ESERCIZI SU RIDUZIONE DI GAUSS E PRODOTTO DI MATRICI

ESERCIZIO 1

$$A_1 = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \xrightarrow{II-3I} \begin{pmatrix} 1 & 3 \\ 0 & -7 \end{pmatrix} \qquad rk(A_1) = 2$$

$$A_5 = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 6 \\ 1 & 6 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 3 & 0 & 6 \end{pmatrix} \xrightarrow{III-3I} \begin{pmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & -18 & -18 \end{pmatrix}$$

$$\xrightarrow{III+18II} \begin{pmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad rk(A_5) = 2$$

$$A_8 = \begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 6 \\ 1 & 6 & 8 \\ 2 & 2 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{III-2I} \begin{pmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & -10 & -14 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{III+10II} \begin{pmatrix} 1 & 6 & 8 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{pmatrix} \qquad rk(A_8) = 3$$

$$A_{11} = \begin{pmatrix} 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 8 \\ 0 & 4 & 8 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 3 & 8 \\ 0 & 2 & 2 & 1 \\ 0 & 4 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[III-4I]{II-2I} \begin{pmatrix} 0 & 1 & 3 & 8 \\ 0 & 0 & -4 & -15 \\ 0 & 0 & -4 & -31 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{III-II} \begin{pmatrix} 0 & 1 & 3 & 8 \\ 0 & 0 & -4 & -15 \\ 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad rk(A_{11}) = 3$$

ESERCIZIO 2

1) $A_{2\times 2}$; $B_{3\times 2}$ $\Rightarrow$ $B\cdot A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 - 1\cdot 2 & 1\cdot 2 + (-1)(-1) \\ 0\cdot 1 + 1\cdot 2 & 0\cdot 2 + 1(-1) \\ 2\cdot 1 + 0\cdot 2 & 2\, 2 + 0(-1) \end{pmatrix}$

$= \begin{pmatrix} -1 & 3 \\ 2 & -1 \\ 2 & 4 \end{pmatrix}$

2) $A_{2\times 3}$; $B_{2\times 2}$ $\Rightarrow$ $B\cdot A = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 3\cdot 3 + 3\cdot 2 & 0 & 3\cdot 1 + 3\cdot 5 \\ 3\cdot 2 + 2\cdot 2 & 0 & 2\cdot 1 + 2\cdot 5 \end{pmatrix}$

$= \begin{pmatrix} 15 & 0 & 18 \\ 10 & 0 & 12 \end{pmatrix}$

3) $A_{3\times 3}$, $B_{3\times 3}$ $\Rightarrow$ $A\cdot B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$B\cdot A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 4 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$

FOGLIO 5

$$\underset{m\times s}{[A]} \cdot \underset{s\times n}{[B]} = \underset{m\times s}{[C]} \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

ESERCIZIO : matrici

NB: $\mathcal{M}_{m,\underset{\downarrow}{s}}(\mathbb{R}) \cdot \mathcal{M}_{\underset{\downarrow}{s},n}(\mathbb{R}) = \mathcal{M}_{\underset{\downarrow}{m},\underset{\downarrow}{n}} = \begin{pmatrix} 1 & 2 & 3 & \cdot \overset{i}{\vdots} \cdot & n \text{ colonne} \\ 2 & & & & \\ \vdots & & & & \\ m \text{ righe} & & & & \end{pmatrix}$

colonne righe righe colonne

- il prodotto tra matrici non è commutativo $A \cdot B \neq B \cdot A$
- vale la prop. associativa : $\underset{n\times s}{A}(\underset{s\times m}{B} \cdot \underset{m\times e}{C}) = (A \cdot B)C$
- " " distributiva : $(A+B)C = A \cdot C + B \cdot C$

a) $A \in \mathcal{M}_{2,2}(\mathbb{R}) \quad B \in \mathcal{M}_{2,3}(\mathbb{R}) \quad \Rightarrow \quad A \cdot B \in \mathcal{M}_{2,3}(\mathbb{R})$

$$A \cdot B = \begin{pmatrix} \overset{a_{11}}{-1} & \overset{a_{12}}{0} \\ \overset{a_{21}}{2} & \overset{a_{22}}{1} \end{pmatrix} \begin{pmatrix} \overset{b_{11}}{2} & \overset{b_{12}}{1} & \overset{b_{13}}{-2} \\ \overset{b_{21}}{-1} & \overset{b_{22}}{2} & \overset{b_{23}}{2} \end{pmatrix}$$

