} + \mathbf{1})^*$$

We use the CFG

$$G = (\{E, I\}, T, P, E)$$

where

- variabile E represents arithmetic expressions
- variabile I represents identifiers

Example

T is defined as

$$\{+,\ *,\ (,\ ),\ a,\ b,\ 0,\ 1\}$$

P contains the following productions

1. $E \rightarrow I$
2. $E \rightarrow E + E$
3. $E \rightarrow E * E$
4. $E \rightarrow (E)$
5. $I \rightarrow a$
6. $I \rightarrow b$
7. $I \rightarrow I\ a$
8. $I \rightarrow I\ b$
9. $I \rightarrow I\ 0$
10. $I \rightarrow I\ 1$

We will later present several examples using this CFG

Compact notation

Usually, productions with a common head are grouped together

Example : Productions $A \rightarrow \alpha_1$, $A \rightarrow \alpha_2$, ..., $A \rightarrow \alpha_n$ can be written in a more compact notation

$$A \rightarrow \alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_n$$

Test

Define a CFG for each of the following languages

- $L = \{a^n b^n \mid n \geqslant 1\}$
- $L = \{a^n b^m \mid n \geqslant m \geqslant 1\}$

Derivation

In order to generate strings using a CFG, we define a binary relation $\underset{G}{\Rightarrow}$ over $(V \cup T)^*$, called **rewrites**

Let $G = (V, T, P, S)$ be a CFG, $A \in V$, $\{\alpha, \beta\} \subset (V \cup T)^*$. If $A \rightarrow \gamma \in P$ then

$$\alpha A \beta \underset{G}{\Rightarrow} \alpha \gamma \beta$$

and we say that $\alpha A \beta$ **derives in one step** $\alpha \gamma \beta$

If G is understood from the context, we use the simplified notation

$$\alpha A \beta \Rightarrow \alpha \gamma \beta$$

Derivation

We define $\stackrel{*}{\Rightarrow}$ as the reflexive and transitive closure of $\Rightarrow$

Base Let $\alpha \in (V \cup T)^*$. Then $\alpha \stackrel{*}{\Rightarrow} \alpha$

Induction If $\alpha \stackrel{*}{\Rightarrow} \beta$ and $\beta \Rightarrow \gamma$, then $\alpha \stackrel{*}{\Rightarrow} \gamma$

Relation $\stackrel{*}{\Rightarrow}$ is called **derivation**

We often write derivations by indicating all of the **intermediate steps**

Example

A possible derivation of $a * (a + b00)$ from E in the CFG for arithmetic expressions :

$$
\begin{aligned}
E &\Rightarrow E * E & &\Rightarrow a * (E + I0)\\
&\Rightarrow E * (E) & &\Rightarrow a * (E + I00)\\
&\Rightarrow I * (E) & &\Rightarrow a * (E + b00)\\
&\Rightarrow a * (E) & &\Rightarrow a * (I + b00)\\
&\Rightarrow a * (E + E) & &\Rightarrow a * (a + b00)\\
&\Rightarrow a * (E + I) & &
\end{aligned}
$$

Contrast with regular expressions, which do not have derivations for individual strings

Example

At each step in a derivation there might be several variables to which we can apply the rewrite relation :

$$I * E \Rightarrow a * E \Rightarrow a * (E)$$
$$I * E \Rightarrow I * (E) \Rightarrow a * (E)$$

Not all choices lead to a derivation of the desired string :

$$I * E \Rightarrow a * E \Rightarrow a * E + E$$

does not lead to a derivation of $a * (a + b00)$

Leftmost derivation

In derivations, we can avoid the choice of variables to be rewritten if we stick to some **canonical** derivation form

The relation $\underset{lm}{\Rightarrow}$ always rewrites the leftmost variable with some production

We also use the reflexive and transitive closure of $\underset{lm}{\Rightarrow}$, written $\overset{*}{\underset{lm}{\Rightarrow}}$, and call it **leftmost derivation**

