Recap

$$\mathcal{H}^s(E) = \sup_{\delta>0}\left[\mathcal{H}^s_\delta(E)\right] =$$

$$= \sup_{\delta>0}\left[\frac{\omega_s}{2^s}\inf_i\left(\Sigma_i(\operatorname{diam} E_i)^s \quad \begin{matrix} E \supseteq \cup_i E_i \\ \operatorname{diam} E_i \leq \delta \end{matrix}\right)\right]$$

$$\left[\begin{matrix} E \subseteq \mathbb{R} \\ \mathcal{H}^1(E) = |E| = \mathcal{H}^1_\delta(E) \quad \delta>0 \end{matrix}\right.$$

$\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$

- $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E) \qquad \forall \lambda>0$
- $\mathcal{H}^s(F(E)) \leq K^s \mathcal{H}^s(E) \qquad F: \mathbb{R}^n \to \mathbb{R}^n$

$$\sup_{x\neq y}\frac{|F(x)-F(y)|}{|x-y|} = K.$$

Proposition Let $0 \le s_1 < s_2$. $\exists\ \bar{d} \ge 1$ (dimensional constant)

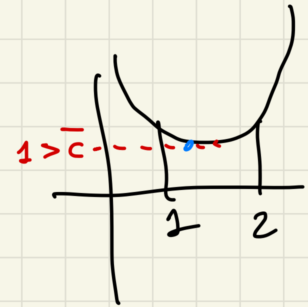
$$\mathcal{H}_\delta^{s_2}(E) \le \bar{d}\, \delta^{s_2 - s_1}\, \mathcal{H}_\delta^{s_1}(E) \qquad \forall E \subseteq \mathbb{R}^n \qquad \delta > 0$$

proof Let $E \subseteq \mathbb{R}^n$ $\quad E_i$ with $\operatorname{diam} E_i < \delta$ $\quad E \subseteq \bigcup_i E_i$

$$\frac{\omega_{s_2}}{2^{s_2}} \sum_i (\operatorname{diam} E_i)^{s_2} = \left(\frac{\sqrt{\pi}}{2}\right)^{s_2} \frac{1}{\Gamma\left(\frac{s_2}{2}+1\right)} \sum_i (\operatorname{diam} E_i)^{s_1} (\operatorname{diam} E_i)^{s_2 - s_1}$$

$$\le \left(\frac{\sqrt{\pi}}{2}\right)^{s_1} \frac{1}{\bar{c}} \frac{1}{\Gamma\left(\frac{s_1}{2}+1\right)} \delta^{s_2 - s_1} \sum_i (\operatorname{diam} E_i)^{s_1}$$

$\frac{1}{\bar{c}} = \bar{d} > 1$



$$\frac{\sqrt{\pi}}{2} < 1 \Rightarrow \left(\frac{\sqrt{\pi}}{2}\right)^{s_2} \le \left(\frac{\sqrt{\pi}}{2}\right)^{s_1}$$

$\Gamma(t)$ is increasing for $t \ge 2$. $\quad \Gamma(1) = \Gamma(2) = 1 \quad$ $\min_{t \ge 0} \Gamma = \bar{c} = \Gamma(\bar{t}) < 1$, $\bar{t} \in (1,2)$

so if $s_2 \ge 2$, $\frac{s_2}{2}+1 \ge 2 \Rightarrow \Gamma\left(\frac{s_2}{2}+1\right) \ge \Gamma\left(\frac{s_1}{2}+1\right)$

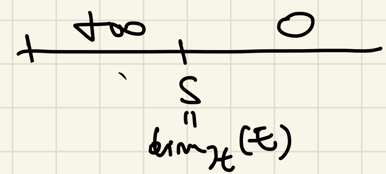
$1 < \frac{s_1}{2}+1 < 2$

$0 < s_1 \le s_2 < 2 \rightarrow \Gamma\left(\frac{s_2}{2}+1\right) \ge \bar{c} \ge \bar{c}\, \Gamma\left(\frac{s_1}{2}+1\right)$ since $\Gamma\left(\frac{s_1}{2}+1\right) \le 1$

$\delta>0$ $$\begin{array}{ccccc} H_\delta^{s_2}(E) & \le & \underbrace{C\,\delta^{s_2-s_1}} & H_\delta^{s_1}(E) & \delta>0 \\ \downarrow \delta\to 0 & & \downarrow \delta\to 0 & \downarrow \delta\to 0 & \\ \mathcal{H}^{s_2}(E) & \le & 0 \cdot & \mathcal{H}^{s_1}(E) & \end{array}$$

so if $H^{s_2}(E)>0 \Rightarrow H^{s_1}(E)=+\infty$ $\quad (s_2>s_1)$

$H^{s_1}(E)<+\infty \Rightarrow H^{s_2}(E)=0$



$\dim_{\mathcal{H}}(E)$ = Hausdorff dimension of E =

$$= \inf\{s\ge 0,\ \mathcal{H}^s(E)=0\} = \sup\{s\ge 0\ \ \mathcal{H}^s(E)=+\infty\}$$

NB It may happen that $\dim_{\mathcal{H}}(E)=r>0$ $\left[\begin{array}{l}\mathcal{H}^r(E)=0 \\ \text{or } \mathcal{H}^r(E)=+\infty\end{array}\right.$

(So it is not necessary that $\mathcal{H}^r(E)\in(0,+\infty)$)

Hausdorff dimension can also be non integer.