$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21}$ $\quad c_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22}$ $\quad c_{13}$

$$= \begin{pmatrix} \boxed{(-1)(2) + 0(-1)} & \boxed{(-1)(1) + 0 \cdot 2} & (-1)(-2) + 0 \cdot 2 \\ 2 \cdot 2 + 1(-1) & 2 \cdot 1 + 1 \cdot 2 & \boxed{2 \cdot (-2) + 1 \cdot 2} \end{pmatrix} = \begin{pmatrix} -2 & -1 & 2 \\ 3 & 4 & -2 \end{pmatrix}$$

$c_{23} = a_{21} \cdot b_{13} + a_{22} \cdot b_{23}$

b) $A \in \mathcal{M}_{4,2}(\mathbb{R}) \quad B \in \mathcal{M}_{3,4}(\mathbb{R}) \quad \Rightarrow \quad B \cdot A \in \mathcal{M}_{3,2}(\mathbb{R})$

$$B \cdot A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 4 & 7 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-4+1 & 4-7 \\ -2 & -8+2 \\ 4 & 7 \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ -2 & -6 \\ 4 & 7 \end{pmatrix}$$

ESERCIZIO **1** : sistemi lineari + metodo di Gauss

a)
$$\begin{cases} x - y + z = 6 \\ 2x + y - z = -3 \\ x - y - z = 0 \end{cases} \xrightarrow{\substack{III - I \\ II - 2I}} \begin{cases} x - y + z = 6 \\ 3y - 3z = -15 \\ -2z = -6 \end{cases}$$

$$\rightarrow \begin{cases} x - y + z = 6 \\ y - z = -5 \\ z = 3 \end{cases} \qquad \begin{cases} x = 6 + y - z = 1 \\ y = -5 + z = -2 \\ z = 3 \end{cases}$$

$$A \cdot x = b \Rightarrow x = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$$

per Gauss i due sistemi sono equivalenti (hanno lo stesso insieme di soluzioni)

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix} \xrightarrow{\text{f. a scala}} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 3 \end{pmatrix}$$

n° pivot = 3 ⇒ rango = 3 ⇒ soluzione unica $S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\}$

n° parametri = $\underset{\text{n° colonne di A}}{\text{n° variabili}}$ − $\underset{\text{rk}}{\text{rango}}$ = $\underset{x,y,z}{3}$ − 3 = 0

rkA = rk A|b = rk massimo

b)
$$\begin{cases} x - y + 4z = 0 \\ x - 2y = 1 \\ 2x - 5y - 4z = 2 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 1 & -2 & 0 & 1 \\ 2 & -5 & -4 & 2 \end{array}\right) \xrightarrow{\substack{II - I \\ III - 2I}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & -1 & -4 & 1 \\ 0 & -3 & -12 & 2 \end{array}\right) \xrightarrow{III - 3II} \left(\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & -1 & -4 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

rk A = 2 ≠ rk(A|b) = 3 ⇒ non ho soluzioni $S = \emptyset$

c) $\begin{cases} x - 2y - z = 0 \\ -x + 2y + 3z = 2 \\ 2x - 4y - 4z = -2 \end{cases}$

$$\underset{A}{\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & 3 \\ 2 & -4 & -4 \end{pmatrix}} \cdot \underset{x}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{b}{\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}}$$

riduco in forma a scala: [il pivot della riga i+1 sta a dx del pivot della riga precedente]

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ -1 & 2 & 3 & 2 \\ 2 & -4 & -4 & -2 \end{array}\right) \xrightarrow[\frac{1}{2}III - I]{I + II} \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

pivot = 1ª elemento non nullo da sx

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$rk\,A = rk\,A|b = 2$ (non massimo $rk_{MAX} = 3$)

$\Rightarrow$ indeterminata

avrò $\underset{\text{inc.}}{3} - \underset{\text{eq}}{2} = 1$ incognite indeterminate che posso scrivere come parametro $\Rightarrow$ impongo $x = t$

$\begin{cases} x - 2y - z = 0 \\ z = 1 \end{cases}$ $\begin{cases} x = t \\ y = t/2 - \frac{1}{2} \\ z = 1 \end{cases}$ $\begin{cases} x = 2u + 1 \\ y = u \quad \leftarrow \text{parametro} \\ z = 1 \end{cases}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \qquad S_{A|b} = \left\{ \begin{pmatrix} x \\ -\frac{1}{2} + \frac{1}{2}x \\ 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + u \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$\hookrightarrow$ generico vettore che è soluzione del sistema lineare c)