Example

Leftmost derivation of $a * (a + b00)$:

$$\begin{aligned} E &\underset{lm}{\Rightarrow} E * E \underset{lm}{\Rightarrow} I * E \underset{lm}{\Rightarrow} a * E \underset{lm}{\Rightarrow} a * (E) \underset{lm}{\Rightarrow} a * (E + E) \\ &\underset{lm}{\Rightarrow} a * (I + E) \underset{lm}{\Rightarrow} a * (a + E) \underset{lm}{\Rightarrow} a * (a + I) \underset{lm}{\Rightarrow} a * (a + I0) \\ &\underset{lm}{\Rightarrow} a * (a + I00) \underset{lm}{\Rightarrow} a * (a + b00) \end{aligned}$$

We conclude that $E \overset{*}{\underset{lm}{\Rightarrow}} a * (a + b00)$

Rightmost derivation

The relation $\underset{rm}{\Rightarrow}$ always rewrites the rightmost variable with the body of a production

We use the reflexive and transitive closure of $\underset{rm}{\Rightarrow}$, written $\overset{*}{\underset{rm}{\Rightarrow}}$, called **rightmost derivation**

Example

Rightmost derivation :

$$
\begin{aligned}
E &\underset{rm}{\Rightarrow} E * E \underset{rm}{\Rightarrow} E * (E) \underset{rm}{\Rightarrow} E * (E + E) \underset{rm}{\Rightarrow} E * (E + I) \\
&\underset{rm}{\Rightarrow} E * (E + I0) \underset{rm}{\Rightarrow} E * (E + I00) \underset{rm}{\Rightarrow} E * (E + b00) \\
&\underset{rm}{\Rightarrow} E * (I + b00) \underset{rm}{\Rightarrow} E * (a + b00) \underset{rm}{\Rightarrow} I * (a + b00) \\
&\underset{rm}{\Rightarrow} a * (a + b00)
\end{aligned}
$$

We conclude that $E \overset{*}{\underset{rm}{\Rightarrow}} a * (a + b00)$

Notation for CFGs

We use the following conventions

- $a, b, c, \ldots$ terminal symbols
- $A, B, C, \ldots$ variables (nonterminal symbols)
- u, v, w, x, y, z terminal strings
- X, Y, Z terminal or nonterminal symbols
- $\alpha, \beta, \gamma, \ldots$ strings over terminal or nonterminal symbols

Language generated by a CFG

Let $G = (V, T, P, S)$ be some CFG. The **generated language** of G is

$$L(G) = \{w \in T^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w\}$$

that is, the set of all strings in T^* that can be derived from the start symbol

$L(G)$ is a **context-free language**, or CFL for short

Example : $L(G_{pal})$ is a CFL

Test

Consider the language L of all strings over "(" and ")" where parentheses are always **well balanced** (assume $\epsilon \notin L$)

- for the following CFG

$$G \quad = \quad (\{S\}, \{(,)\}, P, S)$$

  specify the set P such that $L(G) = L$

- produce a derivation for string

$$w \quad = \quad (\,(\,)\,(\,(\,)\,)\,)$$

Language generated by a CFG

$G_{pal} = (\{P\}, \{0,1\}, A, P)$, where

$$A = \{P \to \epsilon \mid 0 \mid 1 \mid 0P0 \mid 1P1\}$$

Theorem $L(G_{pal}) = \{w \mid w \in \{0,1\}^*,\ w = w^R\}$

Proof ($\supseteq$ part) Assume $w = w^R$. Using induction on $|w|$, we show $w \in L(G_{pal})$

Language generated by a CFG

Base $|w| = 0$ or $|w| = 1$. Then w is $\epsilon, 0$, or else 1. Since $P \rightarrow \epsilon$, $P \rightarrow 0$, and $P \rightarrow 1$ are productions of the grammar, we conclude that $P \overset{*}{\underset{G}{\Rightarrow}} w$

Induction Assume now $|w| \geqslant 2$. Since $w = w^R$, we must have $w = 0x0$ or else $w = 1x1$, with $x = x^R$. From the inductive hypothesis we then have $P \overset{*}{\Rightarrow} x$.

