Analytical Methods for Engineering

Dec. 2021

Exercise 1. Let

$$F(\xi) := \int_0^{+\infty} \frac{e^{-\xi x^2} - e^{-x^2}}{x}\, dx.$$

i) Determine the domain of definition D of F.
ii) Determine for which $\xi \in D$ there exists $\partial_\xi F$ and compute it.
iii) Determine $F(\xi)$ explicitly.

Exercise 2. On $C^1([0,1])$, define

$$\|f\|_* := |f(1)| + \int_0^1 \frac{|f'(x)|}{\sqrt{x}}\, dx.$$

i) Check that $\|\cdot\|_*$ is a well defined norm on V.
ii) Prove that $\|\cdot\|_*$ is stronger than $\|\cdot\|_\infty$.
iii) For $n \geqslant 1$, let

$$f_n(x) := \begin{cases} -\frac{1}{2}n^{3/2}\left(x - \frac{1}{n}\right) + n^{1/2}, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \frac{1}{\sqrt{x}}, & \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$

Check that $(f_n) \subset V$, compute $\|f_n\|_*$ and $\|f_n\|_\infty$ and deduce something on $\|\cdot\|_*$ and $\|\cdot\|_\infty$.

Exercise 3. Let

$$f_a(x) := \frac{1-\cos x}{a^2 + x^2},\ a \geqslant 0.$$

i) Check that $f_a \in L^1(\mathbb{R})$ for every $a \geqslant 0$.
ii) For $a > 0$, compute $\widehat{f_a}$ (it may be useful to remind that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$). Deduce

$$\int_0^{+\infty} f_a(x)\, dx.$$

iii) Use ii) to compute

$$\int_0^{+\infty} \frac{1-\cos x}{x^2}\, dx.$$

iv) (extra question) Compute $\widehat{f_0}$.

JAN. 2022

Exercise 4. Let $f_n(x) := n^{3/2} x e^{-nx}$, $x \in [0, +\infty[$.

i) Is (f_n) point wise convergent on $[0, +\infty[$? If yes, to what?
ii) Is $(f_n) \subset L^\infty([0, +\infty[)$? Is (f_n) convergent in $L^\infty([0, +\infty[)$ and, if yes, to what?
iii) Is $(f_n) \subset L^1([0, +\infty[)$? Is (f_n) convergent in $L^1([0, +\infty[)$ and, if yes, to what?
iv) Is $(f_n) \subset L^2([0, +\infty[)$? Is (f_n) convergent in $L^2([0, +\infty[)$ and, if yes, to what?

Exercise 5. Let

$$F(t) := \int_{-\infty}^{+\infty} e^{-tx^2} \frac{1 - \cos x}{x^2} \, dx.$$

i) Determine the domain of definition of F.
ii) Determine $F'(t)$ for all t for which the derivative exists.

Exercise 6. Let $H := L^2([0, 2])$ equipped with usual scalar product $\langle f, g \rangle := \int_0^2 f(x) g(x) \, dx$. Let

$$U := \{ f \in H \ : \ f(x) = f(2 - x), \ a.e. \ x \in [0, 2] \}.$$

i) Check that U is a closed linear subspace of H.
ii) Check that the orthogonal projection on U, $\Pi_U f$, is

$$\Pi_U f(x) := \begin{cases} f(x), & x \in [0, 1], \\ \\ f(2 - x), & x \in [1, 2] \end{cases}$$

July 2022

Exercise 7. Let

$$f_n(x) := \sqrt{n - n^2 x}\,1_{[0,1/n]}(x).$$

i) Plot quickly the graph of f_n.
ii) Is $(f_n) \subset L^1([0,1])$? Is $(f_n) \subset L^2([0,1])$?
iii) Is (f_n) convergent in $L^1([0,1])$ and, in the case, to what? Is (f_n) convergent in $L^2([0,1])$ and, in the case, to what?

Exercise 8. Let

$$\Phi(s) := \int_0^{+\infty} e^{-sx} \frac{(\sin x)^2}{x}\, dx.$$

i) Show that $\Phi(s)$ is well defined for any $s > 0$.
ii) Show that $\lim_{s \to +\infty} \Phi(s)$ exists and determine its value.
iii) Show that $\exists \Phi'(s)$ for any $s > 0$.
iv) Deduce, by ii) and iii), the value of $\Phi(s)$ for $s \in]0, +\infty[$.

Useful formula: $\int e^{\alpha x} \sin(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha x} \left(\sin(\beta x) - \frac{\beta}{\alpha} \cos(\beta x) \right)$ for $\alpha \neq 0$.

Exercise 9. Give the Definition of the Fourier Transform and of convolution product $f * g$ for $f, g \in L^1(\mathbb{R})$. Prove that

$$\widehat{f * g} = \ldots$$

i) Compute the FT of $e^{-a|x|}$, $a > 0$. Justifying carefully and invoking the necessary theorems, deduce the FT of the Cauchy distribution $\frac{1}{a^2+x^2}$ $(a > 0)$.
ii) Use the previous facts to compute the FT of $\frac{1}{(a^2+x^2)^2}$ $(a > 0)$.

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Exercise 10. Let $H = L^2([-1,1])$ equipped with usual scalar product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Define

$$S := \left\{ f \in H \ : \ \int_{-1}^{0} f(x)\, dx = \int_{0}^{1} f(x)\, dx \right\}.$$

i) Check that S is a well defined and closed subspace of H.
ii) Determine the orthogonal projection $\Pi_S f$ on S of a generic $f \in H$. Compute, in particular, $\Pi_S x$.

Exercise 11. Let

$$g(\xi) := \frac{\sin\xi - \xi\cos\xi}{\xi^3}, \ \xi \in \mathbb{R}\backslash\{0\}.$$

i) Is $g \in L^1(\mathbb{R})$? And in $L^2(\mathbb{R})$? Justify carefully.
ii) Discuss the existence of a Fourier original for g.
iii) Show that $\xi g(\xi) = \partial_\xi \ldots$. Use this to determine a Fourier original for g. Justify carefully the general properties you use to answer.

Exercise 12. On $V := \{f \in \mathscr{C}^1([0,1]) \ : \ f(0) = 0\}$, define

$$\|f\|_* := \sup_{t\in]0,1]} \frac{|f(t)|}{\sqrt{t}}, \quad \|f\|_{**} := \|f'\|_\infty \equiv \max_{t\in[0,1]} |f'(t)|.$$

i) Check that $\|\cdot\|_*$ is well defined norm on V (it might be useful to remind that, for $f \in \mathscr{C}^1([0,1])$, $f(t) = f(0) + f'(0)t + o(t)$ when $t \longrightarrow 0$).
ii) Check that $\|\cdot\|_{**}$ is well defined on V and it fulfils the characteristic properties of a norm.
iii) What relations hold true between $\|\cdot\|_*$ and $\|\cdot\_{**}$? Justify carefully.

January 2023

Exercise 13. Let $H = \{f \in L^2([0,1]) \ : \ \exists f' \in L^2([0,1]),\ f(1) = 0\}$ equipped with the scalar product

$$\langle f, g\rangle_H = \int_0^1 f'(x)g'(x)\,dx.$$

We accept H is a vector space with usual operations of sum and product by scalars on functions.

i) Check that $\langle\cdot,\cdot\rangle_H$ is a well defined scalar product on H with vanishing in the weak form, that is $\langle f, f\rangle_H = 0$ iff $f = 0$ a.e..

ii) Let

$$S := \left\{ f \in H \ : \ \int_0^1 f(x)\,dx = 0 \right\}.$$

Determine $v \in H$ such that $S = \mathrm{Span}(v)^\perp$. (hint: express the characterizing condition of S in terms of the scalar product of H).

iii) We accept H is an Hilbert space. Determine the orthogonal projection of $x - 1$ on S. Justify carefully.

Exercise 14. i) Let $f, g \in L^1(\mathbb{R})$. What is the convolution of f and g? What important property of FT holds in connection with convolution? Write a precise statement and provide a proof of it.

Consider now the equation

$$\lambda f(x) + \int_{\mathbb{R}} \frac{f(y)}{1+(x-y)^2}\,dy = \frac{1}{1+x^2} + \frac{1}{4+x^2},\ x \in \mathbb{R}.$$

ii) Assume $f \in L^1(\mathbb{R})$ be a solution. Determine $\widehat{f}$. Deduce for which values of λ the equation has one and only one solution.

iii) Determine explicitly the solution in the case $\lambda = 2\pi$.

Exercise 15. Let $(f_n) \subset L^1([0,1])$, $f_n \geqslant 0$ for every $n \in \mathbb{N}$. For each of the following statements, say whether it is true or false. In the first case, provide a proof, in the second provide a counterexample.

i) If $\int_0^1 f_n \longrightarrow +\infty$ then $f_n \not\longrightarrow 0$ a.e.

ii) if $f_n \longrightarrow 0$ a.e. then $\int_0^1 f_n \longrightarrow 0$.

iii) if $\int_0^1 f_n \longrightarrow 0$ then $f_n \longrightarrow 0$ a.e.

iv) if $\int_0^1 f_n \longrightarrow +\infty$ then $\|f_n\|_\infty \longrightarrow +\infty$.

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Exercise 16. Compute

$$\lim_{n\to+\infty} n^2 \int_0^n \frac{1-e^{-\frac{x^2}{n^2}}}{x^2(x^2+1)}\,dx.$$

Justify carefully, quoting the general results you use. (it might be useful to know that $e^t \geqslant 1+t$ for every $t \in \mathbb{R}$)

Exercise 17. Let $V := \mathscr{C}^2([0,1])$ the set of real valued continuous functions f with f' and f'' continuous. We accept (trivial) that V is a vector space with usual operations of sum and product by scalars. On V we define

$$\|f\|_V := \|f''\|_\infty + |f'(0)| + |f(0)|.$$

i) Check that $\|\cdot\|_V$ is a well defined norm on V.
ii) On V we can also define the uniform norm $\|\cdot\|_\infty$. Check that $\|\cdot\|_V$ is stronger than uniform norm.
iii) True or false: if $f_n \xrightarrow{\|\cdot\|_V} f$ then $f_n(x) \longrightarrow f(x)$ for every $x \in [0,1]$? Justify carefully.
iv) Consider the sequence $f_n(x) := \frac{1}{n^2}x^n$. Is $(f_n) \subset V$? Is (f_n) convergent in V? What conclusion can you draw on norms $\|\cdot\|_V$ and $\|\cdot\|_\infty$?

Exercise 18. Let

$$g(\xi) := \frac{1}{\xi}\left(\frac{a^2}{a^2+\xi^2} - \frac{b^2}{b^2+\xi^2}\right),$$

where $a, b > 0$ and $a \neq b$.

i) Is $g \in L^1(\mathbb{R})$? Does g have a Fourier original? Justify carefully your answer.
ii) Compute the FT of $\xi g(\xi)$.
iii) Determine the Fourier original of g.

September 2023

Exercise 19. Let $f \in L^1(\mathbb{R})$ be such that $xf(x) \in L^1(\mathbb{R})$. What is the FT of $xf(x)$? Provide a proof of your answer.

Let now

$$g(\xi) := \xi e^{-|\xi|}, \quad \xi \in \mathbb{R}.$$

Discuss the problem of determining a Fourier original for g.

Exercise 20. On $V = \mathscr{C}([0,1])$ we consider

$$\|f\|_* := \int_0^1 \frac{|f(x)|}{\sqrt{x}}\, dx.$$

i) Show that $\|\cdot\|_*$ is a well defined norm on V.
ii) On V we consider also a) the uniform norm $\|\cdot\|_\infty$ and the L^1 norm $\|f\|_1 = \int_0^1 |f(x)|dx$. Establish relations among these norms and $\|\cdot\|_*$ norm, discussing also if they are equivalent or not.

Exercise 21. On $H = L^2([0,1])$, with usual scalar product, consider

$$V := \left\{ f \in H \ : \ \int_0^1 xf(x)\, dx = 0, \ \int_0^1 x^3 f(x)\, dx = 0 \right\}.$$

i) Check that V is a well defined and closed subspace of H.
ii) Determine the orthogonal projection on V of $g(x) = x^2$.

Exercise 22. Let
$$F(\lambda) := \int_0^{+\infty} e^{-\lambda x} \frac{\sin x}{x}\, dx.$$

i) Determine the domain D of definition for F.
ii) Discuss differentiability of F on D and compute $\partial_\lambda F$. It may be helpful to know $\int e^{\alpha x} \sin(\beta x)\, dx = \frac{\alpha}{\alpha^2+\beta^2} e^{\alpha x} \left(\sin(\beta x) - \frac{\beta}{\alpha}\cos(\beta x)\right)$ for $\alpha \neq 0$.
iii) Determine F explicitly.

Exercise 23. Let
$$V := \left\{ f \in \mathscr{C}([0,1]) \ : \ \|f\| := \int_0^1 \frac{|f(x)|}{\sqrt{x}}\, dx < +\infty \right\}.$$
Accept that V is a vector space.

i) Check that $\|\cdot\|$ is a norm on V.
ii) On V it is also defined the uniform norm $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$. Prove that this is stronger than $\|\cdot\|$.
iii) Let
$$f_n(x) := \begin{cases} \sqrt[4]{n}, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \frac{1}{\sqrt[4]{x}}, & \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$
Compute $\|f_n\|$ and $\|f_n\|_\infty$. What can you conclude about norms $\|\cdot\|$ and $\|\cdot\|_\infty$.

Exercise 24. The goal is to compute the FT of $f(x) = \frac{1}{1+x^4}$.

i) Does $\widehat{f}$ exists? If yes, which of the following statements are true/false and why: $\widehat{f} \in L^1(\mathbb{R})$; $\widehat{f} \in L^2(\mathbb{R})$; $\widehat{f} \in \mathscr{C}^1(\mathbb{R})$; $\widehat{f} \in \mathcal{S}(\mathbb{R})$.
ii) By reducing to suitable Cauchy distributions, compute FT of
$$\frac{1}{x^2 \pm \sqrt{2}x + 1}.$$
iii) Noticed that $1 + x^4 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$, express $\frac{1}{1+x^4}$ in terms of $\frac{1}{x^2 \pm \sqrt{2}x+1}$. Use this to determine $\widehat{f}$.

January 2024

Exercise 25. Let
$$F(\xi) := \int_{\mathbb{R}} \frac{1-\cos(\xi x)}{x^2(x^2+1)}\,dx.$$

i) Determine the set $D \subset \mathbb{R}$, domain of definition for F.
ii) Determine the set $D' \subset D$ for which $\partial_\xi F$ and compute it.
iii) Determine the set $D'' \subset D'$ for which there exists $\partial_\xi^2 F(\xi)$ and compute it.
iv) Use FT to express $\partial_\xi^2 F$ and to determine F.

Exercise 26. On $V := \{f \in \mathscr{C}^1([0,1]) \;:\; f(0)=0\}$, we define
$$\|f\| := \max_{t\in[0,1]} t^{1/2}|f'(t)|.$$

i) Check that $\|\cdot\|$ is a norm on V.
ii) Define
$$f_n(x) := \begin{cases} \frac{n^{3/4}}{4}x, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \\ x^{1/4} - \frac{3}{4n^{1/4}}, & \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$
Is $(f_n) \subset V$? If yes, is (f_n) convergent in $\|\cdot\|$ norm?
iii) On V is naturally defined the uniform norm $\|\cdot\|_\infty$. Show that $\|\cdot\|$ is stronger than $\|\cdot\|_\infty$. Are they also equivalent? Justify carefully your answer.

Exercise 27. Let
$$H := \left\{ f : [0,+\infty[\longrightarrow \mathbb{R} \;:\; f \in L(\mathbb{R}),\; \int_0^{+\infty} f(x)^2 e^{-x}\,dx < +\infty \right\}.$$
On H we define
$$\langle f, g\rangle := \int_0^{+\infty} f(x)g(x)e^{-x}\,dx.$$

i) Check that $\langle\cdot,\cdot\rangle$ is a well defined scalar product with vanishing in the form $\langle f, f\rangle = 0$ iff $f = 0$ a.e.

Accept H is Hilbert. Let $U := \{u \in H \;:\; \int_0^{+\infty} u(x)e^{-x}\,dx = 0\}$.

ii) Prove that U a closed subspace of H.
iii) Determine the orthogonal projection on U of $f(x) = e^{-2x}$.

July 2024

Exercise 28. Let

$$f_n(x) := \frac{1}{x^{1+1/n}},\ x \in [1, +\infty[.$$

i) Is $(f_n) \subset L^1([1, +\infty[)$? Is $(f_n) \subset L^2([1, +\infty[)$? Is $(f_n) \subset L^\infty([1, +\infty[)$? Justify carefully.
ii) Discuss convergence of (f_n) in $L^p([1, +\infty[)$ for $p = 1, 2, \infty$.

Exercise 29. Let $\alpha \in \mathbb{C}$ and define

$$f_\alpha(x) := e^{\alpha x} 1_{[0,+\infty[}(x),\ x \in \mathbb{R}.$$

i) Under which conditions on $\alpha \in \mathbb{C}$ is $\widehat{f_\alpha}$ well defined? For such α, compute $\widehat{f_\alpha}$.
ii) Let $\beta \in \mathbb{C}$, $g_\beta(\xi) := \frac{1}{(\xi+\beta)^2}$. Determine under which conditions on $\beta \in \mathbb{C}$, function g_β has an L^1 or L^2 (or both) Fourier original.
iii) For $\operatorname{Im} \beta > 0$, explicitly determine (if any) a Fourier original for g_β.
(hint: $g_\beta = \partial_\xi \ldots$)

Exercise 30. Let

$$F(x) := \int_0^{+\infty} \frac{1 - e^{-xy^2}}{y^2}\, dy.$$

i) Determine the domain D of F. Justify carefully.
ii) [5] Discuss differentiability of F determining $D' \subset D$ for which $\exists \partial_x F(x)$ for every $x \in D'$.
iii) Use ii) to determine F explicitly.

September 2024

Exercise 31. Let

$$g(\xi) = (1-\xi^2)1_{[-1,1]}(\xi), \ \xi \in \mathbb{R}.$$

i) Discuss the problem of existence of a Fourier original f for g with $f \in L^1$ and/or $f \in L^2$. Justify your answers with care.
ii) Determine f explicitly.

