AREA FORMULA

Prop $T : \mathbb{R}^k \to \mathbb{R}^n$ linear map $k < n$

$x \mapsto A \cdot x$ A matrix

then $\mathcal{H}^k(T(E)) = J(T)\,|E|$ $\forall E \in \mathcal{B}(\mathbb{R}^k)$

proof (idea) $J(T) = \sqrt{\det(T^* T)} = \sqrt{\det(A^t A)}$

$T^* : \mathbb{R}^n \to \mathbb{R}^k$ $\quad$ $T^*T : \mathbb{R}^k \to \mathbb{R}^k$

$x \mapsto A^t x$ $\quad$ $x \longrightarrow A^t A x$

$R : \mathbb{R}^n \to \mathbb{R}^n$ rotation $R^* R = id$

such that $\operatorname{Im}(R \circ T) \subseteq \{x \in \mathbb{R}^n \;\; x_{k+1} = 0 = x_{k+2} = \dots = x_n\}$

$R \circ T : \mathbb{R}^k \to \mathbb{R}^n \xrightarrow{\pi_k} \mathbb{R}^k$ $\quad$ $\pi_k : \mathbb{R}^n \to \mathbb{R}^k$

$x \mapsto (x_1 \dots x_k)$ $\quad$ $x \longmapsto (x_1 \dots x_k)$

$J(RT) = J(T)$ change of variable formula

$|\pi_k(R \circ T(E))| = J(T)|E|$

Observation if $B \subseteq \{x \in \mathbb{R}^n \mid x_{k+1} = 0 = x_{k+2} = \cdots = x_n\}$

then $\pi_k(B) = \tilde{B}$ $\quad |\tilde{B}| = \mathcal{H}^k(B)$

$$|\pi_k(R \circ T(E))| = \mathcal{H}^k(R \circ T(E)) = \mathcal{H}^k(T(E)) = J(T)|E| \quad \square$$

~ . ~

Let $f : \mathbb{R}^k \to \mathbb{R}^n$ C^1 $\quad Df(x) : \mathbb{R}^k \to \mathbb{R}^n$ linear map

$$Jf(x) = \sqrt{\det(Df(x)^* Df(x))}$$

$\forall E \subseteq \mathbb{R}^k$ measurable

f injective $\quad \displaystyle\int_E Jf(x)\,dx = \mathcal{H}^k(f(E))$

f NON injective $\quad \displaystyle\int_E Jf(x)\,dx = \int_{\mathbb{R}^n} \mathcal{H}^0(E \cap f^{-1}(y))\,dy$

$\mathcal{H}^0(E \cap f^{-1}(y)) = 0$ if $y \notin f(E)$

$\mathcal{H}^0(E \cap f^{-1}(y)) = m$ if $\exists\, x_1 \ldots x_m \in E$

$f(x_1) = f(x_2) = \cdots = f(x_m) = y$

Let $k < n$ $k \in \mathbb{N}$.

Def. M is a C^1 k-dimensional submanifold of $\mathbb{R}^n$

→ if $\forall x \in M$ $\exists U_x \subseteq \mathbb{R}^n$ neighborhood of x, $V \subseteq \mathbb{R}^k$ open set

f injective of class C^1 $f: V \to U_x$ $f(V) = M \cap U_x$

$D_y f$ injective $\forall y \in V$ $D_y f$ is linear map $: \mathbb{R}^k \to \mathbb{R}^n$

(f is a parametrization of M locally around x)

→ M can be covered by a countable number of parametrizations

By the previous arguments on linear functions

$f: U \subseteq \mathbb{R}^k \to \mathbb{R}^n$ a chart of M

$\underline{A \subseteq f(U) \subseteq M}$ $\mathcal{H}^k(A) = \int_{f^{-1}(A)} Jf\,dx$ surface measure on M.

$\forall g$ meas. $: \mathbb{R}^n \to \mathbb{R}$

$$\int_{A \subseteq M} g(x)\,d\mathcal{H}^k(x) = \int_{f^{-1}(A)} g(f^{-1}(x))\,Jf(x)\,dx$$

Ex

1) $f:\mathbb{R}\to\mathbb{R}^n$ C^1 and injective

$Df = (\dot f_1, \dots \dot f_n)$ $\quad J(f) = |\dot f| = \sqrt{\dot f_1^2 + \dots + \dot f_n^2}$

$C = f([a,b])$ curve in $\mathbb{R}^n$

$$\mathcal{H}^1(C) = \int_{[a,b]} |\dot f|\, dt$$

2) $g:\mathbb{R}^n\to\mathbb{R}$ C^1 $\quad f(x) = (x, g(x))$ $\quad Df = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \frac{\partial g}{\partial x_1} & \dots & \frac{\partial g}{\partial x_n} \end{pmatrix}$ $\quad Jf = \sqrt{1+|Dg|^2}$

$$\mathcal{H}^n(\{(x,g(x))\ x\in U\}) = \int_U (1+|Dg|^2)^{1/2}\, dx .$$

So $\mathcal{H}^k$ defines the measure on lower dimensional manifolds

M C^1 k-dim. submanifold $\Rightarrow \mathcal{H}^k|_M$ is the surface measure on M.