If $w = 0x0$, we can write

$$P \Rightarrow 0P0 \overset{*}{\Rightarrow} 0x0 = w$$

Therefore $w \in L(G_{pal})$

Case $w = 1x1$ can be dealt with similarly

Language generated by a CFG

($\subseteq$ part) Assume now $w \in L(G_{pal})$. We show $w = w^R$

Since $w \in L(G_{pal})$, we have $P \stackrel{*}{\Rightarrow} w$. We use induction on the number of steps of the derivation

Base The derivation $P \stackrel{*}{\Rightarrow} w$ has 1 step. Then w must be ϵ, 0, or 1. All the three generated strings are palindrome

Language generated by a CFG

Induction Let $n \geqslant 2$ be the number of steps in the derivation. At the first step only two cases are possible :

$$P \Rightarrow 0P0 \stackrel{*}{\Rightarrow} 0x0 = w$$

or else

$$P \Rightarrow 1P1 \stackrel{*}{\Rightarrow} 1x1 = w$$

In both cases, the second part of the derivation implies $P \stackrel{*}{\Rightarrow} x$ in $n - 1$ steps (this will be explained later in more detail)

By the inductive hypothesis, x is a palindrome string. Then also w is a palindrome string $\square$

Proofs about CFGs

We need to show that a given CFG generates a desired language

For each variable A in the CFG, define some property $\mathcal{P}_A$ for strings w over the alphabet

Show that, for every A, we have

$A \overset{*}{\Rightarrow} w$ *if and only if* $\mathcal{P}_A(w)$ *holds true*

Proofs about CFGs

If part : if $\mathcal{P}_A(w)$ then $A \overset{*}{\Rightarrow} w$

Use **mutual induction** on $|w|$

- using $\mathcal{P}_A$ definition, choose a factorization $w = x_1 x_2 \cdots x_k$ such that $\mathcal{P}_{B_i}(x_i)$ holds for each i
- use the inductive hypothesis on $\mathcal{P}_{B_i}(x_i)$ to obtain $B_i \overset{*}{\Rightarrow} x_i$, for each i
- choose a production $A \rightarrow B_1 B_2 \cdots B_k$ and obtain

$$\begin{aligned} A &\Rightarrow B_1 B_2 \cdots B_k \\ &\overset{*}{\Rightarrow} x_1 B_2 \cdots B_k \\ &\vdots \\ &\overset{*}{\Rightarrow} x_1 x_2 \cdots x_k = w \end{aligned}$$

Proofs about CFGs

Only if part : if $A \overset{*}{\Rightarrow} w$ then $\mathcal{P}_A(w)$ holds true

Use **mutual induction** on the length of derivation $A \overset{*}{\Rightarrow} w$

- focus on the first production of the derivation

$$
\begin{aligned}
A &\Rightarrow B_1 B_2 \cdots B_k \\
&\overset{*}{\Rightarrow} x_1 B_2 \cdots B_k \\
&\;\vdots \\
&\overset{*}{\Rightarrow} x_1 x_2 \cdots x_k = w
\end{aligned}
$$

- use the inductive hypothesis on $B_i \overset{*}{\Rightarrow} x_i$ to obtain that $\mathcal{P}_{B_i}(x_i)$ holds, for each i
- use $\mathcal{P}_A$ definition to show that $\mathcal{P}_A(w)$ holds true

Sentential form

Let $G = (V, T, P, S)$ be a CFG and let $\alpha \in (V \cup T)^*$

- if $S \stackrel{*}{\Rightarrow} \alpha$ we say that α is a **sentential form**
- if $S \underset{lm}{\stackrel{*}{\Rightarrow}} \alpha$ we say that α is a **left sentential form**
- if $S \underset{rm}{\stackrel{*}{\Rightarrow}} \alpha$ we say that α is a **right sentential form**

