Def $\mu \in \mathcal{M}(\mathbb{R}^n)$, μ IS ABSOLUTELY CONT. W.R.T LEBESGUE

($\mu \ll \mathcal{L}$) if $\forall A \in \mathcal{B}(\mathbb{R}^n)$ with $|A|=0$ it holds $\mu(A)=0$

Prop $\mu \in \mathcal{M}(\mathbb{R}^n)$ $\mu \ll \mathcal{L} \iff \exists f \in L^1_{loc}(\mathbb{R}^n)$ $(f \geq 0)$ such that

$\forall A \in \mathcal{B}(\mathbb{R}^n)$ $\quad \mu(A) = \int_A f(x)\,dx$

f = density of μ (w.r.t. Lebesgue)

For a.e x $\quad f(x) = \lim_{r \to 0^+} \dfrac{\mu(B(x,r))}{\omega_n r^n}$.

(Obs the Radon measures associated to $f \in L^1_{loc}$ $f \geq 0$ are the ABSOLUTELY CONTINUOUS RADON MEASURES)

Def $\mu \in \mathcal{M}(\mathbb{R}^n)$ μ is singular w.r.t. Lebesgue ($\mu \perp \mathcal{L}$) if $\exists A, B \subseteq \mathbb{R}^n$ $A \cap B = \emptyset$ $A \cup B = \mathbb{R}^n$

$\mu(A) = 0$, $|B| = 0$ (A, B Borel).

Ex δ_0 = Dirac delta $\delta_0 \perp \mathcal{L}$ $\mathbb{R}^n = \underbrace{(\mathbb{R}^n \setminus \{0\})}_{A} \cup \underbrace{\{0\}}_{B}$

$\delta_0(A) = 0$ $|B| = 0$.

Prop: $\mu \perp \mathcal{L}$ then for a.e. x $\lim_{r \to 0^+} \frac{\mu(B(x,r))}{\omega_n r^n} = 0$

Theorem (Lebesgue - Radon - Nykodim)

$\mu \in \mathcal{M}(\mathbb{R}^n)$ $\exists!$ $\mu_{ac} \ll \mathcal{L}$ $\mu_{sing} \perp \mathcal{L}$ s. that $\mu = \mu_{ac} + \mu_{sing}$

$\mu(A) = \int_A f(x)\,dx + \mu_{sin}(A)$ f = density of μ = density of μ_{ac}

HAUSDORFF MEASURES (Ch 2 Evans-Gariepy, Appendix Folland)

Introduce Borel measures on $\mathbb{R}^n$ which "can see" set of "low dimension".

$s \geq 0$ $\qquad \omega_s = \dfrac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$ $\qquad \omega_n = |B(0,1)|$

$\omega_0 = 1 \qquad \omega_1 = \dfrac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = 2$

- $\operatorname{diam} E = \operatorname{diam} \bar{E}$
- $\operatorname{diam} E = \operatorname{diam}(\operatorname{co} E)$
- $E \subseteq A_\varepsilon$ open $\operatorname{diam} E + \varepsilon$

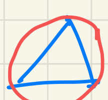
$\delta > 0 \quad E \subseteq \mathbb{R}^n$ subset $\qquad \operatorname{diam} E := \sup_{x,y \in E} |x-y|$

$$H^s_\delta(E) := \frac{\omega_s}{2^s} \inf \Big\{ \sum_{i=1}^{\infty} [\operatorname{diam} E_i]^s \, , \, \underbrace{E \subseteq \bigcup_{i=1}^{\infty} E_i \quad \operatorname{diam} E_i \leq \delta}\Big\}$$

$H^s_\delta : \mathcal{P}(\mathbb{R}^n) \to [0, +\infty]$

DEFINITION DOES NOT CHANGE if we restrict to either E_i open or E_i closed or E_i convex

IT CHANGES if we restrict to balls E_i (Spherical Hausdorff measure)



equilateral triangle $\qquad \operatorname{diam} T = \text{lato} < \operatorname{diam} B$

Fix $s \geq 0$ $\quad \delta_1 > \delta_2$ $\quad$ (diam $E_i \leq \delta_2 < \delta_1$)

$$H^s_{\delta_1}(E) \leq H^s_{\delta_2}(E)$$

$$H^s(E) = \lim_{\delta \to 0^+} H^s_\delta(E) = \sup_{\delta \geq 0} H^s_\delta(E)$$