Exercise 32. On $V := \left\{ f \in \mathscr{C}^1([0,1]) \ : \ \int_0^1 f(x)\,dx = 0 \right\}$ define

$$\|f\|_* := \int_0^1 |f'(x)|\,dx, \ \ \|f\|_{**} := \|f'\|_\infty.$$

i) Check that $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are well defined norms on V.
ii) Let

$$g_n(x) := \begin{cases} \sqrt{n}x, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \\ 2\sqrt{x} - \frac{1}{\sqrt{n}}, & \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$

Determine $c_n \in \mathbb{R}$ in such a way that $f_n := g_n - c_n \in V$. Compute then $\|f_n\|_*$ and $\|f_n\|_{**}$.
iii) Discuss relations between $\|\cdot\|_*$ and $\|\cdot\|_{**}$.

Exercise 33. Let

$$F(x) := \int_0^{+\infty} e^{-xy} \frac{1-\cos y}{y}\,dy.$$

i) Determine the domain of definition of F, that is the set of $x \in \mathbb{R}$ such that $F(x)$ is well defined.
ii) Determine the set of x for which $\exists F'(x)$ and compute it.
iii) Use ii) to explicitly determine F.

November 2024

Exercise 34. Let $X := L^\infty([0,+\infty[)$. For $a > 0$ define

$$\|f\|_a := \int_0^{+\infty} e^{-ax}|f(x)|\,dx.$$

i) Check that $\|\cdot\|_a$ is a well defined norm on X for every $a > 0$.
ii) Show that, for $a < b$, $\|\cdot\|_a$ is stronger than $\|\cdot\|_b$
iii) Are $\|\cdot\|_a$ and $\|\cdot\|_b$ equivalent? Justify carefully.

Exercise 35. Let $(X, \mathscr{F}, \mu)$ be a measure space, and $H := L^2(X)$ be the Hilbert space w.r.t. the usual scalar product

$$\langle f, g\rangle_2 = \int_X fg\,d\mu$$

Let also $E \in \mathscr{F}$ and define

$$U := \{f \in H \;:\; f = 0,\; a.e. \text{ on } E^c\}\,.$$

i) Check that U is a closed linear subspace of H.
ii) Define $\Pi_U f := f 1_E$. Show that $\Pi_U f$ is the orthogonal projection of f on U.

Exercise 36. Let $a, b > 0$ real numbers with $a \neq b$, and

$$g_{a,b}(\xi) := \frac{e^{-a|\xi|} - e^{-b|\xi|}}{\xi},\; \xi \in \mathbb{R}\backslash\{0\}.$$

i) Is $g_{a,b} \in L^1(\mathbb{R})$? Is $g_{a,b} \in L^2(\mathbb{R})$? Determine whether $g_{a,b}$ has a Fourier original $f_{a,b}$ in $L^1(\mathbb{R})$ and/or in $L^2(\mathbb{R})$.
ii) For the case(s) (L^1 and/or L^2) for which there is a Fourier original, determine it.

February 2025

Exercise 37. Let
$$F(\lambda) := \int_0^{+\infty} e^{-\lambda x} \frac{\sin x}{x}\, dx.$$

i) Determine $\lambda \in \mathbb{R}$ for which $F(\lambda)$ makes sense as Lebesgue integral.
ii) Determine $\lambda \in \mathbb{R}$ for which $\exists \partial_\lambda F(\lambda)$ and compute it.
iii) Determine $F(\lambda)$ for λ at i).

Exercise 38. Let
$$f_n(x) := ne^{-nx}(1 - e^{-x}),\ x \in I := [0, +\infty[.$$

i) Is (f_n) point-wise convergent?
ii) Is (f_n) convergent in $L^\infty(I)$?
iii) Is (f_n) convergent in $L^1(I)$?
iv) Is (f_n) convergent in $L^2(I)$?

Exercise 39. Let
$$g(\xi) := \frac{1}{(\xi^2 + a^2)(\xi^2 + b^2)},\ \xi \in \mathbb{R},$$
with $a, b > 0$ and $a \neq b$.

i) Discuss the problem of existence of a Fourier original for g in L^1 and in L^2 cases. Is the Fourier original also L^∞? Justify carefully and determine the Fourier original explicitly.
ii) Show that $\sharp g(\sharp)$ has a Fourier original in L^2 and find it in term of the original f of g. Justify carefully your answer.

July 2025

Exercise 40. Let $V := \left\{ f \in L^1([0,1]) \ : \ \int_0^1 x|f(x)|\,dx < +\infty \right\}$. We accept V is a vector space on $\mathbb{R}$ with usual definitions of sum and product by scalars. On V we define

$$\|f\| := \int_0^1 x|f(x)|\,dx.$$

i) Check that $\|\cdot\|$ is a norm on V with vanishing in the weak form $\|f\| = 0$ iff $f = 0$ a.e..

ii) Define

$$f_n(x) := \begin{cases} n, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \\ \frac{1}{x}, & \frac{1}{n} \leqslant x \leqslant 1. \end{cases}$$

Is $(f_n) \subset V$? Is (f_n) convergent in V respect to $\|\cdot\|$? If yes, to what?

iii) By definition, $V \subset L^1([0,1])$, so on V we can consider also the $\|\cdot\|_1$ norm. Is there any relation between $\|\cdot\|$ and $\|\cdot\|_1$ norms? Justify carefully your answers.

iv) Is V, equipped with $\|\cdot\|$ norm a Banach space?

Exercise 41. Let $H := L^2(\mathbb{R})$ equipped with usual scalar product $\langle f, g\rangle := \int_{\mathbb{R}} f(x)g(x)\,dx$. Let

$$U := \{u \in H \ : \ u(-x) = -u(x),\ a.e.\ x > 0\}.$$

i) Check that U is a closed linear subspace of H.

ii) Check that the orthogonal projection on U, $\Pi_U f$, is

$$[\Pi_U f](x) := \frac{1}{2}\left(f(x) - f(-x)\right),\ x \in \mathbb{R}.$$

Exercise 42. i) What is the convolution $f * g$ and when is it well defined and in $L^1(\mathbb{R})$?

ii) Show s remarkable formula for the FT of $f * g$.

iii) Given $g \in L^1(\mathbb{R})$, consider the equation in the unknown $f \in L^1(\mathbb{R})$:

$$(f * f)(x) = g(x),\ \text{a.e. } x \in \mathbb{R}.$$

Discuss whether or not is solvable and, in this case, determine all the possible solutions when $g(x) = e^{-x^2}$.

September 2025

Exercise 43 (11)**.** Let $x \in \mathbb{R}$ be fixed and set

$$F(y) := \int_{\mathbb{R}} e^{-(x+iy)^2}\, dx.$$

i) [5] Check that F is well defined for every $y \in \mathbb{R}$.
ii) [6] Show that F is differentiable for every $y \in \mathbb{R}$, and compute $\partial_y F(y)$. Use this to educe $F(y)$.

Exercise 44 (11)**.** Let. We define

$$V := \left\{ f \in L([0,+\infty[) \ : \ \|f\| := \int_0^{+\infty} \frac{|f(x)|}{1+x}\, dx < +\infty \right\}.$$

i) [3] Check that $\|\cdot\|$ is a well defined norm on V (that is $\|f\| < +\infty$ for every $f \in V$) with vanishing in the weak form $\|f\| = 0$ iff $f = 0$ a.e..
ii) [2] Check that $L^2([0,+\infty[) \subset V$ and that $\|\cdot\|_2$ is stronger than $\|\cdot\|$.
iii) [4] Let $f_n(x) := \frac{1}{\sqrt{x}} 1_{[1/n,n]}(x)$. Is $(f_n) \subset L^2([0,+\infty[)$? Is (f_n) convergent in $\|\cdot\|_2$? Is $(f_n) \subset V$? Is (f_n) convergent in $\|\cdot\|$? In case of affirmative answer(s), determine also the limit(s) function.
iv) [2] Are $\|\cdot\|_2$ and $\|\cdot\|$ equivalent?

Exercise 45 (11)**.** Let

$$g(\xi) := \frac{3\xi^2 - 1}{(1+\xi^2)^3},\ \xi \in \mathbb{R}.$$

i) [4] Discuss the problem of the existence of a Fourier original of g in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$.
ii) [7] Compute $\partial^2_\xi \frac{1}{1+\xi^2}$ and determine a relation between this derivative and g. Use this to determine a Fourier original of g. (here ∂^2_ξ stands for the second partial derivative respect to ξ)

November 2025

Exercise 46. Let
$$F(x) := \int_0^{+\infty} \frac{\log(1+x^2y^2)}{1+y^2}\, dy.$$
i) Determine the domain D of definition of F.
ii) Determine the domain D' where F' is well defined. Is $D' = D$? Justify carefully.
iii) Use ii) to explicitly determine F.

Exercise 47. Let $X := \{f \in \mathscr{C}^1([0,1]) \ :\ f(0) = 0\}$ and define
$$\|f\| := \int_0^1 \frac{|f(x)|}{x}\, dx, \quad \|f\|_* := \|f'\|_\infty.$$
i) Check that $\|f\|$ and $\|f\|_*$ are well defined norms on X (it might be useful to remind that $f(x) = f(0) + f'(0)x + o(x)$).
ii) Prove that $\|\cdot\|_*$ is stronger than $\|\cdot\|$.
iii) Compute $\|f_n\|$ and $\|f_n\|_*$ for $f_n(x) = x^n$, $n \geqslant 1$. What can be drawn by this calculation?

Exercise 48. .
i) What is the convolution product $f * g$ of two functions f, g? Under which conditions is $f * g$ is well defined? Show that $\widehat{f * g} = \widehat{f}\widehat{g}$.
ii) True or false: if g has a Fourier original and $h := g^2$, then h has a unique Fourier original. Justify your answer.
iii) Consider the equation
$$\int_{-\infty}^{+\infty} f(x-y)e^{-y^2}\, dy = xe^{-ax^2}.$$
Determine for which values of a there exists a unique solution $f \in L^1(\mathbb{R})$. For such a, determine f.

SOLUTIONS

Exercise 1. i) Let

$$f(x,\xi) := \frac{e^{-\xi x^2} - e^{-x^2}}{x},\ x \in]0,+\infty[.$$

We have to check for which ξ we have $f(\sharp,\xi)L^1([0,+\infty[)$. Since $f(\sharp,\xi) \in \mathscr{C}(]0,+\infty[)$ we have to check the behaviour at $x=0$ and $x=+\infty$. Recalling that $e^y = 1+y+o(y)$ we have

$$f(x,\xi) = \frac{(1-\xi x^2 + o(x^2)) - (1-x^2+o(x^2))}{x} = -(\xi-1)x + o(x) \longrightarrow 0,\ x \longrightarrow 0.$$

In particular, $f(\sharp,\xi)$ is integrable at $x=0$ for every $\xi \in \mathbb{R}$. At $+\infty$, for $\xi > 0$ we have

$$|f(x,\xi)| \leqslant e^{-\xi x^2} + e^{-x^2},\ \forall x \geqslant 1,$$

and since both $e^{-\xi x^2}, e^{-x^2}$ are integrable at $+\infty$ we conclude that $f(\sharp,\xi)$ is integrable at $+\infty$. Thus, $f(\sharp,\xi) \in L^1([0,+\infty[)$ for every $\xi > 0$. For $\xi = 0$ we have

$$f(x,0) = \frac{1-e^{-x^2}}{x} \sim_{+\infty} \frac{1}{x},\ \Longrightarrow\ f(\sharp,0) \notin L^1([0,+\infty[).$$

For $\xi < 0$, $f(x,\xi) \longrightarrow +\infty$ for $x \longrightarrow +\infty$ thus certainly $f(\sharp,\xi) \notin L^1([0,+\infty[)$. Conclusion: F is well defined for $\xi \in]0,+\infty[$.

ii) We apply differentiation theorem to deduce

$$\partial_\xi F(\xi) = \int_0^{+\infty} \partial_\xi f(x,\xi)\, dx.$$

To this aim we may notice that

- $\exists \partial_\xi f(x,\xi) = \frac{1}{x} e^{-\xi x^2}(-x^2) = -xe^{-\xi x^2}$, $\forall \xi \in \mathbb{R}$, $\forall x > 0$ (thus a.e. $x \in [0,+\infty[$);
- $|\partial_\xi f(x,\xi)| = xe^{-\xi x^2} \leqslant xe^{-\varepsilon x^2} =: g(x)$ for every $\xi \geqslant \varepsilon > 0$, $\forall x > 0$.

Since $g \in L^1([0,+\infty[)$ we may conclude that $\exists \partial_\xi F(\xi)$ for every $\xi \in [\varepsilon,+\infty[$, and this for every $\varepsilon > 0$. Since $\varepsilon > 0$ is arbitrary, we conclude that, for every $\xi > 0$,

$$\partial_\xi F(\xi) = \int_0^{+\infty} \partial_\xi f(x,\xi)\, dx = \int_0^{+\infty} -xe^{-\xi x^2}\, dx = \frac{1}{2\xi}\int_0^{+\infty} \partial_x e^{-\xi x^2}\, dx = \frac{1}{2\xi}\left[e^{-\xi x^2}\right]_{x=0}^{x=+\infty} = -\frac{1}{2\xi}.$$

iii) Since

$$\partial_\xi F(\xi) = -\frac{1}{2\xi},\ \Longrightarrow\ F(\xi) = -\frac{1}{2}\log\xi + c,\ \forall \xi > 0.$$

Clearly $F(1) = 0$ by which $c = 0$. □

Exercise 2. i) If $f \in \mathscr{C}^1([0,1])$, $f' \in \mathscr{C}([0,1])$ thus, in particular, $|f'(x)| \leqslant \|f'\|_\infty$ for every $x \in [0,1]$. Therefore

$$\int_0^1 \frac{|f'(x)|}{\sqrt{x}}\, dx \leqslant \int_0^1 \frac{\|f'\|_\infty}{\sqrt{x}}\, dx \leqslant \|f'\|_\infty \int_0^1 \frac{1}{\sqrt{x}}\, dx < +\infty.$$

This shows that $\|f\|_*$ is well defined for every $f \in V$. Clearly, $\|f\|_* \geqslant 0$. We have to check the characteristic properties of any norm:

- vanishing: $\|f\|_* = 0$ iff $f(1) = 0$ and $\int_0^1 \frac{|f'(x)|}{\sqrt{x}}\, dx = 0$. By this $f' \equiv 0$ on $[0,1]$, thus f is constant, and since $f(1) = 0$ we conclude $f \equiv 0$.
- homogeneity: $\|\alpha f\|_* = |\alpha f(1)| + \int_0^1 \frac{(\alpha f)'(x)|}{\sqrt{x}}\, dx = |\alpha| \left(|f(1)| + \int_0^1 \frac{|f'(x)|}{\sqrt{x}}\, dx \right) = |\alpha| \|f\|_*$.
- triangular inequality: straightforward.

ii) We have to prove that

$$\exists C > 0, : \ \|f\|_\infty \leqslant C\|f\|_*, \ \forall f \in V.$$

Let $f \in V = \mathscr{C}^1([0,1])$. Notice that, according to the Fundamental Theorem of Integral Calculus,

$$f(1) - f(x) = \int_x^1 f'(y)\, dy, \implies f(x) = f(1) - \int_x^1 f'(y)\, dy.$$

Therefore

$$|f(x)| \leqslant |f(1)| + \int_x^1 |f'(y)|\, dy \leqslant |f(1)| + \int_0^1 |f'(y)| dy \leqslant |f(1)| + \int_0^1 \frac{|f'(y)}{\sqrt{y}}\, dy = \|f\|_*.$$

By this

$$\|f\|_\infty = \max_{x \in [0,1]} |f(x)| \leqslant \|f\|_*.$$

iii) Easily $f_n \in \mathscr{C}([0,1])$ and since

$$f_n'(x) = \begin{cases} -\frac{1}{2} n^{3/2}, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \\ -\frac{1}{2} x^{-3/2}, & \frac{1}{n} < x \leqslant 1, \end{cases}$$

easily we see that $f_n' \in \mathscr{C}([0,1])$, thus, in conclusion, $f_n \in V$ for every n. Computing norms, since $f_n' < 0$ we have

$$\|f_n\|_\infty = \max_{x \in [0,1]} |f_n(x)| = |f_n(0)| = \frac{1}{2} n^{1/2}.$$

On the other hand

$$\begin{aligned} \|f_n\|_* &= 1 + \int_0^1 \frac{|f_n'(x)|}{\sqrt{x}}\,dx = 1 + \tfrac{1}{2}n^{3/2}\int_0^{1/n}\frac{1}{\sqrt{x}}\,dx + \tfrac{1}{2}\int_{1/n}^1 \frac{x^{-3/2}}{x^{1/2}}\,dx \\ &= 1 + n^{3/2}\left[x^{1/2}\right]_{x=0}^{x=1/n} + \tfrac{1}{2}\left[-x^{-1}\right]_{x=1/n}^{x=1} \\ &= 1 + n^{1/2} + \tfrac{1}{2}(n-1). \end{aligned}$$

By this it follows that $\|\cdot\|_\infty$ and $\|\cdot\|_*$ cannot be equivalent. Otherwise, there would be a constant C such that

$$\|f_n\|_* \leqslant C\|f_n\|_\infty,\ \forall n,\ \Longrightarrow\ 1 + n^{1/2} + \frac{1}{2}(n-1) \leqslant \frac{C}{2}n^{1/2},$$

which is manifestly impossible. □

Exercise 3. i) We have

$$|f_a(x)| \leqslant \frac{1}{a^2+x^2} \in L^1(\mathbb{R}),\ \forall a > 0.$$

For $a = 0$ this bound cannot be used globally, but since

$$|f_0(x)| \leqslant \frac{1}{x^2},$$

integrability at $\pm\infty$ is ensured. It remains to check the behaviour at $x = 0$: since $\cos x = 1 - \frac{x^2}{2} + o(x^2)$,

$$f_0(x) = \frac{1 - \left(1 - \frac{x^2}{2} + o(x^2)\right)}{x^2} = \frac{\frac{x^2}{2} + o(x^2)}{x^2} \sim_0 \frac{1}{2},$$

thus f_0 is integrable also at $x = 0$. We conclude $f_0 \in L^1(\mathbb{R})$.

ii) To compute $\widehat{f_a}$ we notice that

$$\widehat{f_a}(\xi) = \widehat{\frac{1}{a^2+\sharp^2}}(\xi) - \widehat{\cos\sharp\frac{1}{a^2+\sharp^2}}(\xi).$$

Recall that

$$\widehat{\frac{1}{a^2+\sharp^2}}(\xi) = \frac{1}{2a}e^{-a|\xi|}.$$

Moreover, $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ thus

$$\widehat{\cos\sharp g}(\xi) = \frac{1}{2}\left(\widehat{e^{i\sharp}g}(\xi) + \widehat{e^{-i\sharp}g}(\xi)\right) = \frac{1}{2}\left(\widehat{g}(\xi-1) + \widehat{g}(\xi+1)\right).$$

