<u>COMPUTABILITY</u> (20/10/2025)

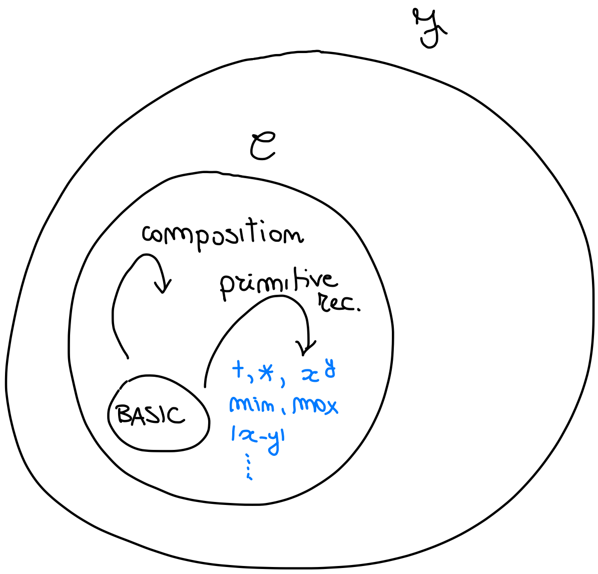
Class $\mathcal{C}$ of URM-computable functions

* contains the BASIC FUNCTIONS

  (a) zero
  (b) successor
  (c) projections

* <u>closed</u> under

  (1) (generalised) composition
  (2) primitive recursion
  (3) (unbounded) minimalisation



OBSERVATION : <u>Definition by cases</u>

Let $f_1, \dots, f_n : \mathbb{N}^k \to \mathbb{N}$ computable total

$Q_1(\vec{x}), \dots, Q_n(\vec{x}) \subseteq \mathbb{N}^k$ decidable predicates $\quad \forall \vec{x} \quad \exists! j \quad Q_j(\vec{x})$

and let $f : \mathbb{N}^k \to \mathbb{N}$

$$f(\vec{x}) = \begin{cases} f_1(\vec{x}) & \text{if } Q_1(\vec{x}) \\ f_2(\vec{x}) & \text{if } Q_2(\vec{x}) \\ \vdots & \\ f_n(\vec{x}) & \text{if } Q_n(\vec{x}) \end{cases}$$

Then f is computable

<u>proof</u>

assumed computable

$$f(\vec{x}) = \underbrace{f_1(\vec{x}) \cdot \overbrace{\chi_{Q_1}(\vec{x})}^{\begin{cases} 1 & \text{if } Q_1(\vec{x}) \\ 0 & \text{otherwise}\end{cases}}}_{\begin{cases} f_1(\vec{x}) & \text{if } Q_1(\vec{x}) \\ 0 & \text{otherwise}\end{cases}} + f_2(\vec{x}) \cdot \chi_{Q_2}(\vec{x}) + \dots + f_n(\vec{x}) \cdot \chi_{Q_n}(\vec{x})$$

proved computable

computable total

by composition. $\square$

<u>Example</u> : $n = 2$ $\quad f_1(x) = x \quad \forall x \qquad Q_1(x) = $ "x even"

$f_2(x) \uparrow \quad \forall x \qquad Q_2(x) = $ "x is odd"

and then
$$f(x) = \begin{cases} f_1(x) = x & \text{if } Q_1(x) \text{ i.e. } x \text{ even} \\ f_2(x) \uparrow & \text{if } Q_2(x), \text{ i.e. } x \text{ odd.} \end{cases}$$

$$\cancel{=} \quad f_1(x) \cdot \chi_{Q_1}(x) + \underbrace{f_2(x)}_{\uparrow \forall x} \cdot \chi_{Q_2}(x) \quad \uparrow \; \forall x$$

Algebra of decidability

Let $Q_1(\vec{x}), Q_2(\vec{x}) \subseteq \mathbb{N}^k$ decidable. Then

1) $\neg Q_1(\vec{x})$
2) $Q_1(\vec{x}) \wedge Q_2(\vec{x})$ are decidable
3) $Q_1(\vec{x}) \vee Q_2(\vec{x})$

proof

1) $$\chi_{\neg Q_1}(x) = \begin{cases} 1 & \text{if } \neg Q_1(x) \text{ is true} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \chi_{Q_1}(x) = 0 \\ 0 & \text{otherwise} \end{cases}$$

$Q_1(\vec{x})$ is false $\Updownarrow$ $\neg Q_1(x)$ is true

$$= \overline{sg}(\chi_{Q_1}(\vec{x}))$$

computable by composition

2) $$\chi_{Q_1 \wedge Q_2}(\vec{x}) = \chi_{Q_1}(\vec{x}) \cdot \chi_{Q_2}(\vec{x})$$