$\mathcal{H}^k$ permits to define a larger class of sets of lower dimension ($k < n$) on which it is possible to perform integration (not only on C^1 k-manifolds)

DEFINITION E is k-RECTIFIABLE if

$\mathcal{H}^k(E) < +\infty$

$\exists (f_i)_i$ countable $f_i : \mathbb{R}^k \to \mathbb{R}^n$ C^1 such that

$$\mathcal{H}^k\left(E \setminus \bigcup_i f_i(\mathbb{R}^k)\right) = 0$$

Def E is COUNTABLY k-RECTIFIABLE if
$\exists (f_i)$ $f_i \in C^1$ $f_i : \mathbb{R}^k \to \mathbb{R}^n$
$\mathcal{H}^k(E \setminus \cup_i f_i(\mathbb{R}^k)) = 0$ (and $\mathcal{H}^k(E) = +\infty$)

Def E is PURELY k-UNRECTIFIABLE if
$\forall$ $f : \mathbb{R}^k \to \mathbb{R}^n$ of class C^1 (or Lipschitz)
$\mathcal{H}^k(E \cap f(\mathbb{R}^k)) = 0$
(that is $\mathcal{H}^k(E \cap M) = 0$ $\forall$ M C^1 k-manifold

$\dim_{\mathcal{H}} E_1 = 1 = \dim_{\mathcal{H}} E_2$

<u>Ex</u> $E_1, E_2 \subseteq \mathbb{R}$ $|E_1| = 0 = |E_2|$ $\Leftrightarrow$ $\mathcal{H}^1(E_1) = 0 = \mathcal{H}^1(E_2)$
$E_1 \times E_2$ is purely 1-unrectifiable in $\mathbb{R}^2$
M C^1 1-manifold is a curve (C^1 curve) $\gamma(t) = (t, y(t))$
locally
$E_1 \times E_2 \cap \gamma \subseteq \{(t, y(t))$ $t \in E_1\}$
$\mathcal{H}^1(E_1 \times E_2 \cap \gamma) \leq \mathcal{H}^1(y(E_1)) \leq \sqrt{1 + \|y'\|_\infty}\, \mathcal{H}^1(E_1) = 0$

REFERENCE: CHAPTER 9 FOLLAND

DISTRIBUTION THEORY

(generalization of the concept of functions and measures L. Schwartz, '50)

$U \subseteq \mathbb{R}^n$ open

$\mathcal{C}_c^\infty(U)$ test functions (with the topology ...)

$(\stackrel{!}{=} \mathcal{D}(U))$

<u>Def</u>: A LINEAR functional $T: \mathcal{C}_c^\infty(U) \longrightarrow \mathbb{R}$ is a DISTRIBUTION if it is (SEQUENTIALLY) CONTINUOUS w.r.t. the convergence defined in $\mathcal{C}_c^\infty(U)$

that is $\underbrace{f_n \longrightarrow f \text{ in } \mathcal{C}_c^\infty(U)} \Longrightarrow T(f_n) \longrightarrow T(f)$ (in $\mathbb{R}$)

REMEMBER $\exists K \subset\subset U$

$\operatorname{supp} f_n, \operatorname{supp} f \subseteq K \quad \|D^\alpha f_n - D^\alpha f\|_\infty \to 0 \ \forall \alpha$

EQUIVALENT DEFINITION $T : \mathcal{C}_c^\infty(U) \to \mathbb{R}$ linear

It is equivalent ① T sequentially continuous

and ② $\forall K \subset\subset U$ $\exists$ $C_K > 0$, $\exists\, p_K \in \mathbb{N}$

such that $\forall \phi \in \mathcal{C}_c^\infty(U)$ $\operatorname{supp} \phi \subseteq K$

it holds $|T(\varphi)| \leq C_K \sup\limits_{|\alpha| \leq p_K} \|D^\alpha \phi\|_\infty$