Hence

$$\widehat{\cos\sharp\frac{1}{a^2+\sharp^2}}(\xi) = \frac{1}{2}\left(\frac{1}{2a}e^{-a|\xi-1|} + \frac{1}{2a}e^{-a|\xi+1|}\right).$$

In conclusion

$$\widehat{f_a}(\xi) = \frac{1}{2a}\left(e^{-a|\xi|} - \frac{1}{2}\left(e^{-a|\xi-1|} + e^{-a|\xi+1|}\right)\right)$$

Finally, being f_a even,

$$\int_0^{+\infty} f_a(x)\ dx = \frac{1}{2}\int_{-\infty}^{+\infty} f_a(x)\ dx = \frac{1}{2}\widehat{f_a}(0) = \frac{1}{4a}\left(1 - \frac{1}{2}\left(e^{-a} + e^{-a}\right)\right) = \frac{1-e^{-a}}{4a}.$$

iii) Ideally, we have to take $a = 0$. We consider

$$\lim_{a\to 0+}\int_0^{+\infty} f_a(x)\ dx = \lim_{a\to 0+}\frac{1-e^{-a}}{4a} = \frac{1}{4}.$$

Let us check that

$$\lim_{a\to 0+}\int_0^{+\infty} f_a(x)\ dx = \int_0^{+\infty}\lim_{a\to 0+} f_a(x)\ dx = \int_0^{+\infty} f_0(x)\ dx.$$

The unique issue concerns the swap between limit and integral. This might be ensured by both monotone or dominated convergence. Clearly

$$\lim_{a\to 0+} f_a(x) = f_0(x),\ \forall x > 0.$$

Moreover

$$|f_a(x)| = \frac{|1-\cos x|}{a^2+x^2} \leqslant \frac{|1-\cos x|}{x^2} =: g(x) \in L^1([0,+\infty[).$$

By this the conclusion follows.

iv). To compute $\widehat{f_0}(\xi)$ we follow the idea exploited in iii). First:

$$\widehat{f_0}(\xi) = \int_{\mathbb{R}} f_0(x)e^{-i\xi x}\ dx = \int_{\mathbb{R}}\lim_{a\to 0+} f_a(x)e^{-i\xi x}\ dx = \lim_{a\to 0+}\int_{\mathbb{R}} f_a(x)e^{-i\xi x}\ dx.$$

Last identity must be justified applying a limit theorem. By what we checked in Q3 we may easily see that dominated convergence applies: indeed

$$\left|f_a(x)e^{-i\xi x}\right| = |f_a(x)| \leqslant g(x).$$

Therefore

$$\begin{aligned}
\widehat{f_0}(\xi) &= \lim_{a\to 0+}\widehat{f_a}(\xi) = \lim_{a\to 0+}\left(\tfrac{1}{2a}\left(e^{-a|\xi|} - \tfrac{1}{2}\left(e^{-a|\xi-1|} + e^{-a|\xi+1|}\right)\right)\right) \\
&\overset{(H)}{=} \tfrac{1}{2}\lim_{a\to 0+}\left(-|\xi|e^{-a|\xi|} - \tfrac{1}{2}\left(-|\xi-1|\,e^{-a|\xi-1|} - |\xi+1|\,e^{-a|\xi-1|}\right)\right) \\
&= -\tfrac{1}{2}\left(|\xi| - \tfrac{1}{2}\left(|\xi-1| + |\xi+1|\right)\right) \\
&= \tfrac{1}{2}\left(1-|\xi|\right)1_{[-1,1]}(\xi). \quad \square
\end{aligned}$$

Exercise 4. i) For $x = 0$, $f_n(0) = 0 \longrightarrow 0$. For $x > 0$,

$$\lim_{n\to+\infty} f_n(x) = x \lim_{n\to+\infty} \frac{n^{3/2}}{e^{nx}} = 0,$$

being $n^{3/2} = o(e^{nx})$. Thus f_n converges point wise to 0 on $[0, +\infty[$.

ii) Clearly $f_n(x) \geqslant 0$ on $[0, +\infty[$. Moreover

$$f_n'(x) = n^{3/2}\left(e^{-nx} - nxe^{-nx}\right) = n^{3/2}e^{-nx}(1 - nx) \geqslant 0, \iff 1 - nx \geqslant 0, \iff x \leqslant \frac{1}{n}.$$

Thus f_n attains a global maximum at $x = \frac{1}{n}$. Therefore

$$\|f_n\|_\infty = \sup_{[0,+\infty[} |f_n(x)| = f_n\left(\frac{1}{n}\right) = n^{1/2}e^{-1} < +\infty,$$

that is $(f_n) \subset L^\infty([0, +\infty[$. However, since $\|f_n\|_\infty = \frac{\sqrt{n}}{e} \longrightarrow +\infty$, (f_n) cannot converge in L^∞.

iii) Let's compute

$$\|f_n\|_1 = \int_0^{+\infty} |f_n(x)|\, dx = \frac{1}{\sqrt{n}} \int_0^{+\infty} nxe^{-nx} d(nx) = \frac{1}{\sqrt{n}} \int_0^{+\infty} ye^{-y}\, dy < +\infty,$$

thus $(f_n) \subset L^1([0, +\infty[)$. By this it follows also that $f_n \stackrel{L^1}{\longrightarrow} 0$ because

$$\|f_n\|_1 = \frac{C}{\sqrt{n}} \longrightarrow 0.$$

iv) Let's compute

$$\|f_n\|_2^2 = \int_0^{+\infty} n^3x^2e^{-2nx}\, dx \stackrel{y=nx}{=} \int_0^{+\infty} y^2e^{-2y}\, dy < +\infty,$$

thus $(f_n) \subset L^2([0, +\infty[)$. If $f_n \stackrel{L^2}{\longrightarrow} f$, then there would be (f_{n_k}) such that $f_{n_k} \longrightarrow f$ a.e. and since $f_n \longrightarrow 0$ everywhere, we conclude $f = 0$ a.e.. Thus the unique possibility is $f_n \stackrel{L^2}{\longrightarrow} 0$. However, $\|f_n - 0\|_2 = \|f_n\|_2 \equiv C > 0 \not\longrightarrow 0$. □

Exercise 9. See LN for the Definitions and proofs, and of i).

ii) We have

$$\frac{1}{a^2 + x^2} = \frac{1}{2a}\widehat{e^{-a|\sharp|}}\left(\frac{x}{2\pi}\right), \implies \frac{1}{(a^2 + x^2)^2} = \frac{1}{4a^2}\widehat{e^{-a|\sharp|} * e^{-a|\sharp|}}\left(\frac{x}{2\pi}\right).$$

Now, recall that in general, for the 1-dim FT

$$\widehat{f\left(\frac{\sharp}{\lambda}\right)}(\xi) = |\lambda|\widehat{f}(\lambda\xi)$$

so

$$\widehat{\frac{1}{(a^2+\sharp^2)^2}}(\xi) = \frac{2\pi}{4a^2}\widehat{\widehat{e^{-a|\sharp|} * e^{-a|\sharp|}}}(2\pi\xi) = \frac{\pi}{2a^2}e^{-a|\sharp|} * e^{-a|\sharp|}(-2\pi\xi),$$

thanks to the inversion formula. To close the calculation, we need to compute the convolution

$$e^{-a|\sharp|} * e^{-a|\sharp|}(x) = \int_{\mathbb{R}} e^{-a|x-y|}e^{-a|y|}\, dy.$$

Easily we see that, by changing x with $-x$ the output doesn't change, so we can consider the case $x \geqslant 0$, for $x < 0$ we will induce the value by symmetry. We have then

$$\begin{aligned} e^{-a|\sharp|} * e^{-a|\sharp|}(x) &= \int_{-\infty}^{0} e^{-a(x-y)}e^{ay}\, dy + \int_0^x e^{-a(x-y)}e^{-ay}\, dy + \int_x^{+\infty} e^{a(x-y)}e^{-ay}\, dy \\ &= e^{-ax}\int_{-\infty}^0 e^{2ay}\, dy + e^{-ax}\int_0^x dy + e^{ax}\int_x^{+\infty} e^{-2ay}\, dy \\ &= e^{-ax}\left[\tfrac{e^{2ay}}{2a}\right]_{y=-\infty}^{y=0} + xe^{-ax} + e^{ax}\left[-\tfrac{e^{-2ay}}{2a}\right]_{y=x}^{y=+\infty} \\ &= e^{-ax}\tfrac{1}{2a} + xa^{-ax} + \tfrac{e^{-ax}}{2a} = \tfrac{e^{-ax}}{a}\left(1+ax\right), \end{aligned}$$

being $a > 0$. Symmetrizing,

$$e^{-a|\sharp|} * e^{-a|\sharp|}(x) = \frac{e^{-a|x|}}{a}\left(1+a|x|\right),$$

so we conclude that

$$\widehat{\frac{1}{(a^2+\sharp^2)^2}}(\xi) = \frac{\pi}{2a^3}e^{-2\pi a|\xi|}\left(1+2\pi a|\xi|\right). \qquad \square$$

Exercise 10. i) We notice that

$$\int_{-1}^{0} f = \int_0^1 f, \quad \iff \quad \int_{-1}^{1} f(1_{[-1,0]} - 1_{[0,1]}) = 0,$$

from which, denoted by $u := 1_{[-1,0]} - 1_{[0,1]}$ we have

$$S = \{f \in H \ : \ \langle f, u\rangle = 0\} = \mathrm{Span}(u)^{\perp}.$$

Since the orthogonal space of any set U is a closed linear sub-space of H we conclude that S is closed.

ii) Since H is Hilbert and S is closed, there exists $\Pi_S f$ for every $f \in H$. Let $U := \mathrm{Span}(u)$. Then

$$\Pi_S f = f - \Pi_U f,$$

and since

$$\Pi_U f = \langle f, \frac{u}{\|u\|}\rangle\frac{u}{\|u\|} = \frac{1}{\|u\|^2}\langle f, u\rangle u,$$

with

$$\|u\|^2 = \int_{-1}^{1} u^2 = \int_{-1}^{1} 1 = 2,$$

we get

$$\Pi_S f = f - \frac{1}{2}\langle f, u\rangle u.$$

In particular,

$$\Pi_S x = x - \frac{1}{2}\langle x, u\rangle u = x - \frac{1}{2}\int_{-1}^{1}(-|x|)\, dx u = x + \frac{1}{2}u. \quad \square$$

Exercise 11. i) Clearly $g \in \mathscr{C}(\mathbb{R}\backslash\{0\})$. At $\xi = 0$, recalling that

$$\sin\xi = \xi - \frac{\xi^3}{6} + o(\xi^3), \quad \cos\xi = 1 - \frac{\xi^2}{2} + o(\xi^2),$$

we have

$$g(\xi) = \frac{\xi - \frac{\xi^3}{6} + o(\xi^3) - \xi + \frac{\xi^3}{2} + o(\xi^3)}{\xi^3} = \frac{1}{3} + \frac{o(\xi^3)}{\xi^3} \overset{\xi\to 0}{\longrightarrow} \frac{1}{3},$$

thus g can be extended by continuity at $\xi = 0$. In other words, we may consider g as a continuous function on $\mathbb{R}$, integrable on every $[a, b] \subset \mathbb{R}$. Hence, for the integrability on $\mathbb{R}$, we have to check the behaviour at $\pm\infty$. Here we have

$$|g(\xi)| \leqslant \frac{1+|\xi|}{|\xi^3|} \sim_{\pm\infty} \frac{1}{|\xi|^2},$$

from which we deduce integrability at $\pm\infty$. In conclusion, $g \in L^1(\mathbb{R})$. From same arguments we have that $f \in L^2([a, b])$, for every $[a, b]$ and since

$$|g(\xi)|^2 \leqslant \left(\frac{1+|\xi|}{|\xi^3|}\right)^2 \sim_{\pm\infty} \frac{1}{|\xi|^4},$$

we deduce that $|g|^2$ is integrable, thus $g \in L^2(\mathbb{R})$.

ii) According to the inversion formula, $g = \widehat{f}$ provided g and $\widehat{g}$ are both $L^1(\mathbb{R})$ functions. We remind that this last follows if $g, g', g'' \in L^1(\mathbb{R})$. About g, this has been checked in Q1. We check for g' (the check for g'' being similar). We have

$$g'(\xi) = \frac{\xi^3(\xi\sin\xi) - 3\xi^2(\sin\xi - \xi\cos\xi)}{\xi^6} = \frac{\xi^2\sin\xi - 3(\sin\xi - \xi\cos\xi)}{\xi^4}.$$

At $\xi = 0$ we have

$$g'(\xi) = \frac{\xi^2\left(\xi - \frac{\xi^3}{6} + o(\xi^3)\right) - 3\left(\xi - \frac{\xi^3}{6} + \frac{\xi^5}{5!} + o(\xi^5) - \xi + \frac{\xi^3}{2} - \frac{\xi^5}{24!}\right)}{\xi^4} = c\xi + o(\xi),$$

thus g' is continuous at $\xi = 0$. At $\pm\infty$, $|g'(\xi)| \leqslant \frac{a|\xi|^4+b}{|\xi|^6} \sim \frac{C}{|\xi|^2}$, that is $g' \in L^1(\mathbb{R})$. A similar check can e made for g''. We conclude that $\widehat{g} \in L^1$, hence inversion formula applies.

The existence of an L^2 Fourier original is much simpler: since $g \in L^2(\mathbb{R})$ then, by the Fourier-Plancherel theorem, $g = \widehat{f}$ for some $f \in L^2(\mathbb{R})$.

iii) Following the hint, we notice that

$$\xi g(\xi) = -\partial_\xi \frac{\sin\xi}{\xi} = -\frac{1}{2}\partial_\xi \widehat{\mathrm{rect}_1}(\xi) = \frac{1}{2}\widehat{i \bullet \mathrm{rect}_1}(\xi) = \frac{i}{2}\widehat{\bullet\mathrm{rect}_1}(\xi).$$

Now, if $g = \widehat{f}$,

$$i\xi g(\xi) = i\xi\widehat{f}(\xi) = \widehat{f}'(\xi),$$

from which

$$\widehat{f}'(\xi) = -\frac{1}{2}\widehat{\bullet\mathrm{rect}_1}(\xi),$$

and, by uniqueness of the FT,

$$f'(x) = -\frac{1}{2}x\mathrm{rect}_1(x).$$

From this, it follows that f is constant on $]-\infty, -1]$ and on $[1, +\infty[$ and since $f \in L^1(\mathbb{R})$, $f \equiv 0$ on these intervals. Moreover, for $x \in [-1, 1]$

$$f(x) = f(x) - f(-1) = \int_{-1}^{x} -\frac{1}{2}y\, dy = -\frac{1}{4}(x^2 - 1).$$

Thus

$$f(x) = \frac{1 - x^2}{4} 1_{[-1,1]}(x). \quad \square$$

Exercise 12. i) Since $f(t) = f(0) + f'(0)t + o(t) = f'(0)t + o(t)$ for $f \in V$, we have

$$\frac{|f(t)|}{\sqrt{t}} = |f'(0)|\sqrt{t} + \frac{o(t)}{\sqrt{t}} \longrightarrow 0,\ t \longrightarrow 0.$$

Thus $\|f\|_*$ is well posed. Clearly positivity holds. Vanishing:

$$\|f\|_* = 0, \iff \sup_{t\in]0,1]} \frac{|f(t)|}{\sqrt{t}} = 0, \iff \frac{|f(t)|}{\sqrt{t}} = 0,\ t \in]0, 1], \iff f(t) = 0,\ t \in]0, 1],$$

and since $f(0) = 0$ ($f \in V$) we conclude $f \equiv 0$ on $[0, 1]$. Homogeneity and triangular inequality are straightforward.

ii) Since $f \in V$ implies, in particular, $f \in \mathscr{C}^1([0, 1])$, $\|f\|_{**} = \|f'\|_\infty$ is well defined ($f' \in \mathscr{C}([0, 1])$). Positivity is evident. Vanishing:

$$\|f\|_{**} = \|f'\|_\infty = 0, \iff f' \equiv 0, \iff f \equiv C,$$

for some constant C. Since $f(0) = 0$ we deduce $C = 0$ and $f \equiv 0$ on $[0,1]$. Homogeneity and triangular inequality are obvious consequences of linearity of the derivative and analogous properties of $\|\cdot\|_\infty$ norm.

iii) We claim that $\|\cdot\|_{**}$ is stronger than $\|\cdot\|_*$ but they are not equivalent. Indeed: from fundamental thm of integral calculus, for $f \in V$,

$$f(t) = f(t) - f(0) = \int_0^t f'(s)\, ds,$$

from which

$$|f(t)| \leqslant \int_0^t |f'(s)|\, ds \leqslant \int_0^t \max_{[0,1]} |f'|\, ds = \|f\|_{**} t,\ \forall t \in [0,1].$$

Thus

$$\max_{t \in]0,1]} \frac{|f(t)|}{\sqrt{t}} \leqslant \max_{t \in]0,1]} \sqrt{t} \|f\|_{**} = 1 \cdot \|f\|_{**}.$$

The vice versa does not hold. Indeed, we may take $f_n(t) = t^n$. Clearly $f_n \in V$ for every $n \geqslant 1$. We have

$$\|f_n\|_* = \max_{t \in]0,1]} \frac{|t^n|}{\sqrt{t}} = \max_{t \in]0,1]} t^{n-1/2} = 1,\quad \|f_n\|_{**} = \max_{t \in [0,1]} |nt^{n-1}| = n.$$

From this we draw that there cannot be a constant c such that $\|f_n\|_{**} \leqslant c\|f\|_*$ for every f, otherwise

$$n = \|f_n\|_{**} \leqslant c\|f_n\|_* = c,\ \forall n \in \mathbb{N},\ n \geqslant 1,$$

which is manifestly impossible. □

Exercise 13. i) We check first that $\langle \cdot, \cdot \rangle$ is well defined. Indeed, this is the standard $L^2([0,1])$ scalar product of f', g'. Since they are assumed to be in H, $f', g' \in L^2([0,1])$, thus $\langle f, g \rangle_H$ makes sense. We check now the characteristic properties of a scalar product:

- (positivity) $\langle f, f \rangle_H = \int_0^1 (f')^2\, dx \geqslant 0$.
- (vanishing) $\langle f, f \rangle_H = 0$ iff $\int_0^1 (f')^2 = 0$. Since $(f')^2 \geqslant 0$, by a well known result, $(f')^2 = 0$ a.e., that is $f' = 0$ a.e., hence f is a.e. constant and since $f(1) = 0$ we conclude $f = 0$ a.e.
- (homogeneity) $\langle \lambda f, g \rangle_H = \int_0^1 (\lambda f)' g' = \int_0^1 \lambda f' g' = \lambda \langle f, g \rangle_H$.
- (symmetry) evident.