Note : $L(G)$ contains all the sentential forms in T^*

Examples

Consider previous CFG G for a fragment of arithmetic expressions. Then $E * (I + E)$ is a sentential form, since

$$E \Rightarrow E * E \Rightarrow E * (E) \Rightarrow E * (E + E) \Rightarrow E * (I + E)$$

This derivation is neither leftmost nor rightmost

$a * E$ is a leftmost sentential form, since

$$E \underset{lm}{\Rightarrow} E * E \underset{lm}{\Rightarrow} I * E \underset{lm}{\Rightarrow} a * E$$

$E * (E + E)$ is a rightmost sentential form, since

$$E \underset{rm}{\Rightarrow} E * E \underset{rm}{\Rightarrow} E * (E) \underset{rm}{\Rightarrow} E * (E + E)$$

Test

Define a CFG for each of the following languages, describing for each variable the set of generated strings

- $L = \{w \mid w = x2x^R,\ x \in \{0,1\}^*\}$
- $L = \{w \mid w = a^i b^j c^k,\ i,j,k \geqslant 1,\ j \neq k\}$

Test

Describe in words the language generated by the following CFG

$$G = (\{S, Z\}, \{0, 1\}, P, S)$$

where

$$P = \{S \to 0S1 \mid 0Z1,\ Z \to 0Z \mid \epsilon\}$$

Derivation composition

We can always compose two derivations $A \overset{*}{\Rightarrow} \alpha B \beta$ and $B \overset{*}{\Rightarrow} \gamma$ into a single derivation

$$A \overset{*}{\Rightarrow} \alpha B \beta \overset{*}{\Rightarrow} \alpha \gamma \beta$$

This follows from the hypothesis about rewriting being **independent** from the context (context-free)

Example

Consider our CFG for generating arithmetic expressions. Starting with

$$E \Rightarrow E + E \Rightarrow E + (E)$$
$$E \Rightarrow I \Rightarrow Ib \Rightarrow ab$$

we can compose at the rightmost occurrence of E, obtaining

$$E \Rightarrow E + E \Rightarrow E + (E) \Rightarrow E + (I) \Rightarrow E + (Ib) \Rightarrow E + (ab)$$

Derivation factorization

Assume $A \Rightarrow X_1 X_2 \cdots X_k \overset{*}{\Rightarrow} w$. We can **factorize** w as $w_1 w_2 \cdots w_k$ such that $X_i \overset{*}{\Rightarrow} w_i$, $1 \leqslant i \leqslant k$

As a special case, we can have $X_i = w_i \in T$

Substring w_i can be identified from derivation $A \overset{*}{\Rightarrow} w$ by considering **only** those derivation steps that rewrite X_i

Example

Consider $E \Rightarrow E * E \stackrel{*}{\Rightarrow} a * b + a$

We have

$$\underbrace{a}_{E} \ \underbrace{*}_{*} \ \underbrace{b + a}_{E}$$

and we can write

$$E \stackrel{*}{\Rightarrow} a$$

$$* \stackrel{*}{\Rightarrow} *$$

$$E \stackrel{*}{\Rightarrow} b + a$$

Parse trees

Parse trees are a graphical representation alternative to derivations

Intuitively, parse trees represent the **syntactic structure** of a string according to the grammar

In compilers, parse trees are the structure of choice when **translating** into executable code

Parse trees

Let $G = (V, T, P, S)$ be a CFG. An ordered tree is a **parse tree** of G if :

- each internal node is labeled with a variable in V
- each leaf node is labeled with a symbol in $V \cup T \cup \{\epsilon\}$; each leaf labeled with ϵ is the only child of its parent
- if an internal node is labeled A and its children (from left to right) are labeled

  $$X_1, X_2, \ldots, X_k$$

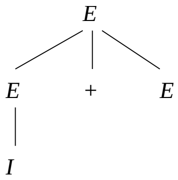
  then $A \to X_1 X_2 \cdots X_k \in P$

Example

CFG for arithmetic expressions and parse tree associated with the derivation $E \Rightarrow E + E \Rightarrow I + E$