Theorem $\mathcal{H}^n(E) = |E| \quad \forall E \in \mathcal{B}$ (for $s=n$ n-dim Hausdorff measure coincide with the Lebesgue measure)
(for $n=1$ already proved)
(proof tomorrow

Corollary (1) H^s is not Radon $\forall\ s<n$

(proof $K \subset\subset \mathbb{R}^n$ $\quad H^n(K) = |K| > 0 \Rightarrow H^s(K) = +\infty \quad \forall s<n$)

2) $H^s \equiv 0$ if $s>n$

(proof $E \in \mathcal{B}(\mathbb{R}^n)$ $\quad |E| < +\infty \Rightarrow H^n(E) = |E| \Rightarrow H^s(E) = 0 \quad \forall s>n$

if $E \in \mathcal{B}(\mathbb{R}^n)$ $\quad |E| = +\infty \quad E = \bigcup_i E_i \quad |E_i| < +\infty \quad E_i = E \cap B(0,i)$

$|E_i| < +\infty \Rightarrow H^s(E_i) = 0 \quad \forall s>n \qquad H^s(E) \leq \sum_i H^s(E_i) = 0$

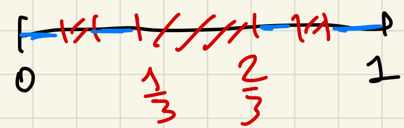
Ex Cantor set in $\mathbb{R}$ is a set with Hausdorff dim. between $0, 1$.

$C_0 = [0,1]$ $\quad C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ $\quad C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{3}, 1]$

$$C_{n+1} = \frac{1}{3} C_n \cup \left(\frac{1}{3} C_n + \frac{2}{3} \right)$$

C_n closed $\forall n$, and contains 2^n intervals of length $\frac{1}{3^n}$ (with distance between intervals $\geq \frac{1}{3^n}$)

$C = \bigcap_{n=0}^{+\infty} C_n$ is the CANTOR SET (CLOSED)



at every step $n \geq 1$ I remove 2^{n-1} intervals of length $\frac{1}{3^n}$

$$|C| = 1 - \sum_{n=1}^{+\infty} \frac{2^{n-1}}{3^n} = 1 - \sum_n \left(\frac{2}{3}\right)^n \frac{1}{3} = 1 - \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}} \right) = 0$$

$|C| = 0 \Longrightarrow \dim_{\mathcal{H}}(C) \leq 1$

since $C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right)$ $\qquad \mathcal{H}^s(C) = 2\mathcal{H}^s\left(\frac{1}{3}C\right) = 2\frac{1}{3^s}\mathcal{H}^s(C)$

if $\exists s$ s.t. $\mathcal{H}^s(C) \neq 0, +\infty \Longrightarrow 2\frac{1}{3^s} = 1 \Rightarrow \boxed{s = \log_3 2 = \frac{\log 2}{\log 3}}$

In particular for $s \neq \log_3 2 \Rightarrow \mathcal{H}^s(C) = 0$ or $\mathcal{H}^s(C) = +\infty$.

1) take $\delta = \frac{1}{3^n}$ $\quad s > 0 \qquad C_n$ is a cover of $C \qquad (C_n^i)_{i = \{1 \dots 2^n\}}$

$$\mathcal{H}^s_{\left(\frac{1}{3}\right)^n}(C) \leq \frac{\omega_s}{2^s} \sum_{i=1}^{2^n} \underbrace{\left(\operatorname{diam} C_n^i\right)^s}_{\left(\frac{1}{3^n}\right)^s} = \frac{\omega_s}{2^s} \left(\frac{1}{3^n}\right)^s \cdot 2^n = \frac{\omega_s}{2^s}\left(\frac{2}{3^s}\right)^n$$

$\left(\frac{1}{3}\right)^n \to 0 \ \underbrace{(n \to +\infty)} \Longrightarrow \mathcal{H}^s(C) \leq \frac{\omega_s}{2^s} \lim_n \left(\frac{2}{3^s}\right)^n = 0$ if $\frac{2}{3^s} < 1$ $\quad \boxed{s > \log_3 2}$

$\Longrightarrow \mathcal{H}^s(C) = 0 \ \forall s > \log_3(2) \Rightarrow \dim_{\mathcal{H}}(C) \leq \log_3(2)$