ii) We notice that

$$\begin{aligned} \int_0^1 f(x)\, dx &= \int_0^1 (x)' f(x)\, dx \overset{parts}{=} [x f(x)]_{x=0}^{x=1} - \int_0^1 x f'(x)\, dx = -\int_0^1 \left(\tfrac{x^2}{2}\right)' f'(x)\, dx \\ &= -\left\langle \tfrac{x^2-1}{2}, f \right\rangle_H . \end{aligned}$$

Thus
$$S = \left\{ f \in H \ : \ \left\langle \sharp^2 - 1, f \right\rangle_H = 0 \right\rangle.$$
Thus $v = x^2 - 1$.

iii) Let $V := \text{Span}\left(x^2 - 1\right) = S^{\perp}$. This is a one dimensional space, hence it is closed. We have
$$f = \Pi_S f + \Pi_V f, \ \forall f \in H, \implies \Pi_S f = f - \Pi_V f.$$
Π_V is easy:
$$\Pi_V f = \langle f, \frac{x^2-1}{\|\sharp^2 - 1\|_H} \rangle_H \frac{x^2-1}{\|\sharp^2 - 1\|_H} = \frac{1}{\|\sharp^2 - 1\|_H^2} \langle x^2 - 1, f \rangle_H x^2.$$
We have
$$\|\sharp^2 - 1\|_H^2 = \langle x^2 - 1, x^2 - 1 \rangle_H = \int_0^1 4x^2 = 4 \left[\frac{x^3}{3} \right]_{x=0}^{x=1} = \frac{4}{3}.$$
Thus
$$\Pi_V f = \frac{\sqrt{3}}{2} \langle f, x^2 - 1 \rangle_H (x^2 - 1).$$
If $f = x - 1 (\in H)$,
$$\langle x - 1, x^2 - 1 \rangle_H = \int_0^1 1 \cdot 2x \, dx = [x^2]_{x=0}^{x=1} = 1.$$
Therefore
$$\Pi_V(\sharp - 1) = \frac{\sqrt{3}}{2}(x^2 - 1),$$
and
$$\Pi_S(\sharp - 1) = x - 1 - \Pi_V(\sharp - 1) = x - 1 - \frac{\sqrt{3}}{2}(x^2 - 1) = (x - 1)\left(1 - \frac{\sqrt{3}}{2}(x + 1)\right). \quad \square$$

Exercise 14. i) See Lecture Notes.

ii) Let $g_a(x) = \frac{1}{a^2+x^2}$ the Cauchy distribution. We may represent the equation under the form
$$\lambda f(x) + f * g_1(x) = g_1(x) + g_2(x).$$
If $f \in L^1(\mathbb{R})$ is a solution,
$$\lambda \widehat{f}(\xi) + \widehat{f}(\xi)\widehat{g_1}(\xi) = \widehat{g_1}(\xi) + \widehat{g_2}(\xi).$$
Now, remind that, for $a > 0$,
$$\widehat{g_a}(\xi) = \frac{\pi}{a} e^{-a|\xi|}.$$
Thus, we obtain the following equation for $\widehat{f}$:
$$\widehat{f}(\xi)\left(\lambda + \pi e^{-|\xi|}\right) = \pi e^{-|\xi|} + \frac{\pi}{2} e^{-2|\xi|} = \pi e^{-|\xi|}\left(1 + \frac{1}{2} e^{-|\xi|}\right),$$

that is

$$\widehat{f}(\xi) = \pi e^{-|\xi|}\frac{1+\frac{1}{2}e^{-|\xi|}}{\lambda+\pi e^{-|\xi|}}.$$

Now,

- if $\lambda = 0$ we obtain
$$\widehat{f}(\xi) = 1 + \frac{1}{2}e^{-|\xi|},$$
in particular $\widehat{f}(\xi) \longrightarrow 1$ when $\xi \longrightarrow \pm\infty$, according to the RL lemma $\widehat{f}$ cannot be a FT of an L^1 function. This means that for $\lambda = 0$ the equation has no solutions in L^1.
- if $\lambda < 0$ we see that the denominator of $\widehat{f}$ vanishes at $\xi = \pm\log(-\frac{\lambda}{\pi})$, in particular $\widehat{f}$ is unbounded at these points, again in contradiction with RL lemma. In particular, no L^1 solution is possible for $\lambda < 0$.
- if $\lambda > 0$ then $\widehat{f} \in \mathscr{C}(\mathbb{R})$, it vanishes at $\pm\infty$ and since
$$\widehat{f}(\xi) \sim_{\pm\infty} \frac{\pi}{\lambda}e^{-|\xi|},$$
we have $\widehat{f} \in L^1$, thus equation makes sense and, by the inversion formula, it has as unique solution
$$f(x) = \frac{1}{2\pi}\widehat{\widehat{f}}(-x) = \frac{1}{4}\widehat{e^{-|\sharp|}\frac{2+e^{-|\sharp|}}{\lambda+\pi e^{-|\sharp|}}}(-x).$$

iii) For $\lambda = 2\pi$ we have

$$f(x) = \frac{1}{4\pi}\widehat{e^{-|\sharp|}\frac{2+e^{-|\sharp|}}{2+e^{-|\sharp|}}}(-x). = \frac{1}{4\pi}\widehat{e^{-|\sharp|}}(-x) = \frac{1}{4\pi}\frac{2}{1+(-x)^2} = \frac{1}{2\pi}\frac{1}{1+x^2}. \qquad \square$$

Exercise 15. i) False: take $f_n = n^2 1_{[0,1/n]}$. Then

$$\int_0^1 f_n = n^2\frac{1}{n} = n \longrightarrow +\infty,$$

but, for every $x > 0$ fixed, as soon as $\frac{1}{n} < x$ (that is $n > [1/x] + 1$) we have $f_n(x) = 0 \longrightarrow 0$.

ii) False: $f_n = \sqrt{n}1_{[0,1/n]}$, same arguments of the previous example.

iii) False: see the example shown in class of a sequence $(f_n) \subset L^1([0,1])$ such that $f_n \xrightarrow{L^1} 0$ but $(f_n(x))$ is not convergent for every $xin[0,1]$.

iv) True: if $\|f_n\|_\infty \not\longrightarrow +\infty$ it means that there exists a constant K such that $\|f_n\|_\infty \leqslant K$ for infinitely many n. For these n,

$$\int_0^1 f_n \leqslant \int_0^1 \|f_n\|_\infty = \|f_n\|_\infty \leqslant K,$$

so, in particular, $\int_0^1 f_n \not\longrightarrow +\infty$. $\square$

Exercise 17. i) Since $f \in V = \mathscr{C}^2([0,1])$, $f'' \in \mathscr{C}([0,1])$ thus uniform norm $\|f''\|_\infty$ makes sense. Also $f'(0)$ makes sense thus $\|f\|_V$ is well defined. Let's check the basic properties of a norm. Vanishing: $\|f\|_V = 0$ iff $\|f''\|_\infty + |f'(0)| + |f(0)| = 0$. Since this is the sum of positive quantities, it can be 0 iff $\|f''\|_\infty = |f'(0)| = |f(0)| = 0$, that is $f'' \equiv 0$ and $f'(0) = 0$, $f(0) = 0$. From $f'' \equiv 0$ we deduce f' constant, but since $f'(0) = 0$ we have $f' \equiv 0$. Therefore f is constant, and since also $f(0) = 0$ we deduce $f \equiv 0$. Homogeneity and triangular inequality are straightforward.

ii) We have to show that

$$\exists C > 0, \ : \ \|f\|_\infty \leqslant C\|f\|_V, \ \forall f \in V.$$

Let's bound $|f(x)|$ by f''. By the fundamental formula of Integral Calculus,

$$f(x) = f(0) + \int_0^x f'(y) \, dy,$$

and applying the same to f',

$$f'(y) = f'(0) + \int_0^y f''(z) \, dz,$$

thus

$$f(x) = f(0) + \int_0^x \left(f'(0) + \int_0^y f''(z) \, dz \right) dy = f(0) + f'(0)x + \int_0^x \int_0^y f''(z) \, dz \, dy.$$

Therefore,

$$|f(x)| \leqslant |f(0)| + |f'(0)|x + \int_0^x \int_0^y |f''(z)| \, dz \, dy$$

and recalling that $\|f\|_\infty = \max_{x \in [0,1]} |f(x)|$, we get

$$\|f\|_\infty \leqslant |f(0)| + |f'(0)| + \int_0^1 \int_0^1 \|f''\|_\infty \, dz \, dy = |f(0)| + |f'(0)| + \|f''\|_\infty = \|f\|_V.$$

iii) True: since $f_n \xrightarrow{\|\cdot\|_V} f$ implies $f_n \xrightarrow{\|\cdot\|_\infty} f$ and this last implies pointwise convergence, we deduce that also V norm convergence implies pointwise convergence.

iv) Clearly $f_n(x) = \frac{1}{n^2}x^n \in \mathscr{C}^2([0,1]) = V$. Notice that $f_n(x) \longrightarrow 0$, for every $x \in [0,1]$. Thus, by iii), if $f_n \xrightarrow{\|\cdot\|_V} f$, necessarily $f = 0$. However,

$$\|f_n - 0\|_V = \|f_n\|_V = |f_n(0)| + |f_n'(0)| + \|f_n''\|_\infty,$$

and since $f_n'(x) = \frac{1}{n}x^{n-1}$ and $f_n''(x) = \frac{n-1}{n}x^{n-2}$, we deduce $f_n(0) = f_n'(0) = 0$ for every $n \geqslant 2$ and

$$\|f_n''\|_\infty = \max_{x \in [0,1]} \frac{n-1}{n}|x^n| = \frac{n-1}{n} \longrightarrow 1,$$

thus (f_n) cannot converge to 0 in V. Since this is the unique possibility, we deduce that (f_n) is not convergent in V. However clearly, $\|f_n\|_\infty = \max_{x\in[0,1]} \frac{1}{n^2}|x^n| = \frac{1}{n^2} \longrightarrow 0$, thus $f_n \stackrel{\|\cdot\|_\infty}{\longrightarrow} 0$. We deduce that the two norm $\|\cdot\|_V$ and $\|\cdot\|_\infty$ are not equivalent, otherwise they would have the same convergent sequences. □

Exercise 21. i) We may notice that, setting $u_1 = x$ and $u_2 = x^3$,

$$V = \{f \in H \ : \ \langle f, u_1\rangle = 0,\ \langle f, u_2\rangle = 0\}.$$

V is clearly a linear subspace of H. It is also closed because of the continuity of scalar product. Indeed, if $(f_n) \subset V$, $f_n \stackrel{H}{\longrightarrow} f$, then

$$\begin{cases} 0 = \langle f_n, u_1\rangle \longrightarrow \langle f, u_1\rangle, \implies \langle f, u_1\rangle = 0, \\ \\ 0 = \langle f_n, u_2\rangle \longrightarrow \langle f, u_2\rangle, \implies \langle f, u_2\rangle = 0, \end{cases} \implies f \in V.$$

ii) It is convenient to determine first the orthogonal projection on $U := \mathrm{Span}(u_1, u_2)$. This is a finite dimensional subspace of H (thus it is closed by a general fact). If (e_1, e_2) is an orthonormal basis for U,

$$\Pi_U f = \sum_{j=1}^{2} \langle f, e_j\rangle e_j.$$

The orthonormal basis can be determined by the Gram-Schmidt algorithm:

$$e_1 = \frac{u_1}{\|u_1\|},\quad e_2 = \frac{u_2 - \langle u_2, e_1\rangle e_1}{\|u_2 - \langle u_2, e_1\rangle e_1\|}.$$

We have

$$\|u_1\|^2 = \int_0^1 u_1^2 = \int_0^1 x^2\,dx = \frac{1}{3},\quad \implies \quad e_1 = \frac{x}{\sqrt{3}},$$

and

$$u_2 - \langle u_2, e_1\rangle e_1 = x^3 - \frac{1}{3}\left(\int_0^1 y^4\,dy\right)x = x^3 - \frac{x}{15},$$

hence,

$$\|u_2 - \langle u_2, e_1\rangle e_1\|^2 = \int_0^1 \left(x^3 - \frac{x}{15}\right)^2 dx = \frac{1}{7} - \frac{2}{75} + \frac{1}{675} = \frac{808}{4725}$$

so

$$e_2 = \sqrt{\frac{4725}{808}}\left(x^3 - \frac{x}{15}\right).$$

Now, to compute $\Pi_V f$ it is enough to set

$$\Pi_V f = f - \Pi_U f.$$

This because $\Pi_V f$ is uniquely characterized by the property

$$\langle f - \Pi_V f, v\rangle = 0,\ \forall v \in V,$$

that is

$$\langle \Pi_U f, v\rangle = 0,\ \forall v \in V.$$

But this is automatically true being $\Pi_U f \in \mathrm{Span}(u_1, u_2)$ and $v \perp u_1, u_2$ when $v \in V$. □

Exercise 23. i) Certainly, $\|\cdot\|_*$ is well defined: indeed $\frac{f(x)}{\sqrt{x}} \in \mathscr{C}(]0,1])$ and since $f \in \mathscr{C}([0,1])$, f is bounded, thus $\left|\frac{f(x)}{\sqrt{x}}\right| \leqslant \frac{\|f\|_\infty}{\sqrt{x}}$ which is integrable on $[0,1]$. Thus $\|f\|_* < +\infty$ for every $f \in V$.

We now check the key properties of a norm:

- positivity: obvious.
- vanishing: $\|f\|_* = 0$ implies $\frac{|f(x)|}{\sqrt{x}} = 0$ a.e. $x \in [0,1]$, thus $f = 0$ a.e. on $[0,1]$. Since f is continuous, $f \equiv 0$ on $[0,1]$.
- homogeneity: straightforward, $\|\lambda f\|_* = \int_0^1 \frac{|\lambda f(x)|}{\sqrt{x}}\,dx = |\lambda| \int_0^1 \frac{|f(x)|}{\sqrt{x}}\,dx = |\lambda|\|f\|_*$.
- triangular inequality: straightforward.

ii) It is easy to verify that the uniform norm is stronger than $\|\cdot\|_*$. Indeed, since $|f(x)| \leqslant \|f\|_\infty$, we have

$$\|f\|_* = \int_0^1 \frac{|f(x)|}{\sqrt{x}}\,dx \leqslant \int_0^1 \frac{\|f\|_\infty}{\sqrt{x}}\,dx = \|f\|_\infty \int_0^1 \frac{1}{\sqrt{x}}\,dx = \|f\|_\infty \left[2\sqrt{x}\right]_{x=0}^{x=1} = 2\|f\|_\infty.$$

Are they equivalent? No! Indeed take

$$f_n(x) := \begin{cases} \sqrt[4]{n}, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \frac{1}{\sqrt[4]{x}}, & \frac{1}{n} < x \leqslant 1. \end{cases}$$

Clearly $f_n \in \mathscr{C}([0,1]) = V$. We have

$$\|f_n\|_\infty = \sqrt[4]{n},$$

while

$$\|f\|_* = \int_0^1 \frac{|f(x)|}{\sqrt{x}}\,dx \leqslant \int_0^1 \frac{1/\sqrt[4]{x}}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{3/4}}\,dx =: M < +\infty.$$

If $\|\cdot\|_*$ is stronger than $\|\cdot\|_\infty$, we have

$$\sqrt[4]{n} = \|f_n\|_\infty \leqslant C\|f_n\|_* \leqslant CM,\ \forall n \in \mathbb{N},$$

which is impossible.

About the $\|\cdot\|_1$ norm, we may notice that, since for $x \in]0,1]$ we have $\frac{1}{\sqrt{x}} \geqslant 1$, then

$$\|f\|_1 = \int_0^1 |f(x)|\, dx \leqslant \int_0^1 \frac{|f(x)|}{\sqrt{x}}\, dx = \|f\|_*,$$

thus $\|\cdot\|_*$ is stronger than $\|\cdot\|_1$. Are they equivalent? No! Here we consider,

$$f_n(x) := \begin{cases} \sqrt{n}, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \frac{1}{\sqrt{x}}, & \frac{1}{n} < x \leqslant 1. \end{cases}$$

Clearly, since $0 \leqslant f(x) \leqslant \frac{1}{\sqrt{x}}$

$$\|f_n\|_1 = \int_0^1 |f_n(x)|\, dx \leqslant \int_0^1 \frac{1}{\sqrt{x}}\, dx =: M < +\infty.$$