$u = \frac{t}{2} \rightarrow 2u = t$

$A \cdot x = b$ m equaz. n incognite $A \in \mathcal{M}_{m,n}(\mathbb{R})$ $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n \times 1}$

$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathcal{M}_{m \times 1}$

Soluz. →
- nessuna: $S_{A|b} = \emptyset$ $rk A \neq rk A|b$
- unica soluz.: $S_{A|b} = \{(x, y, z)\}$ $rk A = rk A|b = rk$ massimo
- infinite: con i parametri $rk A = rk A|b$ non massimo

$S = \{( ) \mid t \in \mathbb{R}\}$

$\Rightarrow$ avrò $n°$ parametri $= \underset{n° \text{ colonne}}{n° \text{ variabili}} - \underset{n° \text{ pivot}}{rk A}$

esempio : sistemi lineari indeterminati dove devo trovare più parametri

$$\begin{cases} 2x - 3y + z - t = 1 \\ 2y - 2z + t = 2 \\ 4x - 2y - 2z = 6 \\ 4y - 4z + 2t = 4 \end{cases}$$

$$\left(\begin{array}{cccc|c} 2 & -3 & 1 & -1 & 1 \\ 0 & 2 & -2 & 1 & 2 \\ 4 & -2 & -2 & 0 & 6 \\ 0 & 4 & -4 & 2 & 4 \end{array}\right) \xrightarrow[\mathrm{IV} - 2\mathrm{II}]{\mathrm{III} - 2\mathrm{I} - 2\mathrm{II}} \left(\begin{array}{cccc|c} 2 & -3 & 1 & -1 & 1 \\ 0 & 2 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

ho $\underset{\text{colonne}}{\overset{\text{inco}}{4}} - \underset{\text{pivot} = rk}{\overset{\text{eq}}{2}} = 2$ variabili indeterminate $\Rightarrow$ 2 parametri

uso z, t come parametri

$$\begin{cases} z = a \\ t = b \\ y = 1 + a - \frac{b}{2} \\ x = 2 + a - \frac{b}{4} \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1/4 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}$$

$$S_{A|b} = \left\{ \begin{pmatrix} 2 + a - \frac{1}{4}b \\ 1 + a - \frac{1}{2}b \\ a \\ b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

ESERCIZIO 2

$$\begin{cases} 2x + y = 1 \\ (k^2-4)y + z = k+2 \\ 2x + y - z = 1 \end{cases}$$

a) $k = 2$

$$\begin{cases} 2x + y = 1 \\ z = 4 \\ 2x + y - z = 1 \end{cases} \quad \left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 2 & 1 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right)$$

$rk A \neq rk(A|b)$

$S = \emptyset$

b) $k = -2$

$$\begin{cases} 2x + y = 1 \\ z = 0 \\ 2x + y - z = 1 \end{cases} \quad \left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

abbiamo $3 - 2 = 1$ variabili indeterminate uso $x = t$

$$\begin{cases} x = t \\ y = 1 - 2t \\ z = 0 \end{cases} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

c) $k = 0$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & -4 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{array}\right) \longrightarrow \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & -4 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{cases} x = \left[1 - \left(-\frac{1}{2}\right)\right] / 2 = \frac{3}{4} \\ y = -\frac{1}{4}(-0+2) = -\frac{1}{2} \\ z = 0 \end{cases} \quad S = \left\{ \begin{pmatrix} \frac{3}{4} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \right\}$$

considero $k \neq \pm 2$ → ho già indagato quest...

d)
$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & k^2-4 & 1 & k+2 \\ 2 & 1 & -1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 1 \\ 0 & k^2-4 & 1 & k+2 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\begin{cases} 2x + y = 1 \\ (k^2-4)y + z = k+2 \\ -z = 0 \end{cases} \rightarrow \begin{cases} x = \left(1 - \dfrac{1}{k-2}\right)/2 = \dfrac{k-3}{2(k-2)} \\ y = \dfrac{k+2}{(k-2)(k+2)} = \dfrac{1}{k-2} \\ z = 0 \end{cases} \quad k \neq \mp 2$$

$$S = \left\{ \begin{pmatrix} (k-3)/2(k-2) \\ 1/(k-2) \\ 0 \end{pmatrix} \right\}$$