3) $$\chi_{Q_1 \vee Q_2}(\vec{x}) = sg(\chi_{Q_1}(\vec{x}) + \chi_{Q_2}(\vec{x}))$$

* Bounded Sum / Product

Given $f(\vec{x}, z)$ $\quad f : \mathbb{N}^{k+1} \to \mathbb{N}$ total computable

define $h : \mathbb{N}^{k+1} \to \mathbb{N}$ $\quad [0, y)$

$$h(\vec{x}, y) = f(\vec{x}, 0) + f(\vec{x}, 1) + \dots + f(\vec{x}, y-1) \quad \text{total computable}$$
$$= \sum_{z<y} f(\vec{x}, z)$$

$$\begin{cases} h(\vec{x}, 0) = 0 \\ h(\vec{x}, y+1) = h(\vec{x}, y) + f(\vec{x}, y) \end{cases}$$

primitive recursion
$\hookrightarrow$ h computable

* $\prod_{z<y} f(\vec{x},z)$ $\qquad$ $\prod_{z<0} f(\vec{x},z) = 1$

$$\prod_{z<y+1} f(\vec{x},y+1) = \left(\prod_{z<y} f(\vec{x},z)\right) * f(\vec{x},y)$$

* Bounded Quantification

Given $Q(\vec{x},z) \subseteq \mathbb{N}^{k+1}$ decidable. Then

① $P_1(\vec{x},y) \equiv \forall z<y.\ Q(\vec{x},z)$

② $P_2(\vec{x},y) \equiv \exists z<y.\ Q(\vec{x},z)$

decidable $\qquad$ EXERCISE

* BOUNDED MINIMALISATION

Given $f: \mathbb{N}^{k+1} \to \mathbb{N}$ total computable

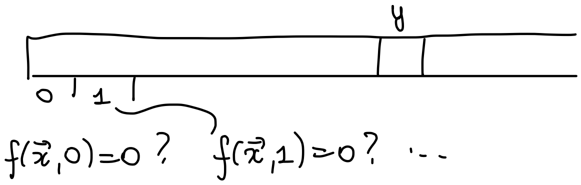
define $h: \mathbb{N}^{k+1} \to \mathbb{N}$

$h(\vec{x},y)$ = " search for minimum $z<y$ s.t. $f(\vec{x},z)=0$ "

$$= \begin{cases} z & \text{minimum } z<y \text{ s.t. } f(\vec{x},z)=0 \\ y & \text{if no such } z \text{ exists} \end{cases} \qquad = \mu z<y.\ f(\vec{x},z) \cancel{=0}$$

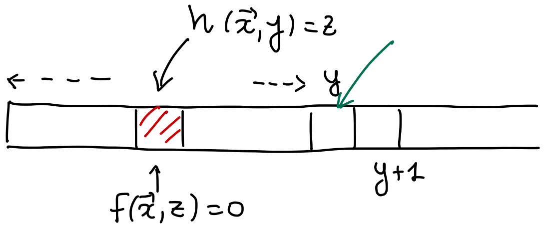
Then h is computable (and total)

proof



primitive recursion

$h(\vec{x},0) = 0$



$$h(\vec{x},y+1) = \begin{cases} h(\vec{x},y)<y & \leadsto\ h(\vec{x},y+1) = h(\vec{x},y) \\ h(\vec{x},y)=y & \leadsto\ \begin{cases} f(\vec{x},y)=0 & \leadsto\ h(\vec{x},y+1) = \underset{\substack{\| \\ h(\vec{x},y)}}{y} \\ f(\vec{x},y)\neq 0 & \leadsto\ h(\vec{x},y+1) = \underset{\substack{\| \\ h(\vec{x},y)+1}}{y+1} \end{cases} \end{cases}$$

$$= h(\vec{x},y) + \underbrace{\overline{sg}(y \dot{-} h(\vec{x},y)) \cdot sg(f(\vec{x},y))}$$

$\uparrow$ 1 if $h(\vec{x},y) = y$ and $f(\vec{x},y) \neq 0$

0 otherwise

is computable by primitive recursion. $\square$

<u>OBSERVATION</u> : The following functions are computable :

* $div : \mathbb{N}^2 \to \mathbb{N}$

$$div(x,y) = \begin{cases} 1 & \text{if } x \text{ divides } y \\ 0 & \text{otherwise} \end{cases}$$

$$= \overline{sg}\,(rm(x,y))$$

* $D(x)$ = number of divisors of x

$$= \sum_{y \leq x} div(y,x)$$

$\uparrow$ $y < x+1$

* $$Pr(x) = \begin{cases} 1 & \text{if } x \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