On the other hand,

$$\|f_n\|_* \geqslant \int_{1/n}^1 \frac{|f(x)|}{\sqrt{x}}\, dx = \int_{1/n}^1 \frac{1}{x}\, dx = \log 1 - \log\frac{1}{n} = \log n,$$

thus, if $\|\cdot\|_1$ were stronger than $\|\cdot\|_*$, we would have that

$$\log n \leqslant \|f_n\|_* \leqslant C\|f_n\|_1 \leqslant CM,\ \forall n \in \mathbb{N}, n \geqslant 1,$$

but this is impossible. □

Exercise 24. i) Clearly $f \in L^1$ thus $\widehat{f}$ is well defined. To check $\widehat{f} \in L^1$, we apply the well known result: if $f, f', f'' \in L^1$ then $\widehat{f} \in L^1$. We already said $f \in L^1$. About f',

$$f'(x) = -\frac{4x^3}{(1+x^4)^2} \in \mathscr{C}(\mathbb{R}),\ f'(x) \sim_{\pm\infty} -4\frac{x^3}{x^8} = \frac{C}{x^5},$$

which is integrable at $\pm\infty$. Similarly for f'':

$$f''(x) = -4\frac{3x^2(1+x^4)^2 - 2x^3(1+x^4)4x^3}{(1+x^4)^4} \in \mathscr{C}(\mathbb{R}),\ f''(x) \sim_{\pm\infty} -4\frac{-5x^{10}}{x^{16}} = \frac{C}{x^6},$$

which is integrable at $\pm\infty$. Is $\widehat{f} \in L^2$? Yes, this because $f \in L^2$ (yet, $f \in \mathscr{C}(\mathbb{R})$ and $|f(x)|^2 \sim_{\pm\infty} \frac{1}{x^8}$ is integrable, thus $\int_{\mathbb{R}} |f|^2 < +\infty$) and the FT maps L^2 into itself. Last: is $\widehat{f} \in \mathcal{S}(\mathbb{R})$? No, this because FT maps the Schwarz space $\mathcal{S}(\mathbb{R})$ into itself, thus $\widehat{f} \in \mathcal{S}(\mathbb{R})$ iff $f \in \mathcal{S}(\mathbb{R})$. Clearly, $f \in \mathscr{C}^\infty$ but, for instance,

$$x^4 f(x) \not\longrightarrow 0,\ |x| \longrightarrow \pm\infty.$$

ii) We may notice that

$$\frac{1}{x^2 \pm \sqrt{2}x + 1} = \frac{1}{\left(x \pm \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}},$$

thus, recalling that $\widehat{g(\sharp + c)} = e^{-ic\xi}\widehat{g}$,

$$\widehat{\frac{1}{\sharp^2 \pm \sqrt{2}\sharp + 1}}(\xi) = \widehat{\frac{1}{\left(\sharp \pm \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}}(\xi) = e^{\mp i\xi/\sqrt{2}}\widehat{\frac{1}{\sharp^2 + \left(\frac{1}{\sqrt{2}}\right)^2}}(\xi) = e^{\mp i\xi/\sqrt{2}}\sqrt{2}\pi e^{-|\xi|/\sqrt{2}}$$

iii) Because $(1 + x^4) = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ we have

$$\frac{1}{1 + x^4} = \frac{1}{(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)} = -\frac{1}{2\sqrt{2}x}\left(\frac{1}{x^2 + \sqrt{2}x + 1} - \frac{1}{x^2 - \sqrt{2}x + 1}\right)$$

thus

$$-2\sqrt{2}x\frac{1}{1 + x^4} = \frac{1}{x^2 + \sqrt{2}x + 1} - \frac{1}{x^2 - \sqrt{2}x + 1}$$

hence

$$-2\sqrt{2}\widehat{(\sharp f)}(\xi) = e^{-i\xi/\sqrt{2}}\sqrt{2}\pi e^{-|\xi|/\sqrt{2}} - e^{+i\xi/\sqrt{2}}\sqrt{2}\pi e^{-|\xi|/\sqrt{2}} = -2i\sqrt{2}\pi e^{-|\xi|/\sqrt{2}}\sin(\xi/\sqrt{2}),$$

that is

$$\widehat{(\sharp f)}(\xi) = -i\pi e^{-|\xi|/\sqrt{2}}\sin(\xi/\sqrt{2}).$$

Now, recalling that

$$\widehat{(-i\sharp) f)} = \partial_\xi \widehat{f},$$

we get,

$$\partial_\xi \widehat{f}(\xi) = -\pi e^{-|\xi|/\sqrt{2}}\sin(\xi/\sqrt{2}).$$

To determine $\widehat{f}$, let's first compute

$$\begin{aligned}\int e^{\alpha\xi}\sin(\beta\xi)\, d\xi &= \tfrac{e^{\alpha\xi}}{\alpha}\sin(\beta\xi) - \int \tfrac{e^{\alpha\xi}}{\alpha}\beta\cos(\beta\xi)\, d\xi \\ &= \tfrac{e^{\alpha\xi}}{\alpha}\sin(\beta\xi) - \tfrac{\beta}{\alpha}\left[\tfrac{e^{\alpha\xi}}{\alpha}\cos(\beta\xi) + \int \tfrac{e^{\alpha\xi}}{\alpha}\beta\sin(\beta\xi)\, d\xi\right],\end{aligned}$$

from which,

$$\int e^{\alpha\xi}\sin(\beta\xi)\, d\xi = \frac{\alpha}{\alpha^2 + \beta^2}e^{\alpha\xi}\left(\sin(\beta\xi) - \frac{\beta}{\alpha}\cos(\beta\xi)\right).$$

Therefore,

$$\widehat{f}(\xi) = \begin{cases} \xi \geqslant 0, & -\pi\int e^{-\xi/\sqrt{2}}\sin(\xi/\sqrt{2})\, d\xi + c_1 = \frac{\pi}{\sqrt{2}}e^{-\xi/\sqrt{2}}\left(\sin(\xi/\sqrt{2}) + \cos(\xi/\sqrt{2})\right) + c_1, \\ \xi < 0, & -\pi\int e^{\xi/\sqrt{2}}\sin(\xi/\sqrt{2})\, d\xi + c_2 = -\frac{\pi}{\sqrt{2}}e^{\xi/\sqrt{2}}\left(\sin(\xi/\sqrt{2}) - \cos(\xi/\sqrt{2})\right) + c_2.\end{cases}$$

Since $f \in L^1(\mathbb{R})$ we have $\widehat{f}(\xi) \longrightarrow 0$ for $\xi \longrightarrow \pm\infty$, from which $c_1 = c_2 = 0$. Thus

$$\widehat{f}(\xi) = \frac{\pi}{\sqrt{2}} e^{-|\xi|/\sqrt{2}} \left(\sin(|\xi|/\sqrt{2}) + \cos(|\xi|/\sqrt{2}) \right). \quad \square$$

Exercise 25. i) Let $f(x,\xi) := \frac{1-\cos(\xi x)}{x^2(x^2+1)}$. Clearly $f \in \mathscr{C}(\mathbb{R}\backslash\{0\})$ for every $\xi \in \mathbb{R}$. We may notice that $f(x,0) \equiv 0$, thus certainly $f(\sharp,0) \in L^1(\mathbb{R})$. For $\xi \neq 0$, recalling that $\cos t = 1 - \frac{t^2}{2} + o(t^2)$ for $t \sim 0$, we have that, for $x \sim 0$,

$$f(x,\xi) = \frac{\frac{xi^2 x^2}{2} + o(x^2)}{x^2(x^2+1)} \sim \frac{1}{2}\frac{\xi^2}{x^2+1} \sim \frac{\xi^2}{2},$$

thus $|f(\sharp,\xi)|$ is integrable at $x = 0$ for every $\xi \in \mathbb{R}$. We need to check integrability at $\pm\infty$. Notice that

$$|f(x,\xi)| \leqslant \frac{2}{x^2(x^2+1)} \sim_{\pm\infty} \frac{2}{x^4},$$

therefore $|f(\sharp,\xi)|$ is integrable at $\pm\infty$ for every $\xi \in \mathbb{R}$. We conclude that $f(\sharp,\xi) \in L^1(\mathbb{R})$ for every $\xi \in \mathbb{R}$, thus $D = \mathbb{R}$.

ii) We apply differentiation theorem. If this holds,

$$\partial_\xi F(\xi) = \int_{\mathbb{R}} \partial_\xi f(x,\xi)\, dx,\ \xi \in D'.$$

To this aim we need to verify hypotheses that are

1. $f(\sharp,\xi) \in L^1(\mathbb{R})$, $\forall \xi \in D'$. This has been checked in i) with $D' = D = \mathbb{R}$.
2. $\exists \partial_\xi f(x,\xi) = \frac{\sin(\xi x)}{x(x^2+1)}$ for every $\xi \in \mathbb{R}$.
3. $\exists g = g(x) \in L^1(\mathbb{R})$ such that $|\partial_\xi f(x,\xi)| \leqslant g(x)$, a.e. $x \in \mathbb{R}$, $\forall \xi \in D'$. Here we may notice that, since $|\sin t| \leqslant |t|$ we have

$$|\partial_\xi f(x,\xi)| = \left| \frac{\sin(\xi x)}{x} \frac{1}{x^2+1} \right| \leqslant |\xi| \frac{1}{x^2+1} \leqslant \frac{R}{x^2+1} =: g_R(x),\ a.e.\ x \in \mathbb{R},\ \forall \xi \in [-R,R].$$

Thus, fixed $R > 0$, we may apply the theorem on $D' = [-R,R]$ and deduce that

$$\partial_\xi F(\xi) = \int_{\mathbb{R}} \frac{\sin(\xi x)}{x(x^2+1)}\, dx,\ \forall \xi \in [-R,R].$$

And since $R > 0$ is arbitrary, we can conclude that the previous identity actually holds for every $\xi \in \mathbb{R}$. Thus, at the end, we can consider $D' = \mathbb{R}$.

iii) Let $g(x,\xi) := \frac{\sin(\xi x)}{x(x^2+1)}$. To compute $\partial^2_\xi F$ we apply again differentiation theorem to

$$\partial_\xi F(\xi) = \int_{\mathbb{R}} g(x,\xi)\, dx.$$

To this aim, we need to verify that

1. $g(\sharp,\xi)\in L^1(\mathbb{R})$, $\forall \xi\in D''$. This is a consequence of 3. of ii), from which we get that $D''=\mathbb{R}$.
2. $\exists \partial_\xi g(x,\xi)=\frac{\cos(\xi x)}{x^2+1}$, a.e. $x\in\mathbb{R}$ (actually, $\forall x\neq 0$) and $\forall\xi\in D''=\mathbb{R}$.
3. $\exists h=h(x)\in L^1(\mathbb{R})$ for which $|\partial_\xi g(x,xi)|\leqslant h(x)$, a.e. $x\in\mathbb{R}$ and $\forall \xi\in D''$. Here we may notice that

$$|\partial_\xi g(x,\xi)|=\left|\frac{\cos(\xi x)}{x^2+1}\right|\leqslant\frac{1}{x^2+1}=:h(x)\in L^1(\mathbb{R}),\ a.e.\,x\in\mathbb{R},\ \forall\xi\in D''=\mathbb{R}.$$

Thus, differentiation theorem applies and

$$\partial^2_\xi F(\xi)=\int_{\mathbb{R}}\frac{\cos(\xi x)}{x^2+1}\,dx,\ \forall\xi\in\mathbb{R}.$$

iv) From previous discussion, and recalling that $\cos t=\mathrm{Re}\,e^{it}$, we have

$$\partial^2_\xi F(\xi)=\mathrm{Re}\int_{\mathbb{R}}\frac{1}{x^2+1}e^{i\xi x}\,dx\overset{y=-x}{=}\mathrm{Re}\int_{\mathbb{R}}\frac{1}{1+y^2}e^{-i\xi y}\,dy=\mathrm{Re}\,\widehat{\frac{1}{1+\sharp^2}}(\xi)=\pi e^{-|\xi|}.$$

Then

$$\partial_\xi F(\xi)=\begin{cases}\int\pi e^{-\xi}\,d\xi+c_1=-\pi e^{-\xi}+c_1, & \xi\geqslant 0,\\ \\ \int\pi e^{\xi}\,d\xi+c_2=\pi e^{\xi}+c_2, & \xi\leqslant 0.\end{cases}$$

Since $\partial_\xi F$ is differentiable it must be continuous, in particular at $\xi=0$. This leads to

$$-\pi+c_1=\partial_\xi F(0)=\pi+c_2,$$

and since also $\partial_\xi F(0)=0$ (trivial) we obtain $c_1=\pi$ and $c_2=-\pi$. In conclusion

$$\partial_\xi F(\xi)=\begin{cases}-\pi e^{-\xi}+\pi, & \xi\geqslant 0,\\ \\ \pi e^{\xi}-\pi, & \xi\leqslant 0.\end{cases}$$

Therefore

$$F(\xi)=\begin{cases}\pi e^{-\xi}+\pi\xi+c_1, & \xi\geqslant 0,\\ \\ \pi e^{\xi}-\pi\xi+c_2, & \xi<0.\end{cases}$$

Again, by continuity at $\xi=0$ and noticed that $F(0)=0$, we have $c_1=-\pi=c_2$, thus

$$F(\xi)=\begin{cases}\pi e^{-\xi}+\pi\xi-\pi, & \xi\geqslant 0,\\ \\ \pi e^{\xi}-\pi\xi-\pi, & \xi<0.\end{cases}=\pi\left(e^{-|\xi|}+|\xi|-1\right).\quad\square$$

Exercise 26. i) Clearly $\|f\|$ is well defined ($f'\in\mathscr{C}([0,1])$, hence $x^{1/2}f'\in\mathscr{C}([0,1])$). Let's check the characteristic properties of a norm:

- positivity: $\|f\|\geqslant 0$, trivial.

- vanishing: $\|f\| = 0$ means $x^{1/2}|f'(x)| \equiv 0$ on $[0, 1]$, thus in particular $f' \equiv 0$ on $]0, 1]$ and because $f' \in \mathscr{C}$, $f' \equiv 0$ on $[0, 1]$. In particular, f is constant and because $f(0) = 0$ ($f \in V$), we conclude $f \equiv 0$.
- homogeneity: $\|\lambda f\| = \max x^{1/2}|(\lambda f)'(x)| = \max x^{1/2}|\lambda||f'(x)| = |\lambda| \max x^{1/2}|f'(x)| = |\lambda|\|f\|$.
- triangular inequality: notice first that if $f, g \in V$ we have

$$|(f+g)'(x)| = |f'(x) + g'(x)| \leqslant |f'(x)| + |g'(x)|,$$

thus

$$x^{1/2}|(f+g)'(x)| \leqslant x^{1/2}|f'(x)| + x^{1/2}|g'(x)| \leqslant \|f\| + \|g\|, \ \forall x \in [0, 1],$$

hence, taxing maximum, $\|f + g\| \leqslant \|f\| + \|g\|$.

ii) We check first that $f_n \in V$. Easily, $f_n \in \mathscr{C}([0, 1])$ and

$$f_n'(x) := \begin{cases} \frac{n^{3/4}}{4}, & 0 \leqslant x < \frac{1}{n}, \\ \\ \frac{1}{4}x^{-3/4}, & \frac{1}{n} < x \leqslant 1 \end{cases}$$

We have that $\lim_{x\to 1/n-} f_n'(x) = \frac{n^{3/4}}{4} = \lim_{x\to 1/n+} f_n'(x)$, thus $\exists f_n'(1/n) = \frac{n^{3/4}}{4}$ and $f_n' \in \mathscr{C}([0, 1])$. And since clearly $f_n(0) = 0$ we have $f_n \in V$ for every $n \geqslant 1$.

To discuss convergence of (f_n) we may notice that

$$x^{1/2}|f_n'(x)| := \begin{cases} \frac{n^{3/4}}{4}x^{1/2}, & 0 \leqslant x \leqslant \frac{1}{n}, \\ \\ \frac{1}{4}x^{-1/4}, & \frac{1}{n} \leqslant x \leqslant 1. \end{cases} \implies \|f_n\| = \max_{t\in[0,1]} x^{1/2}|f_n'(x)| = \frac{1}{4}n^{1/4} \longrightarrow +\infty.$$

Since (f_n) is not even bounded in $\|\cdot\|$, it cannot be convergent.

iii) We have to prove that there exists a universal constant C such that $\|f\|_\infty \leqslant C\|f\|$ for every $f \in X$. We start recalling that

$$f(x) = f(0) + \int_0^x f'(y)\, dy \overset{f\in V,\ f(0)=0}{=} \int_0^x f'(y)\, dy,$$

therefore

$$|f(x)| = \left|\int_0^x f'(y)\, dy\right| \leqslant \int_0^x |f'(y)|\, dy = \int_0^x \frac{1}{y^{1/2}} y^{1/2}|f'(y)|\, dy \leqslant \int_0^x \frac{1}{y^{1/2}}\|f\|\, dy = 2x^{1/2}\|f\|,$$

thus, finally

$$\|f\|_\infty = \max_{x\in[0,1]} |f(x)| \leqslant \max_{x\in[0,1]} 2x^{1/2}\|f\| = 2\|f\|.$$

The two norms are not equivalent. Indeed, if $\|\cdot\|_\infty$ were stronger than $\|\cdot\|$, we would have

$$\|f\| \leqslant C\|f\|_\infty, \ \forall f \in V.$$

Taking $f = f_n$ and noticed that $\|f_n\|_\infty = 1$ (easy), we would have

$$\frac{n^{1/4}}{4} = \|f_n\| \leqslant C\|f_n\|_\infty = C,\ \forall n \geqslant 1,$$

which is clearly impossible. □

Exercise 27. i) We first check that if $f, g \in H$ then $\langle f, g\rangle$ is well defined. We have to show that $f(x)g(x)e^{-x} \in L^1([0, +\infty[)$. The argument is similar to the standard L^2 product:

$$\begin{aligned}\int_0^{+\infty} |f(x)g(x)e^{-x}|\,dx &= \int_0^{+\infty} |f||g|e^{-x} \leqslant \int_0^{+\infty} \tfrac{1}{2}(f^2+g^2)e^{-x} \\ &= \tfrac{1}{2}\left(\int_0^{+\infty} f^2e^{-x} + \int_0^{+\infty} g^2e^{-x}\right) < +\infty,\end{aligned}$$

provided $f, g \in H$. Thus $\langle f, g\rangle$ is well defined. The check of scalar product properties is standard:

- vanishing: $0 = \langle f, f\rangle = \int_0^{+\infty} f^2e^{-x}$ that is $f(x)^2e^{-x} = 0$ a.e., that is $f = 0$ a.e..
- linearity, symmetry: straightforward.

ii) We may notice that $U = \{u \in H\ :\ \langle u, 1\rangle = 0\}$. Indeed,

$$1 \in H, \quad \Longleftrightarrow \quad \int_0^{+\infty} e^{-x}\,dx < +\infty,$$

which clearly true. Therefore, U is closed because of well known continuity properties of the scalar product. Indeed: if $(u_n) \subset U$ is such that $u_n \longrightarrow u$, then, since $\langle u_n, 1\rangle = 0$ for all n and $\langle u_n, 1\rangle \longrightarrow \langle u, 1\rangle$ we conclude that $\langle u, 1\rangle = 0$.

iii) Since U is closed, $\Pi_U f$ is well defined for every $f \in H$. However, U is likely to be infinite dimensional, it seems not easy to determine a basis for U. Nonetheless, U is the space of u perpendicular to 1, so define $V := \mathrm{Span}(1)$. Clearly V is one dimensional, hence Π_V is well defined. Take $e_0 = \frac{1}{\|1\|}$ where $\|1\|^2 = \int_0^{+\infty} 1^2e^{-x}\,dx = \int_0^{+\infty} e^{-x}dx = 1$, that is $e_0 = 1$. Thus $\Pi_V f = \langle f, 1\rangle 1$ and since $f - \Pi_V f \perp 1$, we have $f - \Pi_V f \in U$. We claim $\Pi_U f = f - \Pi_V f$. Indeed:

$$\langle f - \Pi_U f, g\rangle = \langle f - (f - \Pi_V f), g\rangle = \langle \Pi_V f, g\rangle = \langle f, 1\rangle\langle 1, g\rangle = 0,\ \forall g \in U,$$

and since this characterized $\Pi_U f$ we have the conclusion. In particular,

$$\Pi_U e^{-2x} = e^{-2x} - \langle e^{-2x}, 1\rangle 1 = e^{-2x} - \int_0^{+\infty} e^{-3y}\,dy = e^{-2x} - \frac{1}{3}. \quad \square$$