1. $E \to I$
2. $E \to E + E$
3. $E \to E * E$
4. $E \to (E)$

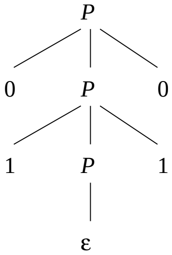
$\vdots$



Example

CFG for palindrome strings and parse tree associated with the derivation $P \Rightarrow 0P0 \Rightarrow 01P10 \Rightarrow 0110$

1. $P \rightarrow \epsilon$
2. $P \rightarrow 0$
3. $P \rightarrow 1$
4. $P \rightarrow 0P0$
5. $P \rightarrow 1P1$



Parse tree terminology

We use the following terms associated with parse trees

- node and arc
- parent node and child node
- ancestor node and descendant node
- root node, inner node (including the root) and leaf node

Recall : For each internal node, the child nodes are **ordered**

Yeld of a parse tree

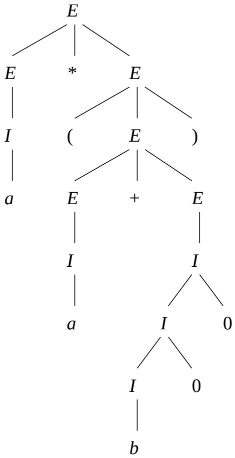
The **yield** of a parse tree is the string obtained by reading the leaves from left to right

Of special importance are the **complete** parse trees, where :

- the yield is a string of terminal symbols
- the root is labeled by the initial symbol

The set of yields of all complete parse trees is the language generated by the CFG

Example



Complete parse tree. The yield is $a * (a + b00)$

Derivations and parse trees

Let $G = (V, T, P, S)$ be a CFG, $A \in V$ and $w \in T^*$. The following statements are equivalent (statements must all be true or must all be false) :

- $A \overset{*}{\Rightarrow} w$
- $A \underset{lm}{\overset{*}{\Rightarrow}} w$
- $A \underset{rm}{\overset{*}{\Rightarrow}} w$
- there exists a parse tree for G with root label A and yield w

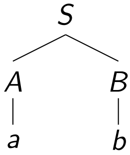
Proof not required for these theorems

Relation between derivations and parse trees **is not** one-to-one (see next slides)

Derivations and parse trees

A parse tree can be associated with **several** derivations

Example : Consider the CFG with productions $S \to AB$, $A \to a$, $B \to b$. The parse tree



is associated with two derivations

$$S \Rightarrow AB \Rightarrow aB \Rightarrow ab$$
$$S \Rightarrow AB \Rightarrow Ab \Rightarrow ab$$

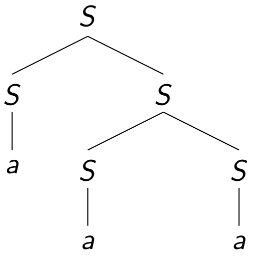
Derivations and parse trees

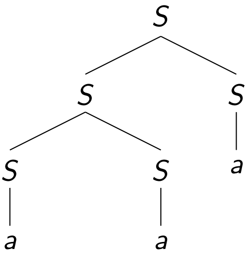
A derivation can be associated with **several** parse trees

Example : Consider the CFG with productions $S \rightarrow SS \mid a$.
The derivation

$$S \Rightarrow SS \Rightarrow SSS \Rightarrow aSS \Rightarrow aaS \Rightarrow aaa$$