(x is prime iff the only divisors of x are 1 and x (and $x \neq 1$)

iff x has exactly two divisors

$$= \overline{sg}\,(\,|D(x)-2|\,)$$

$\uparrow$ $|x-y| = (x \dot{-} y) + (y \dot{-} x)$

* $p_x = x^{th}$ prime number

$p_0 = 0$ $p_1 = 2$ $p_2 = 3$ $p_3 = 5$ $p_4 = 7$ $p_5 = 11$ ---

by primitive recursion

$p_0 = 0$

p_{y+1} = smallest number $z > p_y$ and z prime

$= \mu z \leq \left(\prod_{i=1}^{y} p_i + 1\right) . \overline{sg}\left((z \dot{-} p_y) \cdot \overline{P_r}(z)\right)$

$\quad (z \dot{-} p_y) > 0$ if $z > p_y$ $\qquad \overline{P_r}(z) > 0$ if z prime

we claim $\quad p_{y+1} \leq \left(\prod_{i=1}^{y} p_i\right) + 1$

let p a prime factor of $\prod_{i=1}^{y} p_i + 1$

then $\quad p \neq p_i \quad \forall i = 1..y \quad$ otherwise $\quad p$ divides $\prod_{i=1}^{y} p_i$

p " $\prod_{i=1}^{y} p_i + 1$

$\Downarrow$

p divides 1

$\Rightarrow p = 1$ not a prime number.

Then $\quad p > p_y \rightsquigarrow p \geq p_{y+1}$

hence

$$\prod_{i=1}^{y} p_i + 1 \geq p \geq p_{y+1}$$

$\star \quad (x)_y$ = exponent of p_y in the prime factorisation of x

$x = 20 = 2^2 \cdot 3^0 \cdot 5^1 \cdot 7^0 \cdot \ldots \qquad (20)_1 = 2$

$= p_1^2 \cdot p_2^0 \cdot p_3^1 \cdot p_4^0 \cdot \ldots \qquad (20)_2 = 0$

$(20)_3 = 1$

$(20)_4 = 0$

$\vdots$

$(x)_y = \underline{\max}\ z$ s.t. $\underline{p_y^z \text{ divides } x}$

$= \max z$ s.t. $\operatorname{div}(p_y^z, x) = 1$

$= \min z$. s.t. $\operatorname{div}(p_y^{z+1}, x) = 0$

$= \mu z \leq x \ . \ \operatorname{div}(p_y^{z+1}, x)$ $\qquad$ computable

EXERCISE: All functions obtained from basic functions by using composition and primitive recursion is total

OBSERVATION: Primitive recursion is more general than you would think

Fibonacci

$$\begin{cases} f(0) = 1 \\ f(1) = 1 \\ f(n+2) = fib(n) + fib(n+1) \end{cases}$$

not primitive recursion (strictly speaking)

$g: \mathbb{N} \to \mathbb{N}^2$

$g(n) = (f(n), f(n+1))$

$D = \mathbb{N}^2$

$\pi: \mathbb{N}^2 \to \mathbb{N}$ bijective "effective"

$\pi^{-1}: \mathbb{N} \to \mathbb{N}^2$ effective

$\pi^{-1}(n) = (\pi_1(n), \pi_2(n))$

$\pi_1, \pi_2: \mathbb{N} \to \mathbb{N}$ computable

$\pi(x, y) = 2^x \cdot (2y+1) \dot{-} 1$ computable

$\pi^{-1}(n) = (\pi_1(n), \pi_2(n))$

$\pi_1(n) = (n+1)_1$

$\pi_2(n) = \left(\frac{n+1}{2^{\pi_1(n)}} - 1\right) / 2$

computable

$n = 2^x (2y+1) \dot{-} 1$

$n+1 = 2^x (2y+1)$

$g: \mathbb{N} \to \mathbb{N}$

$g(n) = \pi(f(n), f(n+1))$

$$\begin{cases} g(0) = \pi(f(0), f(0+1)) = \pi(1,1) = 2^1 \cdot (2 \cdot 1 + 1) - 1 = 5 \\[2ex] g(n+1) = \pi(\underset{\pi_1(g(n))}{\underset{\uparrow}{f(n+1)}}, \underset{\substack{f(n)+f(n+1) \\ = \pi_1(g(n)) + \pi_2(g(n))}}{\underset{\uparrow}{f(n+2)}}) = \pi(\pi_1(g(n)), \pi_1(g(n)) + \pi_2(g(n))) \end{cases}$$

hence g is computable

Thus $f(n) = \pi_1(g(n))$ is computable by composition.