② $\Rightarrow$ ① Take $f_n \to f$ in $\mathcal{C}_c^\infty(U) \Rightarrow \exists\, K \subset\subset U$ such

that $\operatorname{supp} f_n, \operatorname{supp} f \subseteq K \subset\subset U$. $\|D^\alpha (f_n - f)\|_\infty \to 0\ \forall \alpha$

by 2 applied to $f_n - f$ $\quad |T(f_n - f)| \leq C_K \sup\limits_{|\alpha| \leq p_K} \|D^\alpha (f_n - f)\|_\infty \longrightarrow 0$

$\Rightarrow |T(f_n - f)| \to 0 \Rightarrow T f_n \to T f$

1 $\Rightarrow$ 2 By contradiction: assume 2 does not hold and 1 holds

So $\exists K \subset\subset U$ $\forall j \in \mathbb{N}$ $\exists \phi_j \in C_c^\infty(U)$ with $\operatorname{supp} \phi_j \subseteq K$

$$|T(\varphi_j)| \geq j \cdot \sup_{|\alpha| \leq j} \|D^\alpha \phi_j\|_\infty$$

Def $\psi_j = \dfrac{\phi_j}{|T(\varphi_j)|} \in C_c^\infty(U)$ $\quad \operatorname{supp} \psi_j \subseteq K$, $|T\psi_j| = 1$ by linearity

$$j \cdot \sup_{|\alpha| \leq j} \|D^\alpha \psi_j\|_\infty = j \sup_{|\alpha| \leq j} \frac{\|D^\alpha \phi_j\|_\infty}{|T(\varphi_j)|} \leq 1$$

$\sup_{|\alpha| \leq j} \|D^\alpha \psi_j\|_\infty \leq \frac{1}{J} \Rightarrow \forall n \in \mathbb{N}$ $\psi_j \in (C^m(K), \|\cdot\|_{m,\infty})$ $\quad \|f\|_{m,\infty} = \sup_{|\alpha| \leq m} \|D^\alpha f\|_\infty$

$\forall n$ fixed $\psi_j \to 0$ in $C^n(K)$

$\|D^\alpha \psi_j\|_\infty \to 0$ $\forall \alpha \Rightarrow \psi_j \to 0$ in $C_c^\infty(U)$. So by 1 $T\psi_j \to 0$

$|T\psi_j| = 1 \longrightarrow 0$ CONTRADICTION

Def If in the definition p_K can be chosen INDEPENDENT of K ($p_K = p \ \forall K$)

$$\left[\begin{array}{l}\text{that is } \forall K \subset\subset U \ \exists c_K > 0 \ \exists p \in \mathbb{N} \text{ s.th.} \\ \forall \phi \in C_c^\infty(U) \ \operatorname{supp} \phi \subseteq K \quad \sup_{|\alpha| \le p} \|D^\alpha \phi\|_\infty \le c_K.\end{array}\right]$$

then p = ORDER of the DISTRIBUTION.

(Equivalently if $\sup_{K \subset\subset U} p_K = p \in \mathbb{N}$).

p order of the DISTRIBUTION – if it exists – is the MINIMUM natural number such that

$$\forall K \subset\subset U \quad \forall \phi \in C_c^\infty(U) \ \operatorname{supp} \phi \subseteq K \qquad |T(\varphi)| \le c_K \sup_{|\alpha| \le p} \|D^\alpha \phi\|_\infty$$

Ex $f \in L^1_{loc}(U)$ is associated to a distribution

$$T_f(\varphi) = \int_U \phi(x) f(x)\,dx \quad \forall \phi \in C^\infty_c(U)$$

DISTR. of ORDER 0

$$|T_f(\phi)| \le \int_{K = \operatorname{supp}\phi} |f|\,|\varphi|\,dx \le \|\phi\|_\infty \cdot \underbrace{\int_K |f|\,dx}$$

$\forall K \subset\subset U \quad \exists\, C_K = \int_K |f(x)|\,dx \quad$ s.t. $\quad |T_f(\varphi)| \le C_K \|\phi\|_\infty$

$p = 0$!

Ex μ Radon measure $\mathcal{M}(U)$ is associated to a distribution $\quad T_\mu(\phi) = \int_U \varphi(x)\,d\mu(x)$

$$|T_\mu(\varphi)| \le \left| \int_{K = \operatorname{supp}\phi} \phi(x)\,d\mu(x) \right| \le \int_K |\phi(x)|\,d\mu \le \|\phi\|_\infty\, \mu(K)$$

$\forall K \subset\subset U$
$\exists\, C_K = \mu(K) \quad$ s.t. $\quad |T_\mu(\varphi)| \le C_K \|\varphi\|_\infty$

$p = 0$

DISTR of ORDER 0