Exercise 29. i) To have FT, f_α needs to be either $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$. In the first case, writing $\alpha = a + ib$,

$$\int_{\mathbb{R}} |f_\alpha(x)|\,dx = \int_0^{+\infty} |e^{\alpha x}|\,dx = \int_0^{+\infty} |e^{ax}e^{ibx}|\,dx = \int_0^{+\infty} e^{ax}\,dx < +\infty, \quad \Longleftrightarrow \quad a = \mathrm{Re}\,\alpha < 0.$$

Same conclusion for L^2. Thus: f has a FT iff Re $\alpha < 0$. For such α we have

$$\widehat{f_\alpha}(\xi) = \int_0^{+\infty} e^{\alpha x} e^{-i\xi x}\, dx = \int_0^{+\infty} e^{(\alpha - i\xi)x}\, dx = \left[\frac{e^{(\alpha-i\xi)x}}{\alpha - i\xi} \right]_{x=0}^{x=+\infty}.$$

Now, since Re $\alpha = a < 0$,

$$\left| e^{(\alpha-i\xi)x} \right| = |e^{ax} e^{i(b-\xi)x}| = e^{ax} \longrightarrow 0, \quad x \longrightarrow +\infty$$

we have

$$\widehat{f_\alpha}(\xi) = -\frac{1}{\alpha - i\xi} = \frac{1}{i\xi + \alpha}.$$

ii) To have an L^1 Fourier original, according to RL lemma, $g_\beta \in \mathscr{C}(\mathbb{R})$ and it must be bounded. Now, in order g_β be continuous, we need that $\xi + \beta \neq 0$ for every $\xi \in \mathbb{R}$. If $\beta \in \mathbb{R}$ this is impossible, because $\xi + \beta = 0$ at $\xi = -\beta$. If $\beta \in \mathbb{C}\backslash\mathbb{R}$ however, $\xi + \beta \neq 0$ for every $\xi \in \mathbb{R}$, thus $g_\beta \in \mathscr{C}(\mathbb{R})$. Clearly g_β would be also bounded in this case. However, g_β continuous and bounded is not sufficient to have a Fourier original in L^1. As well known, a sufficient condition is $\widehat{g_\beta} \in L^1$. To ensure this, a sufficient condition is $g_\beta, g_\beta', g_\beta'' \in L^1(\mathbb{R})$. $g_\beta \in \mathscr{C}(\mathbb{R})$ and since

$$|g_\beta| \sim_{\pm\infty} \frac{1}{|\xi|^2},$$

we deduce $g_\beta \in L^1(\mathbb{R})$ for every $\beta \in \mathbb{C}\backslash\mathbb{R}$. For g_β' the check is similar being

$$g_\beta'(\xi) = -\frac{2}{(\xi + \beta)^3}$$

thus $g_\beta' \in \mathscr{C}(\mathbb{R})$ and $|g_\beta'| \sim_{\pm\infty} \frac{2}{|\xi|^3}$, integrable at $\pm\infty$, thus $g_\beta' \in L^1(\mathbb{R})$ for every $\beta \in \mathbb{C}\backslash\mathbb{R}$. Same check for g_β''.

For L^2 inversion, the discussion is much more easy: it suffices to verify $g_\beta \in L^2(\mathbb{R})$. Since $g_\beta \in \mathscr{C}(\mathbb{R})$ and $|g_\beta|^2 \sim_{\pm\infty} \frac{1}{|\xi|^4}$, we deduce that $g_\beta \in L^2(\mathbb{R})$ for every $\beta \in \mathbb{C}\backslash\mathbb{R}$.

iii) Following the hint,

$$g_\beta(\xi) = -\partial_\xi \frac{1}{\xi + \beta} = -i\partial_\xi \frac{1}{i\xi + i\beta} = -i\partial_\xi \frac{1}{i\xi + \alpha},$$

where $\alpha = i\beta$. Now, Re $\alpha = -$Im $\beta < 0$. Thus, by i),

$$\frac{1}{i\xi + \alpha} = \widehat{e^{\alpha\sharp} 1_{[0,+\infty[}}(\xi) = \widehat{e^{i\beta\sharp} 1_{[0,+\infty[}},$$

whence

$$g_\beta(\xi) = -i\partial_\xi \widehat{e^{i\beta\sharp} 1_{[0,+\infty[}}(\xi) = -i\left(-i\sharp\widehat{e^{i\beta\sharp} 1_{[0,+\infty[}}\right) = -\sharp\widehat{e^{i\beta\sharp} 1_{[0,+\infty[}}(\xi).$$

Therefore, the Fourier original of g_β is $xe^{i\beta x} 1_{[0,+\infty[}(x)$. $\square$

Exercise 30. Let $f(x,y) := \frac{1-e^{-xy^2}}{y^2}$, $y \in]0,+\infty[$. The domain of F is

$$D := \left\{ x \in \mathbb{R} \; : \; f(x,\sharp) \in L^1([0,+\infty[) \right\}$$

We notice that $f(0,y) \equiv 0$. If $x \neq 0$, $f(x,\sharp) \in \mathscr{C}(]0,+\infty[)$, so we need to check the asymptotic behavior of $f(x,y)$ when $y \to 0+,+\infty$. Recalling of $e^t = 1 + t + o(t)$ when $t \to 0$, we have

$$f(x,y) = \frac{1-(1-xy^2+o(y^2))}{y^2} = x + o(1) \longrightarrow x, \; y \to 0+,$$

so $f(x,\sharp)$ can be extended by continuity at $y = 0$, in particular, $f(x,\sharp)$ is integrable at $y = 0$. When $y \to +\infty$ we have

$$f(x,y) \begin{cases} \longrightarrow -\infty, & x < 0 \implies \nexists \int^{+\infty} f(x,y)\,dy. \\ \sim_{y\to+\infty} \frac{1}{y^2}, & x > 0, \implies \exists \int^{+\infty} f(x,y)\,dy. \end{cases}$$

Conclusion: $D = [0,+\infty[$.

ii) We apply the differentiation under integral sign,

$$\partial_x F(x) = \int_0^{+\infty} \partial_x f(x,y)\,dy. \; (\star)$$

To this aim we notice that:

- $\exists \partial_x f(x,y) = \frac{-e^{-xy^2}(-y^2)}{y^2} = e^{-xy^2}$, $\forall y > 0$ (so a.e. $y \in [0,+\infty[$), $\forall x \geqslant 0$.
- $|\partial_x f(x,y)| = e^{-xy^2} \leqslant e^{-\varepsilon y^2} =: g(y) \in L^1([0,+\infty[)$, $\forall y > 0$ (a.e. $y \in [0,+\infty[$) and $\forall x \geqslant \varepsilon$.

Let $D_\varepsilon := [\varepsilon,+\infty[$ with $\varepsilon > 0$. The previous facts say that we can apply the differentiation theorem on D_ε, so $(\star)$ holds for every $x \geqslant \varepsilon$. Since this ε is an arbitrary positive number, this means that $(\star)$ actually holds for every $x > 0$. Conclusion: F is differentiable on $]0,+\infty[$ and

$$\partial_x F(x) = \int_0^{+\infty} e^{-xy^2}\,dy.$$

iii) $\partial_x F(x)$ is basically a Gaussian integral

$$\partial_x F(x) = \frac{1}{2}\sqrt{\frac{\pi}{x}},$$

from which

$$F(x) = \sqrt{\pi x} + c.$$

To determine the value of the constant c we notice that F is continuous at $x = 0$. This because

- $f(\sharp,y) \in \mathscr{C}([0,+\infty[)$ for all $y > 0$ (thus a.e. $y \in [0,+\infty[$),
- $0 \leqslant f(x,y) = \frac{1-e^{-xy^2}}{y^2} \leqslant \frac{1-e^{-y^2}}{y^2} =: g(y) \in L^1([0,+\infty[)$, $\forall x \in [0,1]$.

So we can apply continuity under integral sign to get that F is continuous on $[0,1]$ and, in particular, at $x=0$. Therefore

$$\lim_{x\to 0+} F(x) = F(0) = 0, \implies c = 0.$$

Therefore $F(x) = \sqrt{\pi x}$ for all $x \geqslant 0$. From this it is also evident that F cannot be differentiable at $x=0$. □

Exercise 35. i) To check U is closed we have to prove that if $(f_n) \subset U$ is convergent (in H) to some $f \in H$ then $f \in U$. So, assume $f_n \longrightarrow f$ in $L^2(X)$. By extracting a subsequence, $f_{n_k}(x) \longrightarrow f(x)$ $\mu-$a.e. $x \in X$. That is, modulo a measure zero set N, $(\mu(N)=0)$,

$$f_{n_k}(x) \longrightarrow f(x),\ \forall x \in X\backslash N.$$

Now, each $f_{n_k} = 0$ a.e. on E^c that is $f_{n_k}(x) = 0$ for all $x \in E^c\backslash N_k$ with $\mu(N_k) = 0$. In particular,

$$f_{n_k}(x) = 0,\ \forall x \in E^c\backslash \bigcup_k N_k,$$

and since $M := N \cup \bigcup_k N_k$ is a $\mu-$ null set (a union of null sets), we have, for $x \in E^c\backslash M$

$$0 \longleftarrow 0 \equiv f_{n_k}(x) \longrightarrow f(x),$$

thus $f(x) = 0$ for every $x \in E^c\backslash M$, that is $f \in U$.

In alternative: since $f_n \stackrel{L^2}{\longrightarrow} f$, then easily also $f_n 1_{E^c} \stackrel{L^2}{\longrightarrow} f 1_{E^c}$ ($\|f_n 1_{E^c} - f 1_{E^c}\|_2^2 = \int_X (f_n - f)^2 1_{E^c}\, d\mu \leqslant \|f_n - f\|_2^2$), so $\|f_n 1_{E^c}\|_2 \longrightarrow \|f 1_{E^c}\|_2$. But $\|f_n 1_{E^c}\|_2^2 = \int_{E^c} f_n^2\, d\mu = 0$ for every n; so $\|f 1_{E^c}\|_2^2 = 0$ from which $f = 0$ a.e. on E^c. From this the conclusion follows.

ii) Recall that the orthogonal projection is characterised by the orthogonality relation

$$\langle u, f - \Pi_U f\rangle = 0,\ \forall u \in U,\ \forall f \in H.$$

Now, since $f - \Pi_U f = f - 1_E f = (1 - 1_E) f = 1_{E^c} f$ we have

$$\langle u, f - \Pi_U f\rangle = \int_X u 1_{E^c} f\, d\mu = \int_{E^c} u f\, d\mu = 0$$

because $u \in U$ is $= 0$ $\mu-$a.e. $x \in E^c$. □

Exercise 36. i) Let

$$g_{a,b}(\xi) := \frac{e^{-a|\xi|} - e^{-b|\xi|}}{\xi},\ \xi \in \mathbb{R}\backslash\{0\}.$$

Here $a, b > 0$ are fixed. We notice that, being $e^t = 1 + t + o(t)$,

$$g_{a,b}(\xi) = \frac{1 - a|\xi| + o(\xi) - (1 - b|\xi| + o(\xi)}{\xi} = \frac{(b-a)|\xi| + o(\xi)}{\xi} \sim_0 (b-a)\mathrm{sgn}(\xi),$$

which is integrable at $\xi = 0$. At $\pm\infty$ we could say that

$$|g_{a,b}(\xi)| \leqslant \left(e^{-a|\xi|} + e^{-b|\xi|}\right), \ \forall |\xi| \geqslant 1,$$

thus $g_{a,b}$ is integrable at $\pm\infty$. In conclusion $g_{a,b} \in L^1(\mathbb{R})$. Similarly, $|g_{a,b}|^2 \sim_0 (b-a)^2$ is integrable at $\xi = 0$ while, as above,

$$|g_{a,b}(\xi)|^2 \leqslant \left(e^{-a|\xi|} + e^{-b|\xi|}\right)^2, \ \forall |\xi| \geqslant 1,$$

thus easily $|g_{a,b}|^2$ is integrable at $\pm\infty$. In conclusion, $g_{a,b} \in L^2(\mathbb{R})$.

Let us discuss the inversion problem. Since $g_{a,b} \in L^2(\mathbb{R})$, then $g_{a,b}$ has a Fourier original in $L^2(\mathbb{R})$. The same does not apply for an L^1 original. Indeed, if $g_{a,b} = \widehat{f_{a,b}}$ for some $f_{a,b} \in L^1$ then $g_{a,b} \in \mathscr{C}(\mathbb{R})$. However, $g_{a,b}(\xi) \sim_0 (b-a)\mathrm{sgn}(\xi)$ which is not continuous at $\xi = 0$.

ii) If $f_{a,b} \in L^2$ is such that $\widehat{f_{a,b}} = g_{a,b}$ then, according to inversion formula,

$$\widehat{g_{a,b}}(x) = \widehat{\widehat{f_{a,b}}}(x) = 2\pi f_{a,b}(-x),$$

that is $f_{a,b}(x) = \frac{1}{2\pi}\widehat{g_{a,b}}(-x)$. We compute then $\widehat{g_{a,b}}$. To this aim notice that

$$\widehat{\sharp g_{a,b}}(x) = \widehat{e^{-a|\sharp|} - e^{-b|\sharp|}}(x) = \frac{2a}{a^2+x^2} - \frac{2b}{b^2+x^2}.$$

Recalling that $\widehat{(-i\sharp) g_{a,b}} = \partial_x \widehat{g_{a,b}}$ we deduce that

$$\partial_x \widehat{g}_{a,b} = -i\left(\frac{2a}{a^2+x^2} - \frac{2b}{b^2+x^2}\right).$$

Thus,

$$\begin{aligned} \widehat{g_{a,b}}(x) &= -i\left(\tfrac{2}{a}\int \tfrac{1}{1+(\frac{x}{a})^2}\,dx - \tfrac{2}{b}\int \tfrac{1}{1+(\frac{x}{b})^2}\,dx\right) + c \\ &= -i\left(\arctan\left(\tfrac{x}{a}\right) - \arctan\left(\tfrac{x}{b}\right)\right) + c, \end{aligned}$$

where c is a suitable constant. Finally, to determine the value of c, we may notice that letting $x \longrightarrow +\infty$, we have

$$\widehat{g_{a,b}}(x) \longrightarrow -i\left(\frac{\pi}{2} - \frac{\pi}{2}\right) + c = c,$$

and because we already know that $\widehat{g_{a,b}} \in L^2$, this is possible only if $c = 0$. By this we finally obtain that the original of $g_{a,b}$ is

$$f_{a,b}(x) = \frac{i}{2\pi}\left(\arctan\left(\frac{x}{b}\right) - \arctan\left(\frac{x}{a}\right)\right).$$

Notice that we can check that $f_{a,b} \notin L^1(\mathbb{R})$. Clearly, $f_{a,b} \in \mathscr{C}(\mathbb{R})$ so the integrability depends on the behavior at $\pm\infty$. Recall of the remarkable identity

$$\arctan t + \arctan\frac{1}{t} = \frac{\pi}{2}, \ \forall t > 0,$$

so, for $x \longrightarrow +\infty$,

$$\arctan\frac{x}{b} - \arctan\frac{x}{a} = \arctan\frac{a}{x} - \arctan\frac{b}{x},$$

ans since $\arctan u = u + o(u)$ for $u \longrightarrow 0$, we deduce

$$\frac{2\pi}{i} f_{a,b}(x) = \arctan\frac{a}{x} - \arctan\frac{b}{x} = \frac{a-b}{x} o\left(\frac{1}{x}\right) \sim_{+\infty} \frac{C}{x} \notin L^1.$$

This confirms once more that $g_{a,b}$ cannot have a Fourier original in L^1. □

Exercise 37. i) Let $f(t,x) := e^{-\lambda x}\frac{\sin x}{x}$. Because $\sin x \sim_0 x$, we may consider f well defined and continuous at $x = 0$, thus $f(\lambda, \sharp)$ is integrable at $x = 0$ for every $\lambda \in \mathbb{R}$. At $x = +\infty$, because $|\sin x| \leqslant |x|$, we have

$$|f(\lambda, x)| \leqslant e^{-\lambda x} \in L^1([0, +\infty[), \ \forall \lambda > 0.$$

For $\lambda = 0$,

$$F(0) = \int_0^{+\infty} \frac{\sin x}{x}\, dx$$

exists (as generalized integral but not in L^1 sense). Thus, we may still consider F well defined at $\lambda = 0$.

ii) We wish to apply differentiation under integral, that is

$$\partial_\lambda F = \int_0^{+\infty} \partial_\lambda f(\lambda, x)\, dx.$$

To ensure this for every $\lambda \in \Lambda$ we need to check a) $f(\lambda, \sharp) \in L^1([0, +\infty[)$ for every $\lambda \in \Lambda$. This is true with $\Lambda =]0, +\infty[$. b) $\exists \partial_\lambda f(\lambda, x) = -xe^{-\lambda x}\frac{\sin x}{x} = -e^{-\lambda x}\sin x$, for every $\lambda \in]0, +\infty[$, a.e. $x \in [0, +\infty[$. c) there exists $g \in L^1([0, +\infty[)$ such that

$$|\partial_\lambda f(\lambda, x)| \leqslant g(x), \forall \lambda \in \Lambda, \text{ a.e. } x \in [0, +\infty[.$$

Now,

$$|\partial_\lambda f(\lambda, x)| \leqslant e^{-\lambda x} \leqslant e^{-\lambda_0 x} \in L^1([0, +\infty[), \ \forall \lambda \in [\lambda_0, +\infty[.$$

Thus, on $\Lambda = [\lambda_0, +\infty[$ with $\lambda_0 > 0$, we can conclude

$$\partial_\lambda F(\lambda) = \int_0^{+\infty} -e^{-\lambda x}\sin x\, dx, \ \forall \lambda \geqslant \lambda_0,$$

and because λ_0 can be chosen arbitrarily > 0, we conclude the previous holds true for every $\lambda > 0$. Recalling that

$$\int e^{\alpha x}\sin(\beta x)\, dx = \frac{\alpha}{\alpha^2 + \beta^2} e^{\alpha x}\left(\sin(\beta x) - \frac{\beta}{\alpha}\cos(\beta x)\right)$$

we have

$$\partial_\lambda F(\lambda) = \frac{\lambda}{\lambda^2 + 1}\left[e^{-\lambda x}\left(\sin x + \frac{1}{\lambda}\cos x\right)\right]_{x=0}^{x=+\infty} = -\frac{\lambda}{\lambda^2+1}\frac{1}{\lambda} = -\frac{1}{1+\lambda^2}.$$

iii) By last calculation,

$$F(\lambda) = -\arctan \lambda + c,$$

where c is a constant. The value of c can be determined letting $\lambda \longrightarrow +\infty$ and computing

$$\lim_{\lambda \to +\infty} F(\lambda) = \lim_{\lambda \to +\infty} \int_0^{+\infty} e^{-\lambda x} \frac{\sin x}{x}\, dx.$$

We can invert limit with integral applying the Dominated Convergence noticing that

- $\lim_{\lambda \to +\infty} f(\lambda, x) = 0$, for all $x > 0$;
- $|f(\lambda, x)| \leqslant e^{-\lambda x} \leqslant e^{-x}$ for every $\lambda \geqslant 1$, a.e. $x \in [0, +\infty[$.