(s-dimensional Hausdorff measure)

- Observe that $A, B \subseteq \mathbb{R}^n$ $\quad H^s_\delta(A \cup B) = H^s_\delta(A) + H^s_\delta(B)$
  if $\operatorname{dist}(A,B) > 2\delta$

Proposition
(NO PROOF)

1) $H^s_\delta \cdot$ is a OUTER MEASURE on $\mathbb{R}^n$

2) H^s is a OUTER MEASURE on $\mathbb{R}^n$

$$H^s(A \cup B) = H^s(A) + H^s(B) \quad \text{if } \operatorname{dist}(A,B) > 0$$

Caratheodory : H^s is a Borel measure

(also $\forall A \subseteq \mathbb{R}^n$ $\exists E \in \mathcal{B}(\mathbb{R}^n)$ $A \subseteq E$ $\quad H^s(A) = H^s(E)$)

NB H^s is NOT RADON MEASURE (except that for $s = n$ = dim $\mathbb{R}^n$)
(we will prove it)

Proposition

1) $\mathcal{H}^0$ is the COUNTING MEASURE $\quad \mathcal{H}^0(E) = \#\{x \in E\} = \text{card}\, E$

$\mathcal{H}^0_\delta(E) = \inf\{\# E_i \,,\ \cup_i E_i \supseteq E \ \text{diam}\, E_i \leq \delta\}$

2) $\mathcal{H}^1 = \mathcal{L}$ on $\mathbb{R}$ i.e. $\mathcal{H}^1_\delta(A) = \mathcal{H}^1(A) = |A| \quad \forall A \in \mathcal{B}(\mathbb{R})$

Let $A \in \mathcal{B}(\mathbb{R})$ $\quad |A| = \inf\{\sum_i (\text{diam}\, C_i)\,,\ A \subseteq \cup_i C_i \ \ C_i \text{ open}\}$

$\leq \inf\{\sum_i (\text{diam}\, C_i)^1,\ A \subseteq \cup_i C_i \ \ C_i \text{ open} \ \text{diam}\, C_i \leq \delta\}$

$= \mathcal{H}^1_\delta(A)$ $\qquad |A| \leq \mathcal{H}^1_\delta(A)$

$I_k = [k\delta, (k+1)\delta] \qquad \text{diam}\,(C_i \cap I_k) \leq \delta \qquad \forall k \in \mathbb{Z}$

$\text{diam}\, C_i \geq \sum_k (\text{diam}\, C_i \cap I_k)$

$|A| = \inf\{\sum_i (\text{diam}\, C_i),\ A \subseteq \cup_i C_i\} \geq \inf\{\sum_i \sum_k (\text{diam}\, (C_i \cap I_k)) \,,\ A \subseteq \cup_i C_i\}$

$\geq \inf \{ \sum_i (\operatorname{diam} E_i) \quad A \subseteq \cup_i E_i \quad \operatorname{diam} E_i \leq \delta \}$

$= \mathcal{H}^1_\delta (A)$

$|A| \geq \mathcal{H}^1_\delta (A) \quad \Rightarrow \quad \mathcal{L} = \mathcal{H}^1 = \mathcal{H}^1_\delta$ in $\mathbb{R}$.

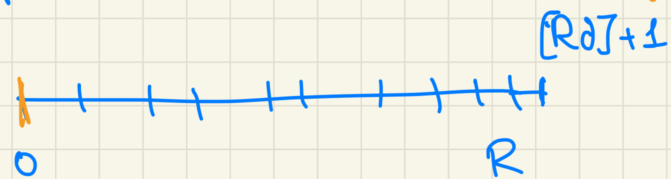
n. on

on $\quad \sum_k \operatorname{diam}(C_i \cap I_k) \leq \operatorname{diam} C_i$

if $\operatorname{diam} C_i = +\infty$ it is finished

if $\operatorname{diam} C_i = R < +\infty \Rightarrow$ up to translations $C_i \subseteq [0,R]$

$$C_i = \bigcup_{k=0}^{[R\delta]+1} (C_i \cap I_k)$$



$$R \geq \sum_{k=0}^{[R\delta]+1} \operatorname{diam}(C_i \cap I_k)$$