is associated with two parse trees





Ambiguous CFGs

In the CFG

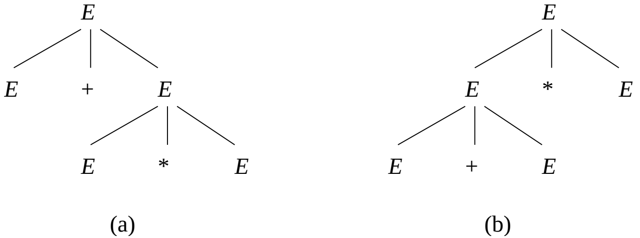
1. $E \rightarrow I$
2. $E \rightarrow E + E$
3. $E \rightarrow E * E$
4. $E \rightarrow (E)$
5. $I \rightarrow a$
6. $I \rightarrow b$
7. $I \rightarrow I\ a$
8. $I \rightarrow I\ b$
9. $I \rightarrow I\ 0$
10. $I \rightarrow I\ 1$

the sentential form $E + E * E$ has two derivations

$$E \Rightarrow E + E \Rightarrow E + E * E$$
$$E \Rightarrow E * E \Rightarrow E + E * E$$

Ambiguous CFGs

Associated parse trees for the derivations of $E + E * E$



The two derivations correspond to different **precedences** for operators sum and multiplication

Ambiguous CFGs

The existence of different derivations for a string is not problematic, if these correspond to a single parse tree

Example : In our CFG for arithmetic expressions, the string $a + b$ has at least two derivations

$$E \Rightarrow E + E \Rightarrow I + E \Rightarrow a + E \Rightarrow a + I \Rightarrow a + b$$

$$E \Rightarrow E + E \Rightarrow E + I \Rightarrow I + I \Rightarrow I + b \Rightarrow a + b$$

However, the associated parse trees are the same, and string $a + b$ is **not** ambiguous

Ambiguous CFGs

Let $G = (V, T, P, S)$ be a CFG. G is **ambiguous** if there exists a string in $L(G)$ with more than one parse tree

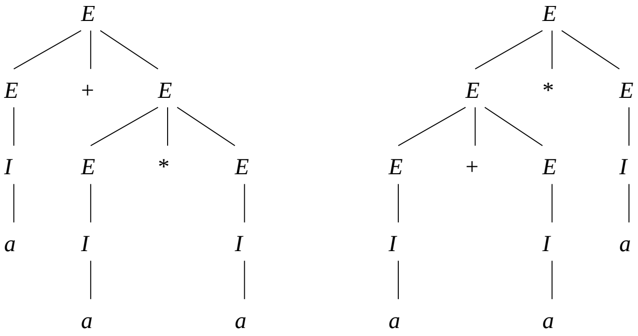
If every string in $L(G)$ has only one parse tree, G is said to be **unambiguous**

The ambiguity is **problematic** in many applications where the syntactic structure of a string is used to interpret its meaning

Example: compilers for programming languages

Example

In the CFG for arithmetic expressions, the terminal string $a + a * a$ has two parse trees



(a) (b)

Canonical derivations

A parse tree is associated with a **unique** leftmost derivation

A leftmost derivation is associated with a **unique** parse tree

More than one leftmost derivations always imply more than one parse trees

Similary for rightmost derivations

Inherent ambiguity

A CFL L is **inherently ambiguous** when every CFG such that $L(G) = L$ is ambiguous

Example : Let us consider the language

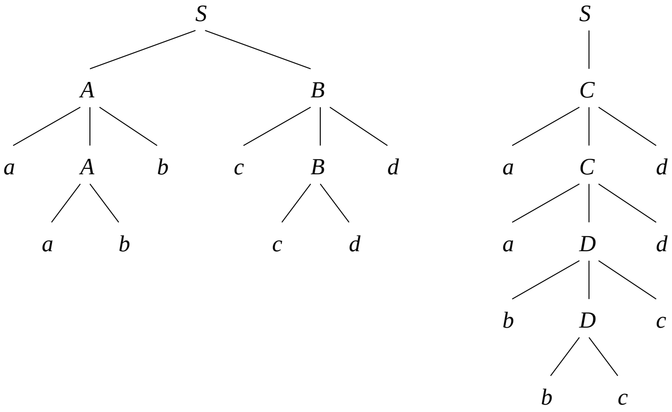
$$L = \{a^n b^n c^m d^m \mid n \geqslant 1,\ m \geqslant 1\} \cup \{a^n b^m c^m d^n \mid n \geqslant 1,\ m \geqslant 1\}$$

