<u>Theorem</u> $\mathcal{H}^n$ is Lebesgue. (for $n=1$ proved already)

To prove the theorem we pass through an important inequality.

ISODIAMETRIC INEQUALITY $\forall E \in \mathcal{B}(\mathbb{R}^n)$

$$|E| \leq \omega_n \left(\frac{\operatorname{diam}(E)}{2}\right)^n$$ (= volume of the ball with ~~diameter~~ diam(E))

that is:
(among sets of given diameter, the ball has maximal volume / among sets of given volume the ball has minimal diameter.

<u>proof</u> $n=1$ trivial $|E| \leq \operatorname{diam} E$

$n>1$ proof by **STEINER SYMMETRIZATION** (method used in several other context in minimization pb)

STEINER SYMMETRIZATION PROCEDURE

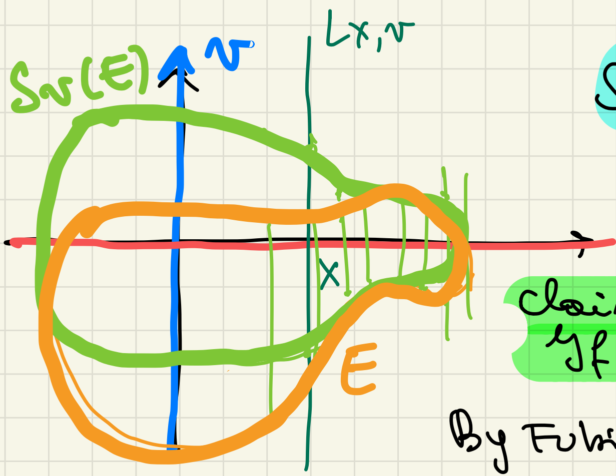
Let $E \subseteq \mathbb{R}^n$. Fix $v \in S^{n-1}$ ($|v|=1$) "a direction"

π_v = hyperplane $\perp v$ passing through the origin

$\stackrel{!}{=} \{ x \in \mathbb{R}^n \quad x \cdot v = 0 \}$

↑ scalar product

$\forall x \in \pi_v \quad L_{x,v} = x + tv \quad t \in \mathbb{R}$ (straight line $\perp \pi_v$)



$$S_v(E) = \bigcup_{x \in \pi_v} \{ x + tv, \ |t| \le \tfrac{1}{2} \mathcal{H}^1(E \cap L_{x,v}) \}$$

↓ Steiner symmetrization of E w.r.t.

claim

If E is measurable $\to S_v(E)$ is measurable

By Fubini-Tonelli $\boxed{|S_v(E)| = |E|}$

claim: $\text{diam } S_v(E) \le \text{diam } E$

Claim
if E is measurable $\Rightarrow S_v(E)$ is measurable

SUBGRAPH OF A MEASURABLE FUNCTION IS MEASURABLE

Thm $f : \mathbb{R}^n \to [0, +\infty)$ measurable ($\mathcal{L}$-meas.)

$A = \{ (x, y) \quad y \le f(x) \}$ in measurable in $\mathbb{R}^{n+1}$

proof: $F : (x,y) \xrightarrow{\text{meas}} (f(x), y) \xrightarrow{\text{Borel}} f(x) - y$ $\quad A = \{F(x,y) \ge 0\}$ measurable

measurable

Take $v = e_1$ $\qquad \mathcal{H}^1 = \mathcal{L}$ on $\mathbb{R}$ $\qquad \pi_v = \{0\} \times \mathbb{R}^{n-1} \; (\cong \mathbb{R}^{n-1})$

$f : \underset{\pi_v^{\shortparallel}}{\mathbb{R}^{n-1}} \to \mathbb{R}$ $\quad$ By Fubini the map $f : x \longmapsto \mathcal{H}^1(E \cap L_{xv})$ is measurable $|E| = \int_{\mathbb{R}^n} \chi_E \, dx = \int_{\mathbb{R}^{n-1}} f(x) \, dx = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(E \cap L_{xv}) \, dx$

$$S_v(E) = \left\{ (y_1, x) \quad -\frac{f(x)}{2} \le y_1 \le \frac{f(x)}{2} \right\}$$

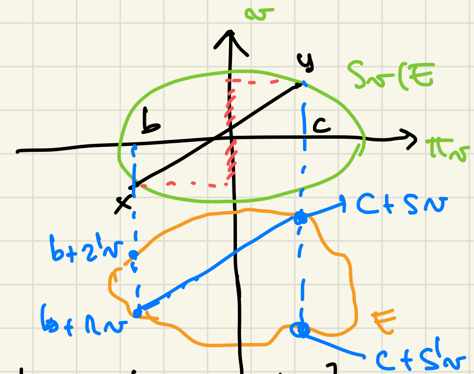
Therefore $|E| = |S_v(E)|$

Claim: diam $S_v(E) \le$ diam E.