ESERCIZIO 4

$$\begin{cases} 2x + 4y + kz = -2 \\ x + (k+1)y + z = 1 \\ x + 2ky + 2z = 3 \end{cases}$$

a) $k \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} 2 & 4 & k & -2 \\ 1 & k+1 & 1 & 1 \\ 1 & 2k & 2 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 4 & k & -2 \\ 0 & 2k-2 & 2-k & 4 \\ 0 & 4k-4 & 4-k & 8 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & 4 & k & -2 \\ 0 & 2(k-1) & 2-k & 4 \\ 0 & 0 & k & 0 \end{array}\right)$$

a) per $k = 1$ $\rightarrow$ $\left(\begin{array}{ccc|c} 2 & 4 & 1 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 2 & 4 & 1 & -2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 \end{array}\right)$

$S_{A|b} = \emptyset \qquad rkA \neq rk\,A|b \qquad 0 \neq 4$

b) per $k = 0$ $\left(\begin{array}{ccc|c} 2 & 4 & 0 & -2 \\ 0 & -2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{cases} x + 2y = -1 \\ -y + z = 2 \end{cases}$

$\begin{cases} x = -1 - 2y \\ y = z - 2 \end{cases} \qquad \begin{cases} x = -1 - 2(z-2) = -1 - 2z + 4 = -2z + 3 \\ y = z - 2 \end{cases}$

$S_{A|b} = \left\{ \begin{pmatrix} -2z+3 \\ z-2 \\ z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

c) $k = -1$ $\left(\begin{array}{ccc|c} 2 & 4 & 1 & -2 \\ 0 & -4 & 3 & 4 \\ 0 & 0 & -1 & 0 \end{array}\right) \qquad \begin{cases} 2x + 4y + z = -2 \\ -4y + 3z = 4 \\ -z = 0 \end{cases} \qquad \begin{cases} x = 1 \\ y = -1 \\ z = 0 \end{cases}$

$S = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

d) ...

e) $\begin{cases} 2x + 4y + kz = -2 \\ 2(k-1)y + (2-k)z = 4 \\ kz = 0 \end{cases} \xrightarrow{k \neq 1, 0} \begin{cases} x = \left(-2 - 4\dfrac{2}{k-1}\right)/2 = \dfrac{-k-3}{k-1} = \dfrac{k+3}{1-k} \\ y = \dfrac{2}{k-1} \\ z = 0 \end{cases}$

ES 1

$$\begin{cases} x-y+z=6 \\ 2x+y-z=-3 \\ x-y-z=0 \end{cases} \longrightarrow A\cdot x = b \longrightarrow \overbrace{\begin{array}{c|c} \begin{matrix} I \\ II \\ III \end{matrix} & \left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 2 & 1 & -1 & -3 \\ 1 & -1 & -1 & 0 \end{array}\right) \end{array}}^{(A|b)}$$

A
A|b

1) rk A = rk A|b = rk è max 1 sol.

2) $\begin{cases} \text{rk A} = \text{rk A|b} \to \text{no max} \\ n - rk = t \end{cases}$ ∞ soluzioni

3) rk A ≠ rk A|b 0 soluzioni

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 3 & -3 & -15 \\ 0 & 0 & -2 & -6 \end{array}\right)$$

$x - y + z = 6 \rightarrow x = y - z + 6 = -2-3+6 = 1$

$3y - 3z = -15 \rightarrow y = -\frac{15}{3} + \frac{3}{3}z = -\frac{15}{3} + \frac{\cancel{3}}{\cancel{3}}\cdot(+3) = -2$

$-2z = -6 \rightarrow z = +3$

$S = \left\{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right\}$ $AA^{-1} = I$ $A_{n\times n}$ rk = n ⇒ inv.

$A_{3\times 3}$ rk = 3 ⇒ inv.

$$\underbrace{\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right)}_{(A \mid I)} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & -3 & -2 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & +1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right) \longrightarrow \underbrace{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & +\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right)}_{(I \mid A^{-1})}$$

$$Ax = b \rightarrow x = A^{-1}\cdot b$$

$$= \begin{pmatrix} +\frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 6 \\ -3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} +\frac{1}{3}\cdot 6 + \frac{1}{3}(-3) + 0\cdot 0 \\ -\frac{1}{6}\cdot 6 + \frac{1}{3}(-3) + \left(-\frac{1}{2}\right)\cdot 0 \\ \frac{1}{2}\cdot 6 + 0(-3) + \left(-\frac{1}{2}\right) 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