L can be generated by a CFG with the following productions

$$\begin{aligned}
S &\rightarrow AB \mid C \\
A &\rightarrow aAb \mid ab \\
B &\rightarrow cBd \mid cd \\
C &\rightarrow aCd \mid aDd \\
D &\rightarrow bDc \mid bc
\end{aligned}$$

Inherent ambiguity

There are two parse trees for the string *aabbccdd*



(a) (b)

Inherent ambiguity

Associated leftmost derivations

$$S \underset{lm}{\Rightarrow} AB \underset{lm}{\Rightarrow} aAbB \underset{lm}{\Rightarrow} aabbB \underset{lm}{\Rightarrow} aabbcBd \underset{lm}{\Rightarrow} aabbccdd$$
$$S \underset{lm}{\Rightarrow} C \underset{lm}{\Rightarrow} aCd \underset{lm}{\Rightarrow} aaDdd \underset{lm}{\Rightarrow} aabDcdd \underset{lm}{\Rightarrow} aabbccdd$$

It is possible to show that **every** CFG generating L provides a similar ambiguity for the string $aabbccdd$ (not in the textbook)

Language L is therefore inherently ambiguous

Exercises

- Provide an example showing that the CFG with productions

$$S \rightarrow aS \mid aSbS \mid \epsilon$$

  is ambiguous. **Hint**: consider some string of length 3
- Provide an example showing that the CFG with productions

$$S \rightarrow aSbS \mid bSaS \mid \epsilon$$

  is ambiguous. **Hint**: consider some string of length 4

Reguar languages and CFL

A regular language is **always** a CFL

From a regular expression or from an FA we can aways construct a CFG generating the same language

This is not in the textbook!

From regular expression to CFG

Let E be any regular expression. We use a variable for E (start symbol) and a variable for each subexpression of E

We use **structural induction** on the regular expression to build the productions of our CFG

- if $E = \boldsymbol{a}$, then add production $E \rightarrow a$
- if $E = \epsilon$, then add production $E \rightarrow \epsilon$
- if $E = \emptyset$, then production set is empty
- if $E = F + G$, then add production $E \rightarrow F \mid G$
- if $E = FG$, then add production $E \rightarrow FG$
- if $E = F^*$, then add production $E \rightarrow FE \mid \epsilon$
- if $E = (F)$, then add production $E \rightarrow F$

Example

Regular expression : $\mathbf{0}^*\mathbf{1}(\mathbf{0}+\mathbf{1})^*$

Use left-associativity for concatenation

CFG :

$$E \rightarrow AR$$
$$R \rightarrow BC$$
$$A \rightarrow 0A \mid \epsilon$$
$$B \rightarrow 1$$
$$C \rightarrow DC \mid \epsilon$$
$$D \rightarrow 0 \mid 1$$

From FA to CFG

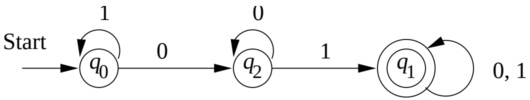
We use a variable Q for each state q of the FA. Initial symbol is Q_0

For each transition from state p to state q under symbol a, add production $P \rightarrow a\ Q$

If q is a final state, add production $Q \rightarrow \epsilon$

Example

Automaton :



CFG :

$$Q_0 \rightarrow 1Q_0 \mid 0Q_2$$
$$Q_2 \rightarrow 0Q_2 \mid 1Q_1$$
$$Q_1 \rightarrow 0Q_1 \mid 1Q_1 \mid \epsilon$$

String 1101 is accepted by the automaton. In the equivalent CFG, 1101 has the following derivation :

$$Q_0 \Rightarrow 1Q_0 \Rightarrow 11Q_0 \Rightarrow 110Q_2 \Rightarrow 1101Q_1 \Rightarrow 1101$$