Moreover $s = \log_3 2$ $\mathcal{H}^{\log_3 2}(C) \leq \frac{\omega_{\log_3 2}}{2^{\log_3 2}}$

② Reverse much more difficult (possible reference FALCONER Geometry of Fractal sets

$$\mathcal{H}^{\log_3 2}(C) \geq 3^{-s} \frac{\omega_s}{2^s} > 0$$

$\downarrow$

$\dim_{\mathcal{H}} C = \log_3 2$ $\mathcal{H}^{\log_3 2}(C) = \frac{\omega_{\log_3 2}}{2^{\log_3 2}}$

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DEVIL'S STAIRCASE (C has the same cardinality of $[0,1]$)

1 OBSERVATION

→ Every $x \in C$ can be written in base 3 as

$x =_3 0, x_1 x_2 x_3 x_4 \ldots$ where $x_i \in \{0, 2\}$

Indeed $\quad C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

Observe $\quad 0 =_3 0 \qquad \frac{1}{3} =_3 0,1 =_3 0,0\overline{2}$ (we choose $0.0\overline{2}$)

$\frac{2}{3} =_3 0,2 \qquad 1 =_3 1 =_3 0,\overline{2}$ (we choose $0,\overline{2}$)

$\frac{1}{3} < x < \frac{2}{3} \implies x = 0,1\,x_2 x_3 x_4 \dots$ (and we eliminate them)

$C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

$\frac{1}{9} =_3 0,01 = 0,00\overline{2} \qquad \frac{2}{9} =_3 0,02 \qquad \frac{1}{9} < x < \frac{2}{9} \to x = 0,01\,x_3 x_4 \dots$

...

$$f : C \longrightarrow [0,1]$$

$0,x_1x_2x_3\ldots \longrightarrow 0,\frac{x_1}{2}\frac{x_2}{2}\frac{x_3}{2}\ldots \rightarrow$ this coincide with a number written in base 2 and in $[0,1]$.

(number written in base 3 only with $x_i=\{0,2\}$) $\quad$ $\left(\frac{x_i}{2}\in\{0,1\}\right)$

$f(0)=0 \quad \underbrace{f(0,\overline{2})}_{=1} = 0,\overline{1}_2 = 1 \qquad (f(0)=0 \quad f(1)=1)$

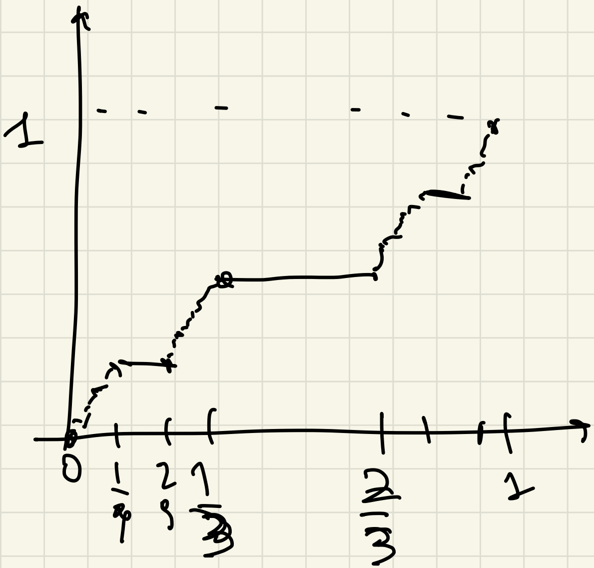
$$f\left(\tfrac{1}{3}\right) = f(0,0\overline{2}) = 0,0\overline{1} =_2 0,1 =_2 \tfrac{1}{2} =_2 0,1 = f(0,2) = f\left(\tfrac{2}{3}\right)$$

$f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right) = \frac{1}{2} \rightarrow$ on the interval $\left[\frac{1}{3},\frac{2}{3}\right]$ f is constant $=\frac{1}{2}$

$$f\left(\tfrac{1}{9}\right) = f(0,00\overline{2}) = 0,00\overline{1} = 0,01 = \tfrac{1}{4} = f(0,02) = f\left(\tfrac{2}{9}\right)\ldots$$

$f(0)=0 \quad f(1)=1$

$f : C \to [0,1]$ is monotone non decreasing, surjective and CONSTANT on every interval I which is not contained in C.



graph of f is the DEVIL'S STAIRCASE

NB in every interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\left(\frac{7}{9}, \frac{8}{9}\right)$, $\left(\frac{1}{27}, \frac{2}{27}\right)$, $\left(\frac{25}{27}, \frac{26}{27}\right)$

f is CONSTANT

$\Downarrow$

$f' = 0$ ALMOST EVERYWHERE (because $|C|=0$)

and is NON DECREASING (NON CONSTANT)