Therefore

$$\lim_{\lambda \to +\infty} F(\lambda) = \lim_{\lambda \to +\infty} \int_0^{+\infty} e^{-\lambda x} \frac{\sin x}{x}\, dx = \int_0^{+\infty} \lim_{\lambda \to +\infty} e^{-\lambda x} \frac{\sin x}{x}\, dx = \int_0^{+\infty} 0\, dx = 0.$$

On the other hand,

$$\lim_{\lambda \to +\infty} F(\lambda) = \lim_{\lambda \to +\infty} (-\arctan \lambda + c) = -\frac{\pi}{2} + c,$$

thus $c = \frac{\pi}{2}$ and

$$F(\lambda) = \frac{\pi}{2} - \arctan \lambda. \quad \square$$

Exercise 38. i) Let $f_n(x) := ne^{-nx}(1 - e^{-x})$. We notice that $f_n(0) = 0 \longrightarrow 0$. For $x > 0$, clearly $\frac{n}{e^{nx}} \longrightarrow 0$ thus $f_n(x) \longrightarrow 0$ for every $x > 0$. We conclude that (f_n) goes to $f = 0$ point wise.

ii) Since $f_n \longrightarrow 0$ point wise and since if $f_n \longrightarrow f$ uniformly, that is in sup norm, it implies $f_n \longrightarrow f$ point wise, the unique possibility is $f_n \longrightarrow 0$ uniformly. To check if this is true we have to verify if

$$\|f_n - 0\|_\infty = \|f_n\|_\infty = \sup_{x \in [0, +\infty[} |f_n(x)| \longrightarrow 0.$$

We compute the supremum. Since $f_n \geqslant 0$ and it is a regular function, we discuss if f_n has a maximum on $[0, +\infty[$. To this aim we may notice that

$$f_n'(x) = -n^2 e^{-nx}(1 - e^{-x}) + ne^{-nx}e^{-x} = ne^{-nx}\left(-n(1 - e^{-x}) + e^{-x}\right) = ne^{-nx}\left(-n + (n+1)e^{-x}\right).$$

Hence

$$f_n' \geqslant 0, \quad \Longleftrightarrow \quad e^{-x} \geqslant \frac{n}{n+1}, \quad \Longleftrightarrow \quad x \leqslant -\log \frac{n}{n+1} = \log \frac{n+1}{n}.$$

Thus f_n has a global maximum point at $x = \log \frac{n+1}{n}$ with maximum value

$$\|f_n\|_\infty = f_n\left(\log \frac{n+1}{n}\right) = ne^{-n \log(1+\frac{1}{n})}\left(1 - \frac{n}{n+1}\right) = \frac{n}{n+1}\left(1 + \frac{1}{n}\right)^{-n} \longrightarrow e^{-1} \neq 0.$$

Thus confute $f_n \longrightarrow 0$ uniformly.

iii) Recall that if $f_n \stackrel{L^1}{\longrightarrow} f$ then, extracting a sub-sequence, $f_{n_k} \longrightarrow f$ point wise a.e.. By i) we already know $f_n \longrightarrow 0$ point wise everywhere. Thus the unique possible candidate to be a limit is $f = 0$. To check if this is the case we must verify if

$$\|f_n - 0\|_1 = \|f_n\|_1 = \int_0^{+\infty} |f_n(x)|\, dx \longrightarrow 0.$$

Since $f_n \geqslant 0$ we have

$$\begin{aligned}
\|f_n\|_1 &= \int_0^{+\infty} ne^{-nx}(1-e^{-x})\, dx = n\int_0^{+\infty}\left(e^{-nx} - e^{-(n+1)x}\right)\, dx \\
&= n\left(\left[\tfrac{e^{-nx}}{-n}\right]_{x=0}^{x=+\infty} - \left[\tfrac{e^{-(n+1)x}}{-(n+1)}\right]_{x=0}^{x=+\infty}\right) \\
&= n\left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) = 1 - \tfrac{n}{n+1} \longrightarrow 0.
\end{aligned}$$

By this we conclude that $f_n \stackrel{L^1}{\longrightarrow} 0$.

iv) The calculation is similar to iii). The unique possibility is $f_n \stackrel{L^2}{\to} 0$, so we compute $\|f_n\|_2$. We have

$$\begin{aligned}
\|f_n\|_2^2 &= n^2\int_0^{+\infty} e^{-2nx}(1=e^{-x})^2\, dx = n^2\int_0^{+\infty}\left(e^{-2nx} - 2e^{-(2n+1)x} + e^{(2n+2)x}\right)\, dx \\
&= n^2\left(\left[\tfrac{e^{-2nx}}{-2n}\right]_{x=0}^{x=+\infty} - 2\left[\tfrac{e^{-(2n+1)x}}{-(2n+1)}\right]_{x=0}^{x=+\infty} + \left[\tfrac{e^{-(2n+2)x}}{-(2n+2)}\right]_{x=0}^{x=+\infty}\right) \\
&= n^2\left(\tfrac{1}{2n} - 2\tfrac{1}{2n+1} + \tfrac{1}{2n+2}\right) = n^2\left(\left(\tfrac{1}{2n} - \tfrac{1}{2n+1}\right) - \left(\tfrac{1}{2n+1} - \tfrac{1}{2n+2}\right)\right) \\
&= n^2\left(\tfrac{1}{2n(2n+1)} - \tfrac{1}{(2n+1)(2n+2)}\right) \longrightarrow \tfrac{1}{4} - \tfrac{1}{4} = 0. \quad \square
\end{aligned}$$

Exercise 39. i) Clearly $g \in L^2(\mathbb{R})$. Indeed $g \in \mathscr{C}(\mathbb{R})$ thus $f \in L^2(I)$ for every closed and bounded interval I. Moreover, $|g(\xi)| \leqslant \frac{1}{\xi^4}$ thus $|g(\xi)|^2 \leqslant \frac{1}{\xi^8}$, thus $|g|^2$ is integrable at $\pm\infty$. Conclusion: $g \in L^2(\mathbb{R})$. According to Fourier–Plancherel theorem, g has a Fourier original $f = \check{g}$. Since $\check{g}(x) = \widehat{g}(-x)$, we compute $\widehat{g}$. To compute this last, we first notice that, since $a^2 \neq b^2$,

$$g(\xi) = \frac{1}{b^2 - a^2}\left(\frac{1}{\xi^2 + a^2} - \frac{1}{\xi^2 + b^2}\right),$$

thus

$$\widehat{g}(x) = \frac{1}{b^2 - a^2}\left(\widehat{\frac{1}{\sharp^2 + a^2}}(x) + \widehat{\frac{1}{\sharp^2 + b^2}}(x)\right) = \frac{1}{b^2 - a^2}\left(\frac{\pi}{a}e^{-2\pi a|x|} - \frac{\pi}{b}e^{-2\pi b|x|}\right),$$

therefore

$$f(x) = \widehat{g}(-x) = \frac{1}{b^2 - a^2}\left(\frac{\pi}{a}e^{-2\pi a|x|} - \frac{\pi}{b}e^{-2\pi b|x|}\right).$$

ii) Again, clearly $\xi g(\xi) \in C(\mathbb{R})$, thus $\sharp g$ is integrable on every closed and bounded interval I. Since $|\xi g(\xi)|^2 \leqslant \frac{1}{|\xi|^7}$, $\sharp g \in L^2(\mathbb{R})$. Thus also $\sharp g$ has a Fourier original $\widetilde{f}(x) = \check{\sharp g}(x) = \widehat{\sharp g}(-x)$. Since also $\sharp g \in L^1(\mathbb{R})$ ($|\xi g(\xi)| \leqslant \frac{1}{|\xi|^3}$) we have

$$\widehat{-i2\pi\sharp g}(x) = \partial_x \widehat{g}(x) = \partial_x f(x) = -2\pi^2 \frac{\operatorname{sgn} x}{b^2 - a^2}\left(e^{-2\pi a|x|} - e^{-2\pi b|x|}\right).$$

By this we get

$$\widehat{\sharp g}(x) = -i\pi \frac{\operatorname{sgn} x}{b^2 - a^2}\left(e^{-2\pi a|x|} - e^{-2\pi b|x|}\right). \quad \square$$

Exercise 40. i) Clearly $\|\cdot\|$ is well defined on V. We have $\|f\| = 0$ iff $\int_0^1 x|f(x)|\,dx = 0$ that is, $x|f(x)| = 0$ a.e. $x \in [0,1]$ and this is equivalent to $f = 0$ a.e. $x \in [0,1]$. Homogeneity and triangular inequality are straightforward.

ii) First $f_n \in \mathscr{C}([0,1]) \subset L([0,1])$. Moreover

$$\|f_n\| = \int_0^1 x|f_n(x)|\,dx = \int_0^{1/n} xn\,dx + \int_{1/n}^1 1\,dx = n\left[\frac{x^2}{2}\right]_{x=0}^{x=1/n} + 1 - \frac{1}{n} = 1 - \frac{1}{2n},$$

thus $(f_n) \subset V$. However, this $f \notin V$ (this because $f \notin L^1([0,1])$), thus (f_n) cannot be convergent to f in V. Is it possible $f_n \xrightarrow{V} g$ for some other $g \in V$? The answer is no: indeed, $f_n \xrightarrow{V} g$ iff $0 \longleftarrow \|f_n - g\| = \int_0^1 |xf_n(x) - xg(x)|\,dx = \|xf_n - xg\|_1$, that is $xf_n \xrightarrow{L^1} xg$. In particular, for a suitable subsequence $xf_{n_k} \xrightarrow{a.e.} xg$. But $xf_n \xrightarrow{a.e.} 1 = x\frac{1}{x}$ thus, necessarily, $g = \frac{1}{x}$ a.e.. Since such $g \notin V$ this says (f_n) cannot converge in V.

iii) Clearly

$$\|f\| = \int_0^1 x|f(x)|\,dx \leqslant \int_0^1 |f(x)|\,dx = \|f\|_1,$$

so $\|\cdot\|_1$ s stronger than $\|\cdot\|$. The vice versa is false: if there exists C such that $\|f\|_1 \leqslant C\|f\|$, then $\|f_n\|_1 \leqslant C\|f_n\| = C\left(1 - \frac{1}{2n}\right)$. But

$$\|f_n\|_1 = \int_0^1 |f_n(x)|\,dx = \int_0^{1/n} n\,dx + \int_{1/n}^1 \frac{1}{x}\,dx = 1 + [\log x]_{x=1/n}^1 = 1 - \log\frac{1}{n} = 1 + \log n,$$

thus we should have $1 + \log n \leqslant C\left(1 - \frac{1}{2n}\right)$ which is clearly impossible.

iv) No: take (f_n) as in ii). We claim it is a Cauchy sequence. Indeed for $m > n$,

$$\begin{aligned}\|f_n - f_m\| &= \int_0^{1/m} x|n-m|\,dx + \int_{1/m}^{1/n} x\left|n - \tfrac{1}{x}\right|\,dx = (m-n)\tfrac{1}{2m^2} + \int_{1/m}^{1/n} (1-xn)\,dx \\ &= \tfrac{1}{2m}\left(1 - \tfrac{n}{m}\right) + \left(\tfrac{1}{n} - \tfrac{1}{m}\right) - n\left(\tfrac{1}{2n^2} - \tfrac{1}{2m^2}\right) \longrightarrow 0,\ n, m \longrightarrow +\infty.\end{aligned}$$

Since (f_n) cannot be convergent, we have an example of a Cauchy sequence not having limit in V. □

Exercise 41. i) Let $(u_n) \subset U$ be such that $u_n \overset{L^2}{\longrightarrow} u$. The goal is to prove $u \in U$ that is $u(-x) = -u(x)$ a.e. x. Since $u_n \in U$,

$$u_n(-x) = -u_n(x), \text{ a.e. } x.$$

Now, since $u_n \overset{L^2}{\longrightarrow} u$, also $u_n(-\sharp) \overset{L^2}{\longrightarrow} u(-\sharp)$ and $-u_n \overset{L^2}{\longrightarrow} -u$, thus $u(-\sharp) = -u$ a.e., that is $u \in U$.

ii) The characteristic property of $\Pi_U f$ is the unique element of U such that

$$\langle f - \Pi_U f, u\rangle = 0,\ \forall u \in U.$$

To check that $\Pi_U f(x) = \frac{1}{2}(f(x) - f(-x))$ we first notice that $\frac{1}{2}(f(x) - f(-x)) \in U$ (trivial check). Therefore, to be the orthogonal projection of f on U we have to check that

$$\int_{\mathbb{R}} \left(f(x) - \frac{1}{2}(f(x) - f(-x))\right) u(x)\,dx = 0,\ \forall u \in U.$$

We notice that

$$\begin{aligned}\int_{\mathbb{R}} \left(f(x) - \tfrac{1}{2}(f(x) - f(-x))\right) u(x)\,dx &= \tfrac{1}{2}\int_{\mathbb{R}} (f(x) + f(-x))\,u(x)\,dx \\ &= \tfrac{1}{2}\left(\int_{\mathbb{R}} f(x)u(x)\,dx + \int_{\mathbb{R}} f(-x)u(x)\,dx\right).\end{aligned}$$

Because

$$\int_{\mathbb{R}} f(-x)u(x)\,dx \overset{y=-x}{=} \int_{\mathbb{R}} f(y)u(-y)\,dy \overset{u\in U}{=} -\int_{\mathbb{R}} f(y)u(y)\,dy \equiv -\int_{\mathbb{R}} f(x)u(x)\,dx,$$

the conclusion follows. □

Exercise 42. I0 and ii) see notes.

iii) We consider the equation

$$f * f(x) = e^{-x^2}.$$

By computing the FT to both sides we get

$$\widehat{f * f}(\xi) = \widehat{e^{-\sharp^2}}(\xi) = \sqrt{\pi}e^{-\frac{1}{4}\xi^2}, \quad \Longleftrightarrow \quad \widehat{f}(\xi)^2 = \sqrt{\pi}e^{-\frac{1}{4}\xi^2}.$$

Recall now that $\widehat{f}$ is a pointwise well defined continuous function. Therefore, for each $\xi \in \mathbb{R}$, we have

$$\widehat{f}(\xi) = \phi(\xi)\pi^{1/4}e^{-\frac{1}{8}\xi^2},$$

where $\phi(\xi) = \pm 1$ for every $\xi \in \mathbb{R}$. Since we can always write

$$\phi(\xi) = \pi^{-1/4}e^{\frac{1}{8}\xi^2}\widehat{f}(\xi) \in \mathscr{C}(\mathbb{R}),$$

and ϕ takes values in $\{-1, +1\}$, we deduce that either $\phi \equiv +1$ or $\phi \equiv -1$. We deduce that, necessarily,

$$\widehat{f}(\xi) = +\pi^{1/4}e^{-\frac{1}{8}\xi^2}, \text{ or } \widehat{f}(\xi) = -\pi^{1/4}e^{-\frac{1}{8}\xi^2}$$

this yielding to

$$f(x) = +\pi^{1/4}e^{-2x^2}, \text{ or } \widehat{f}(\xi) = -\pi^{1/4}e^{-2\xi^2}$$

The conclusion is: the proposed equation has exactly two L^1 solutions. □

Exercise 43. i) Let $f(x, y) = e^{-(x+iy)^2} = e^{-(x^2-y^2+i2xy)}$. Clearly $f(\sharp, y) \in \mathscr{C}(\mathbb{R})$ for every $y \in \mathbb{R}$, thus $f(\sharp, y) \in L^1([-R, R])$ for every $R > 0$. We have to check the behaviour at $\pm\infty$: since

$$|f(x, y)| = e^{-(x^2-y^2)} = e^{-x^2}e^{y^2}$$

it is integrable in x at $\pm\infty$. We conclude F is well defined for every $y \in \mathbb{R}$.

ii) We wish to deduce

$$\partial_y F(y) = \int_{-\infty}^{+\infty} \partial_y f(x, y)\, dx. \;(\star)$$

To do this we need:

- $\partial_y f$ exists: we have $\partial_y f(x, y) = e^{-(x^2-y^2+i2xy)}(2y + i2x)$, this for every $y \in \mathbb{R}$ almost every $x \in \mathbb{R}$;
- $|\partial_y f(x, y)| = e^{-x^2}e^{y^2} \leqslant e^{R^2}e^{-x^2} =: g(x) \in L^1(\mathbb{R})$ for every $y \in [-R, R]$.