Take $z, y \in S_v(E)$; $|z-y|^2 \ge (\text{diam } S_v(E))^2 - \varepsilon$ $\quad \varepsilon$ small

$z \in S_v(E)$ $\quad z = x_1 + tv \quad b \in \pi_v \to t = z v \quad (b\cdot v = 0!)$

$$z = x_1 + (z\cdot v)\, v$$



$y \in S_v(E)$ $\quad y = x_2 + (y\cdot v)\, v \quad c \in \pi_v$

$$(\text{diam } S_v(E))^2 - \varepsilon \le |z-y|^2 = |x_1 - x_2|^2 + |z\cdot v - y\cdot v|^2$$

$s := \sup\{t \mid x_2 + tv \in E\}$ $\quad s' = \inf\{t \mid x_2 + tv \in E\}$

$r := \inf\{t \mid x_1 + tv \in E\}$ $\quad r' = \sup\{t \mid x_1 + tv \in E\}$

$|y\cdot v| \le \dfrac{s-s'}{2} \qquad |z\cdot v| = \dfrac{r'-r}{2}$

Assume without loss of generality $\boxed{s - r \ge r' - s'}$

$$|z\cdot v - y\cdot v| \le \frac{s-s'}{2} + \frac{r'-r}{2} = \frac{s-r}{2} + \frac{r'-s'}{2} \le \frac{r'-s'}{2} + \frac{r'-s'}{2} = r'-s'$$

$$|z-y|^2 = |x_1-x_2|^2 + (z\cdot v - y\cdot v)^2 \le |x_1-x_2|^2 + |r'-s'|^2 =$$

$$= |(x_1+r'v) - (x_2+s'v)|^2 \le (\mathrm{diam}\, E)^2 \qquad \left(\begin{array}{l} x_1+r'v \in \overline{E} \\ x_2+s'v \in \overline{E} \end{array}\right)$$

$$(\mathrm{diam}\, S_v(E))^2 - \varepsilon \le (\mathrm{diam}\, E)^2$$

$\xrightarrow{\varepsilon \to 0}$ $\mathrm{diam}\, S_v(E) \le \mathrm{diam}(E)$.

Observ. $S_v(E)$ symmetric w.r.to π_v.

If E is symmetric w.r.to π_w, with $w \perp v$ $(w\cdot v = 0)$

then $S_v(E)$ is symmetric w.r.to π_v, π_w.

$E_1 = S_{e_1}(E)$ $\quad e_1 = (1, 0 \ldots 0)$ $\quad$ symm w.r.t $\pi_{e_1} = \{x_1 = 0\}$

$E_2 = S_{e_2}(E_1)$ $\quad \to$ symm w.r.to π_{e_1}, π_{e_2}

.

$E^* = S_{e_n}(E_{n-1})$ $\to$ symmetric w.r.to $\pi_{e_1}, \pi_{e_2}, \ldots \pi_{e_n}$

$|E^*| = |E|$ $\quad$ $\mathrm{diam}\, E^* \le \mathrm{diam}\, E$.

E^* is symmetric w.r.t all coordinate axis.
(STEINER SYMMETRIZATION of E).

$$\downarrow \quad E^* \subseteq \overline{B\left(0, \frac{\operatorname{diam} E^*}{2}\right)} \subseteq \overline{B\left(0, \frac{\operatorname{diam} E}{2}\right)}$$

$$|E| = |E^*| \leq \omega_n \left(\frac{\operatorname{diam} E}{2}\right)^n \qquad \square.$$

Theorem $\mathcal{H}^n(E) = \mathcal{H}^n_\delta(E) = |E| \qquad \forall\, E \in \mathcal{B}(\mathbb{R}^n)$