ES 8 → Foglio 6 - MATRICI

$$A_K = \begin{pmatrix} 1 & 0 & 1 \\ K & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

a) $$\begin{pmatrix} 1 & 0 & 1 \\ K & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ K & 1 & 2 \end{pmatrix} \xrightarrow{III-KI} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2-K \end{pmatrix} \dashrightarrow$$

$$\xrightarrow{2III-II} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3-2K \end{pmatrix}$$

per $K = \frac{3}{2}$ $\operatorname{rk} A_K = 2 \neq 3$

$\Longrightarrow$ la matrice non è invertibile

b) $$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$ $(A_1 \mid \mathbb{I}) \longrightarrow (\mathbb{I} \mid A_1^{-1})$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{II-I} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \dashrightarrow$$

$$\xrightarrow{III-2II} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & +2 & -2 & +1 \end{array}\right) \xrightarrow[III\cdot(-1)]{\substack{II+III \\ I+III}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right)$$

$$\Longrightarrow A_1^{-1} = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

ES 6

$$A_k = \begin{pmatrix} 2 & 0 & 5 \\ -4k & 4k-1 & k-2 \\ 0 & 1-4k & 0 \end{pmatrix}$$

per quali k A_k é invertibile ? $\rightarrow$ Riduco in forma a scala e verifico per quali k $rk A_k = rk$ massimo $= 3$

$$\xrightarrow{II \to III} \begin{pmatrix} 2 & 0 & 5 \\ 0 & 1-4k & 0 \\ -4k & 4k-1 & k-2 \end{pmatrix} \xrightarrow{III + 2kI} \begin{pmatrix} 2 & 0 & 5 \\ 0 & 1-4k & 0 \\ 0 & 4k-1 & -2+11k \end{pmatrix}$$

$$\xrightarrow{III + II} \begin{pmatrix} 2 & 0 & 5 \\ 0 & 1-4k & 0 \\ 0 & 0 & -2+11k \end{pmatrix}$$

A_k é invertibile $\Leftrightarrow \begin{cases} 1-4k \neq 0 \\ 11k-2 \neq 0 \end{cases} \Rightarrow$ per $k \neq \frac{1}{4}, \frac{2}{11}$

per $k = \frac{1}{4}$ $A_{k=\frac{1}{4}} = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 3/4 \end{pmatrix} \Rightarrow rk A_{\frac{1}{4}} = 2$

per $k = \frac{2}{11}$ $A_{k=\frac{2}{11}} = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 3/11 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow rk A_{\frac{2}{11}} = 2$

ES 1 e) FOGLIO 5

$$\begin{cases} 2x + y - 2t = 0 \\ 3x + z = 1 \\ y - 2z + t = 2 \\ 2x - 7y + 4z + 4t = -1 \end{cases}$$

$$\left(\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 2 & -7 & 4 & 4 & -1 \end{array}\right)$$

$$\xrightarrow[IV - I]{II - \frac{3}{2} I} \left(\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 0 & -3/2 & 1 & 3 & 1 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & -8 & 4 & 6 & -1 \end{array}\right) \xrightarrow[IV - \frac{16}{3} II]{III + \frac{2}{3} II} \left(\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 0 & -3/2 & 1 & 3 & 1 \\ 0 & 0 & -4/3 & 3 & 8/3 \\ 0 & 0 & -4/3 & -10 & -19/3 \end{array}\right)$$

$$\xrightarrow{IV - III} \left(\begin{array}{cccc|c} 2 & 1 & 0 & -2 & 0 \\ 0 & -3/2 & 1 & 3 & 1 \\ 0 & 0 & -4/3 & 3 & 8/3 \\ 0 & 0 & 0 & -13 & -9 \end{array}\right)$$

$rkA = rkA|b = 4 \leftarrow$ è massimo

$\Rightarrow \exists$ 1 soluzione

$$\begin{cases} 13t = 9 \\ -\frac{4}{3}z + 3t = \frac{8}{3} \\ -\frac{3}{2}y + z + 3t = 1 \\ 2x + y - 2t = 0 \end{cases} \qquad \begin{cases} t = \frac{9}{13} \\ z = -\frac{3}{4}\left(\frac{8}{3} - 3 \cdot \frac{9}{13}\right) = -\frac{23}{52} \\ y = -\frac{2}{3}\left(1 - 3 \cdot \frac{9}{13} - \left(-\frac{23}{52}\right)\right) = \frac{11}{26} \\ x = \frac{1}{2}\left(2 \cdot \frac{9}{13} - \frac{11}{26}\right) = \frac{25}{52} \end{cases}$$

$$S_{A|b} = \left\{ \begin{pmatrix} 25/52 \\ 11/26 \\ -23/52 \\ 9/13 \end{pmatrix} \right\}$$