Thus, $(\star)$ holds true for every $y \in [-R, R]$, for any $R > 0$, and because this last is arbitrary, we deduce that $(\star)$ holds true for every y. In particular,

$$\partial_y F(y) = \int_{\mathbb{R}} e^{-(x+iy)^2}2(x + iy)i\, dx = i\int_{\mathbb{R}} \partial_x e^{-(x+iy)^2}\, dx = i\left[e^{-(x+iy)^2}\right]_{x=-\infty}^{x=+\infty} = 0.$$

Thus $\partial_y F(y) \equiv 0$ hence $F(y) \equiv C$. Since $F(0) = \int_{\mathbb{R}} e^{-x^2}\, dx = \sqrt{\pi}$ we deduce $F(y) \equiv \sqrt{\pi}$. □

Exercise 44. i) Let $f \in L^2([0, +\infty[)$. By the Cauchy–Schwarz inequality we have

$$\|f\| = \int_0^{+\infty} \frac{1}{1+x}|f(x)|\, dx \leqslant \left(\int_0^{+\infty} \frac{1}{(1+x)^2}\, dx\right)^{1/2}\left(\int_0^{+\infty} |f(x)|^2\, dx\right)^{1/2} = 1 \cdot \|f\|_2.$$

Thus $\|f\|$ is well defined for $f \in V$ and clearly $\|f\| \geqslant 0$. We check vanishing:

$$\|f\| = 0, \iff \int_0^{+\infty} \frac{|f(x)|}{1+x}\, dx = 0, \iff \frac{|f(x)|}{1+x} = 0, \text{ a.e.}, \iff f = 0, \text{ a.e.}.$$

Homogeneity and triangular inequality are straightforward.

ii) In i) we proved $\|f\| \leqslant \|f\|_2$ for every $f \in V$: this says that $\|\cdot\|_2$ is stronger than $\|\cdot\|$.

iii) Let

$$f_n(x) := \frac{1}{\sqrt{x}} 1_{[1/n,n]}(x).$$

We have $f_n \in L^2 \subset V$ since

$$\|f_n\|_2 = \int_0^{+\infty} |f_n(x)|^2\, dx = \int_{1/n}^{n} \frac{1}{x}\, dx = [\log x]_{x=1/n}^{x=n} = 2\log n < +\infty.$$

In particular, since $\|f_n\|_2 \longrightarrow +\infty$ we conclude that $(f_n) \subset L^2([0,+\infty[)$ is not convergent in L^2. We also notice that, if $x \in]0,+\infty[$, we have $\frac{1}{n} \leqslant x \leqslant n$ for $n \geqslant N$. Therefore, $f_n(x) = \frac{1}{\sqrt{x}} \longrightarrow \frac{1}{\sqrt{x}} =: f(x)$ for every $x > 0$. We notice that such $f \in V$ because

$$\|f\| = \int_0^{+\infty} \frac{1}{1+x}\frac{1}{\sqrt{x}}\, dx \overset{y=\sqrt{x},\ x=y^2}{=} \int_0^{+\infty} \frac{1}{1+y^2}\frac{1}{y} 2y\, dy = 2\int_0^{+\infty} \frac{1}{1+y^2}\, dy =$$

$$= 2\,[\arctan y]_{y=0}^{y=+\infty} = 2\frac{\pi}{2} = \pi.$$

Furthermore, being $f_n(x) = \frac{1}{\sqrt{x}} = f(x)$ for $x \in [1/n, n]$, we have

$$\|f_n - f\| = \int_0^{+\infty} \frac{1}{1+x}|f_n(x) - f(x)|\, dx = \int_0^{1/n} \frac{1}{1+x}\frac{1}{\sqrt{x}}\, dx + \int_n^{+\infty} \frac{1}{1+x}\frac{1}{\sqrt{x}}\, dx$$

$$= 2\,[\arctan y]_{y=0}^{y=1/\sqrt{n}} + 2\,[\arctan y]_{y=n}^{y=+\infty} = 2\arctan\frac{1}{\sqrt{n}} + \left(\frac{\pi}{2} - \arctan n\right) \longrightarrow 0.$$

We conclude that $f_n \overset{V}{\longrightarrow} f$.

iv) Since

$$\|f_n\| = \int_{1/n}^{n} \frac{1}{\sqrt{x}(1+x)}\, dx = 2\,[\arctan y]_{x=1/\sqrt{n}}^{x=\sqrt{n}} = 2\left(\arctan\sqrt{n} - \arctan\frac{1}{\sqrt{n}}\right),$$

if $\|\cdot\|$ and $\|\cdot\|_2$ were equivalent, we should have

$$\exists C > 0\ :\ \|f\|_2 \leqslant C\|f\|,\ \forall f \in V,\ \implies\ 2\log n \leqslant 2C\left(\arctan\sqrt{n} - \arctan\frac{1}{\sqrt{n}}\right),\ \forall n \in \mathbb{N}.$$

Letting $n \to +\infty$ we would have $+\infty \leqslant 2C\frac{\pi}{2} < +\infty$, which is impossible. We conclude that the two norms are not equivalent. □

Exercise 45. i) We notice that $g \in \mathscr{C}(\mathbb{R})$. Furthermore,

$$|g(\xi)| \sim_{\pm\infty} \frac{3\xi^2}{\xi^6} = \frac{3}{\xi^4},$$

so, in particular, $g^2 \sim_{\pm\infty} \frac{9}{\xi^8}$ is integrable at $\pm\infty$. Thus, $g \in L^2(\mathbb{R})$ and, according to the Fourier–Plancherel theorem, it has a Fourier original $f \in L^2(\mathbb{R})$,

$$f(x) = \frac{1}{2\pi}\widehat{g}(-x).$$

The same conclusion holds with the L^1 FT provided inversion formula applies, that is $g, \widehat{g} \in L^1(\mathbb{R})$. Without computing $\widehat{g}$, it is sufficient to check if $g, \partial_\xi g, \partial_\xi^2 g \in L^1$. For example

$$\partial_\xi g(\xi) = \frac{6\xi(1+\xi^2)^3 - (3\xi^2-1)3(1+\xi^2)^2 2\xi}{(1+\xi^2)^6} \sim_{\pm\infty} \frac{6\xi^7 - 18\xi^7}{\xi^{12}} = \frac{-12}{\xi^5},$$

from which $\partial_\xi g \in L^1$ at $\pm\infty$, and since it is also continuous we conclude that $\partial_\xi g \in L^1(\mathbb{R})$. Same check for $\partial_\xi^2 g$. Therefore, $\widehat{g} \in L^1(\mathbb{R})$ and inversion formula applies, so g has a Fourier original also in L^1.

ii) Following the hint, we have

$$\begin{aligned}\partial_\xi^2 \frac{1}{1+\xi^2} &= \partial_\xi\left(-\frac{2\xi}{(1+\xi)^2}\right) = \frac{(-2)(1+\xi^2)^2 + 2\xi\cdot 2(1+\xi^2)2\xi}{(1+\xi^2)^4} = \frac{-2(1+\xi^2)+8\xi^2}{(1+\xi^2)^3} \\ &= \frac{6\xi^2-2}{(1+\xi^2)^3} = 2g(\xi).\end{aligned}$$

Therefore,

$$\widehat{g}(x) = \frac{1}{2}\widehat{\partial_\xi^2 \frac{1}{1+\sharp^2}}(x) = \frac{1}{2}(ix)^2\widehat{\frac{1}{1+\sharp^2}}(x) = -\frac{x^2}{2}\frac{1}{2}e^{-|x|} = -\frac{x^2}{4}e^{-|x|}.$$

From this, we conclude that the $L^1 \cap L^2$ Fourier original of g is

$$f(x) = -\frac{x^2}{8\pi}e^{-|x|}. \quad \square$$

Exercise 46. i) Let $f(x,\xi) := \frac{\log(1+\xi^2x^2)}{1+x^2}$. We have to discuss $f(\sharp,\xi) \in L^1([0,+\infty[)$. Clearly $f(\sharp,\xi) \in \mathscr{C}([0,+\infty[)$, thus the unique problem is to check the behaviour at $x = +\infty$. Clearly, $f(x,\xi) \longrightarrow 0$ for $x \longrightarrow +\infty$, $\forall \xi \in \mathbb{R}$, but this is not sufficient to conclude integrability. We may notice that $f(x,0) \equiv 0$ thus $f(\sharp,0) \in L^1([0,+\infty[)$, while for $\xi \neq 0$,

$$f(x,\xi) \sim_{+\infty} \frac{\log x^2}{x^2} = \frac{2\log x}{x^2}$$

and since $\log x \leqslant C\sqrt{x}$ for suitable C, the r.h.s is integrable at $+\infty$. Thus $f(\sharp,\xi) \in L^1([0,+\infty[)$ for every $\xi \in \mathbb{R}$.

ii) We apply the differentiation theorem:

$$\partial_\xi F(\xi) = \int_0^{+\infty} \partial_\xi f(x,\xi)\ dx$$

provided

- $\partial_\xi f(x,\xi)$ exists: indeed, $\partial_\xi f(x,\xi) = \frac{2\xi x^2}{1+\xi^2 x^2}\frac{1}{1+x^2}$, $\forall \xi \in \mathbb{R}$, a.e. $x \in [0,+\infty[$;
- there exists $g = g(x) \in L^1([0,+\infty[)$ such that $|\partial_\xi f(x,\xi)| \leqslant g(x)$: indeed,

$$|\partial_\xi f(x,\xi)| = \frac{2|\xi| x^2}{1+\xi^2 x^2}\frac{1}{1+x^2}$$

Notice that, for $\xi \neq 0$,

$$|\partial_\xi f(x,\xi)| = \frac{1}{|\xi|}\frac{2|\xi|^2 x^2}{1+\xi^2 x^2}\frac{1}{1+x^2} \leqslant \frac{2}{|\xi|}\frac{1}{1+x^2} \leqslant \frac{2}{\varepsilon}\frac{1}{1+x^2} \in L^1([0,+\infty[),\ \forall |\xi| \geqslant \varepsilon.$$

Thus, for $|\xi| \geqslant \varepsilon$ we may differentiate under integral sign. Since $\varepsilon > 0$ is arbitrary we conclude that $\partial_\xi F$ exists for every $\xi \neq 0$.

Thus

$$\begin{aligned}
\partial_\xi F &= \frac{2\xi}{\xi^2+1}\int_0^{+\infty} \frac{(\xi^2+1)x^2}{(1+\xi^2x^2)(1+x^2)}\ dx = \frac{2\xi}{\xi^2-1}\int_0^{+\infty}\left(\frac{1}{1+x^2} - \frac{1}{1+\xi^2x^2}\right)\ dx \\
&= \frac{2\xi}{\xi^2-1}\left([\arctan x]_{x=0}^{x=+\infty} - \frac{1}{\xi}\,[\arctan(\xi x)]_{x=0}^{x=+\infty}\right) \\
&= \begin{cases} \frac{2\xi}{\xi^2-1}\left(\frac{\pi}{2} - \frac{1}{\xi}\frac{\pi}{2}\right) = \pi\frac{\xi}{\xi^2-1}\frac{\xi-1}{\xi} = \frac{\pi}{\xi+1}, & \xi > 0, \\ \\ \frac{2\xi}{\xi^2-1}\left(\frac{\pi}{2} + \frac{1}{\xi}\frac{\pi}{2}\right) = \pi\frac{\xi}{\xi^2-1}\frac{\xi+1}{\xi} = \frac{\pi}{\xi-1}, & \xi < 0. \end{cases}
\end{aligned}$$

We may notice that $\partial_\xi F(0\pm) = \pm\pi$ thus in particular F is not differentiable at $\xi = 0$.

iii) We have

$$F(\xi) = \begin{cases} \pi\log(\xi+1) + c_1, & \xi > 0, \\ \\ \pi\log(1-\xi) + c_2, & \xi < 0. \end{cases}$$

In particular $F(0+) = c_1$, $F(0-) = c_2$. On the other side we may compute $\lim_{\xi\to 0} F(\xi)$: to this aim we wish to do

$$\lim_{\xi\to 0} F(\xi) = \int_0^{+\infty} \lim_{\xi\to 0} f(x,\xi)\ dx = \int_0^{+\infty} 0\ dx = 0,$$

by which it would follow $c_1 = c_2 = 0$. To justify the passage of limit inside integral we invoke dominated convergence. Clearly $\lim_{\xi\to 0} f(x,\xi) = f(x,0) = 0$, and moreover, for $|\xi| \leqslant 1$

$$|f(x,\xi)| = \frac{\log(1+\xi^2x^2)}{1+x^2} \leqslant \frac{\log(1+x^2)}{1+x^2} =: g(x) \in L^1([0,+\infty[). \quad \square$$

Exercise 47. i) Let $f \in X$. Since $f \in \mathscr{C}^1([0,1])$, $f' \in \mathscr{C}([0,1])$ thus $\|f\|_* = \|f'\|_\infty$ is well defined. To check that also $\|f\|$ is well defined we have to check that $g(x) := \frac{|f(x)|}{x}$ is integrable on $[0,1]$. Clearly $g \in \mathscr{C}(]0,1])$. Since $f(x) = f(0)+f'(0)x+o(x) \overset{f\in X,\implies f(0)=0}{=} f'(0)x+o(x)$ thus $g(x) = |f'(0)| + \frac{o(x)}{x} \longrightarrow |f'(0)| \in \mathbb{R}$ when $x \longrightarrow 0+$. In particular $g \in \mathscr{C}([0,1])$ it is integrable and $\|f\|$ is well defined.

Let's now check that $\|\cdot\|$ is a norm. Clearly $\|f\| \geqslant 0$. Vanishing: if $\|f\| = 0$ that is $\int_0^1 \frac{|f(x)|}{x}\,dx = 0$, by a well known lemma $\frac{|f(x)|}{x} = 0$ for all $x \in]0,1]$ thus, in particular, $f \equiv 0$ on $]0,1]$. By continuity $f \equiv 0$ on $[0,1]$. Homogeneity and triangular inequality are straightforward.

Finally, let's check that also $\|\cdot\|_*$ is a norm. Clearly $\|f\|_* \geqslant 0$. Vanishing: if $0 = \|f\|_* = \|f'\|_\infty$ then $f' \equiv 0$ on $[0,1]$. Therefore, $f \equiv C$ and since $f(0) = 0$ we conclude $f \equiv 0$ on $[0,1]$. Homogeneity and triangular inequality are straightforward.

ii) We have to prove that there exists a constant C such that $\|f\| \leqslant C\|f\|_*$. Notice first that, according to the fundamental theorem of integral calculus,

$$f(x) = f(0)+\int_0^x f'(y)\,dy = \int_0^x f'(y)\,dy, \implies |f(x)| \leqslant \int_0^x |f'(y)|\,dy \leqslant \int_0^x \|f'\|_\infty\,dy = x\|f\|_*,$$

thus

$$\|f\| = \int_0^1 \frac{|f(x)|}{x}\,dx \leqslant \int_0^1 \frac{x\|f\|_*}{x}\,dx = \|f\|_*,\ \forall f \in X.$$

iii) Let $f_n(x) = x^n$ with $n \geqslant 1$. Clearly $f_n \in X$. We have

$$\|f_n\| = \int_0^1 \frac{x^n}{x}\,dx = \int_0^1 x^{n-1}\,dx = \left[\frac{x^n}{n}\right]_{x=0}^{x=1} = \frac{1}{n},$$

while

$$\|f_n\|_* = \|f_n'\|_\infty = \max_{x\in[0,1]} |nx^{n-1}| = n.$$

We may conclude that $\|\cdot\|$ is not stronger than $\|\cdot\|_*$ (hence the two are not equivalent). Indeed, if this were the case, there would be a constant C such that $\|f\|_* \leqslant C\|f\|$ for every $f \in X$. In particular then

$$n = \|f_n\|_* \leqslant C\|f_n\| = \frac{C}{n},\ \forall n \geqslant 1,$$

which is clearly impossible. $\square$

Exercise 48. i) About the convolution and its properties as well as proof see Lecture Notes.

ii) Let g have a Fourier original, that is, $g = \widehat{f}$ for an $f \in L^1(\mathbb{R})$. Then $g^2(\xi) = (\widehat{f}(\xi))^2 = \widehat{f}(\xi)\widehat{f}(\xi) = \widehat{f * f}(\xi)$ because $f \in L^1(\mathbb{R})$, so $f * f$ is well defined and it belongs to $L^1(\mathbb{R})$. Therefore, $f * f$ is a Fourier original of g^2, and this shows existence. The Fourier original of g^2 is unique because of the injectivity of the FT.

iii) Consider the equation

$$\int_{-\infty}^{+\infty} f(x-y)e^{-y^2}\, dy = xe^{-ax^2}.$$

This is a convolution equation

$$f * e^{-\sharp^2} = xe^{-ax^2}.$$

We start noticing that a must bene > 0. Indeed, if $f \in L^1(\mathbb{R})$ is a solution, since $e^{-\sharp^2} \in L^1(\mathbb{R})$, $f * e^{-\sharp^2} \in L^1(\mathbb{R})$ by Young's theorem. So $\sharp e^{-a\sharp^2} \in L^1(\mathbb{R})$ and this is possible iff $a > 0$.

So, let $a > 0$. We recall that

$$\widehat{e^{-\frac{\sharp^2}{2\sigma^2}}}(\xi) = \sqrt{2\pi\sigma^2}e^{-\frac{1}{2}\sigma^2\xi^2}, \implies \widehat{e^{-a\sharp^2}}(\xi) = \sqrt{\frac{\pi}{a}}e^{-\frac{1}{4a}\xi^2}$$

Since of course $xe^{-ax^2} \in L^1(\mathbb{R})$, we have

$$\widehat{-i\sharp e^{-a\sharp^2}}(\xi) = \partial_\xi \widehat{e^{-a\sharp^2}}(\xi) = \partial_\xi \sqrt{\frac{\pi}{a}}e^{-\frac{1}{4a}\xi^2} = \sqrt{\frac{\pi}{a}}\left(-\frac{1}{2a}\xi\right)e^{-\frac{1}{4a}\xi^2},$$

from which

$$\widehat{\sharp e^{-a\sharp^2}}(\xi) = -i\sqrt{\frac{\pi}{a}}\frac{\xi}{2a}e^{-\frac{1}{4a}\xi^2}.$$

Therefore, $f \in L^1(\mathbb{R})$ is a solution iff

$$\widehat{f}(\xi)\sqrt{\pi}e^{-\frac{\xi^2}{4}} = -i\sqrt{\frac{\pi}{a}}\frac{\xi}{2a}e^{-\frac{1}{4a}\xi^2}, \iff \widehat{f}(\xi) = -\frac{1}{2a^{3/2}}i\xi e^{-\frac{1}{4}\left(\frac{1}{a}-1\right)\xi^2}.$$

In order the r.h.s. be the FT of an L^1 function, according to RL lemma, it must be $\widehat{f}(\pm\infty) = 0$, which is possible iff $\frac{1}{a} - 1 > 0$, that is $a < 1$. In this case, being

$$\sqrt{\pi\left(\frac{1}{a}-1\right)}e^{-\frac{1}{4}\left(\frac{1}{a}-1\right)\xi^2} = \widehat{e^{-\frac{a}{1-a}\sharp^2}}(\xi),$$

so

$$i\xi\sqrt{\pi\left(\frac{1}{a}-1\right)}e^{-\frac{1}{4}\left(\frac{1}{a}-1\right)\xi^2} = (i\xi)\widehat{e^{-\frac{a}{1-a}\sharp^2}}(\xi) = \widehat{\partial_x e^{-\frac{a}{1-a}\sharp^2}}(\xi)$$

then

$$\widehat{f}(\xi) = -\frac{1}{2a^{3/2}}\sqrt{\frac{a}{(1-a)\pi}}\widehat{\partial_x e^{-\frac{a}{1-a}\sharp^2}}(\xi),$$

from which, finally,

$$f(x) = -\frac{1}{\sqrt{4\pi a^2(1-a)}}\partial_x e^{-\frac{a}{1-a}x^2} = \frac{1}{\sqrt{\pi(1-a)^3}} x e^{-\frac{a}{1-a}x^2}. \qquad \square$$