Step 1 Let E_i open with $\operatorname{diam} E_i \leq \delta$, $E \subseteq \bigcup_i E_i$

$$|E| \leq \sum_i |E_i| \underset{(\text{ISODIAM.})}{\leq} \sum_i \omega_n \left(\frac{\operatorname{diam} E_i}{2}\right)^n = \underbrace{\frac{\omega_n}{2^n} \sum_i (\operatorname{diam} E_i)^n}_{\leq\, \mathcal{H}^n_\delta(E)}$$

$$|E| \leq \mathcal{H}^n_\delta(E) \leq \mathcal{H}^n(E) \qquad \forall \delta > 0.$$

Step 2 Let Q a cube in $\mathbb{R}^n$ $Q = (a,b)^n = (a,b)\times(a,b)\cdots\times(a,b)$

$$|Q| = \left(\frac{\operatorname{diam} Q}{\sqrt{n}}\right)^n \qquad \left(\text{edge of } Q \text{ is } \frac{\operatorname{diam} Q}{\sqrt{n}}\right)$$

$$\mathcal{H}^n_\delta(E) \le \inf\left\{ \frac{\omega_n}{2^n} \sum_i (\operatorname{diam} Q_i)^n \ \begin{matrix} \operatorname{diam} Q_i \le \delta \\ Q_i \text{ cube} \quad E \subseteq \cup_i Q_i \end{matrix} \right\} \le$$

$$\le \inf\left\{ \frac{\omega_n}{2^n} \sum_i n^{\frac{n}{2}} \cdot |Q_i| \right\} \le \inf\left\{ \frac{\omega_n}{2^n} n^{n/2} \sum_i |Q_i| \ \begin{matrix} Q_i \text{ cube} \\ E \subseteq \cup_i Q_i \\ \operatorname{diam} Q_i \le \delta \end{matrix} \right\}$$

$$\le \frac{\omega_n}{2^n} n^{n/2} |E|$$

$\forall E \in \mathcal{B}(\mathbb{R}^n)$

Letting $\delta \to 0^+$ $\mathcal{H}^n(E) \le \frac{\omega_n}{2^n} n^{n/2} |E| \implies$ $\boxed{\begin{matrix} \mathcal{H}^n_\delta \ll \mathcal{L} \\ \mathcal{H}^n \ll \mathcal{L} \end{matrix}}$

Step 3 Fix $\varepsilon > 0$ Q_i cubes of diam δ such that

$E \subseteq \cup_i Q_i$ $\qquad |E| \le \sum_i |Q_i| \le |E| + \varepsilon$

By Vitali covering theorem, $\forall$ Q_i° (interior of Q_i) there exist a countable covering of closed disjoint balls of diameter $\leq \delta$

$|Q_i \setminus \bigcup_k B_i^k| = 0$ $\qquad \bigcup_i B_i^k \subseteq Q_i^\circ$

$\downarrow$

$\mathcal{H}^n(Q_i \setminus \bigcup_k B_i^k) = 0$ since $\mathcal{H}^n \ll \mathcal{L}$, $\mathcal{H}^n_\delta(Q_i \setminus \bigcup_k B_i^k) = 0$

$$\mathcal{H}^n_\delta(E) \leq \sum_i \mathcal{H}^n_\delta(Q_i) = \sum_i \mathcal{H}^n_\delta\left(\bigcup_k B_i^k\right) \leq \sum_i \sum_k \underbrace{\mathcal{H}^n_\delta(B_i^k)}$$

$$\leq \sum_i \sum_k \frac{\omega_n}{2^n}(\operatorname{diam} B_i^k)^n = \sum_i \sum_k |B_i^k| = \sum_i |Q_i| \leq |E| + \varepsilon$$

$\uparrow$
$\operatorname{diam} B_i^k < \delta$!
CHOOSE COVERING OF B_i^k as B_i^k ITSELF

$$\mathcal{H}^n_\delta(E) \leq |E| + \varepsilon \implies \mathcal{H}^n_\delta(E) \leq |E|$$

$\downarrow$ $\mathcal{H}^n(E) \leq |E|$

We conclude

$$\boxed{|E| \leq \mathcal{H}^n_\delta(E) \leq \mathcal{H}^n(E) \leq |E|}$$

$\forall E \in \mathcal{B}(\mathbb{R}^n)$