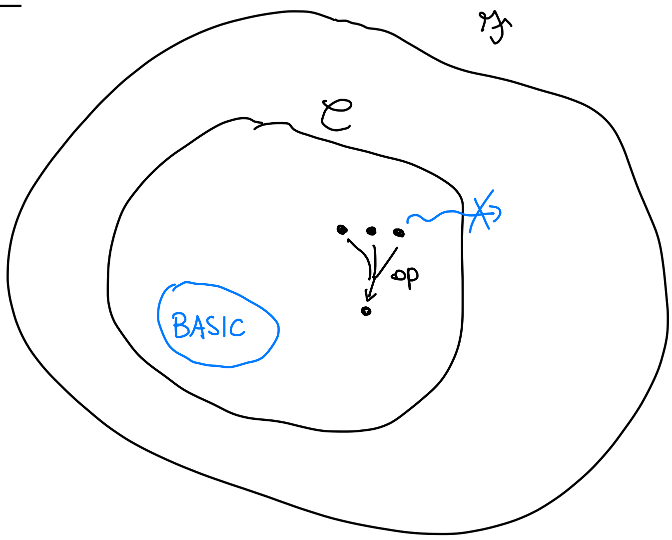
COMPUTABILITY (14/10/2025)

* Generation of computable functions

$\mathcal{C}$ is closed under

- composition
- primitive recursive
- unbounded minimalisation



* BASIC FUNCTIONS

$\vec{x} = (x_1, \dots, x_k)$

1. constant zero $\quad z : \mathbb{N}^k \to \mathbb{N} \quad z(\vec{x}) = 0 \quad \forall \vec{x} \in \mathbb{N}^k$
2. successor $\quad s : \mathbb{N} \to \mathbb{N} \quad s(x) = x+1 \quad \forall x \in \mathbb{N}$
3. projection $\quad U_j^k : \mathbb{N}^k \to \mathbb{N} \quad U_j^k(\vec{x}) = x_j \quad \forall \vec{x} \in \mathbb{N}^k$

They are in $\mathcal{C}$ as they are computed by

1. $Z(1)$
2. $S(1)$
3. $T(j,1)$

* Notation

given a program P

- $\rho(P) = \max\{n \mid \text{register } R_n \text{ is referred in } P\}$
- $\ell(P)$ = number of instructions in P
- P is in <u>standard form</u> if whenever it terminates, it does at instruction index $\ell(P)+1$
- <u>concatenation</u> : given P, Q programs

$$\begin{matrix} P \\ Q \end{matrix} \quad \rightsquigarrow \quad \begin{matrix} P \\ Q' \end{matrix} \longleftarrow J(m,n,t) \rightsquigarrow J(m,n,t+\ell(P))$$

- given a program P

$$P[i_1, i_2, \dots, i_k \to i] \qquad (*)$$

program taking the input from $R_{i_1}, \dots, R_{i_k}$ and outputs in R_i without assuming that register different from input are $\emptyset$.

$$\left.\begin{array}{l} T(i_1, 1) \\ \vdots \\ T(i_k, k) \\ Z(k+1) \\ \vdots \\ Z(\rho(P)) \\ P \\ T(1, i) \end{array}\right\} (*)$$

problem $\quad P[2,1 \to 1]$

$\boxed{x \mid y \mid \cdots}$ $\overset{\text{we want}}{\rightsquigarrow}$ $\boxed{y \mid x \mid \cdots}$

$T(2,1)$
$T(1,2)$
P

$\overset{\text{we get}}{\rightsquigarrow}$ $\boxed{y \mid y \mid}$

<u>EXERCISE</u> : Solve the problem !

* <u>COMPOSITION</u>

Given $\quad f : \mathbb{N}^k \to \mathbb{N}, \quad g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$

you define $\quad h : \mathbb{N}^m \to \mathbb{N}$

$$h(\vec{x}) = \begin{cases} f(g_1(\vec{x}), \dots, g_k(\vec{x})) & \text{if } g_1(\vec{x})\downarrow, \dots, g_k(\vec{x})\downarrow \\ & \quad f(g_1(\vec{x}), \dots, g_k(\vec{x}))\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

E.g. $\quad Z(x) = 0 \quad \forall x \qquad\qquad Z(\emptyset(x))\uparrow \quad \forall x$

$\quad \emptyset(x)\uparrow \quad \forall x$

$\quad U_1^2(x,y) = x \qquad\qquad U_1^2(x, \emptyset(y))\uparrow \quad \forall x, y$

Proposition: $\mathscr{C}$ is closed under composition

proof

Given $f : \mathbb{N}^k \to \mathbb{N}$, $g_1, \dots, g_k : \mathbb{N}^n \to \mathbb{N}$ computable functions in $\mathscr{C}$

then

$$h : \mathbb{N}^n \to \mathbb{N}$$

$$h(\vec{x}) = f(g_1(\vec{x}), \dots, g_k(\vec{x})) \quad \text{is in } \mathscr{C}$$

Let $F, G_1, \dots, G_k$ be programs (std form) for $f, g_1, \dots, g_k$

1	....	n		m	m+1		m+n	m+n+1		m+n+k
x_1	...	x_n			x_1	---	x_n	$g_1(\vec{x})$	...	$g_k(\vec{x})$

$$m = \max\{\rho(F), \rho(G_1), \dots, \rho(G_k), k, n\}$$

$T(1, m+1)$

$\vdots$

$T(n, m+n)$

$G_1[m+1 \dots m+n \to m+n+1]$

$\vdots$

$G_k[m+1 \dots m+n \to m+n+k]$

$F[m+n+1 \dots m+n+k \to 1]$

is the program for h, hence $h \in \mathscr{C}$ $\square$

Example: $f(x_1, x_2) = x_1 + x_2$ is known to be in $\mathscr{C}$

$$g(x_1, x_2, x_3) = x_1 + x_2 + x_3$$

$$g(x_1, x_2, x_3) = f(\underbrace{\underbrace{f(\underbrace{U_1^3(\vec{x})}_{\mathbb{N}^3 \to \mathbb{N}}, \underbrace{U_2^3(\vec{x})}_{\mathbb{N}^3 \to \mathbb{N}})}_{\mathbb{N}^3 \to \mathbb{N}}, \underbrace{U_3^3(\vec{x})}_{\mathbb{N}^3 \to \mathbb{N}}}_{\mathbb{N}^3 \to \mathbb{N}})$$

$$= f(f(U_1^3(\vec{x}), U_2^3(\vec{x})), U_3^3(\vec{x}))$$

<u>Example</u> : Let f be computable and total

$$Q_f(x,y) = \text{“ } f(x) = y \text{ ”} \qquad \text{decidable ?}$$

We know

$$\chi_{Eq} : \mathbb{N}^2 \to \mathbb{N}$$

$$\chi_{Eq}(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \text{computable}$$

hence Q_f is decidable since

$$\chi_{Q_f}(x,y) = \chi_{Eq}(f(x), y) \qquad \text{computable by composition}$$

* <u>Primitive Recursion</u>

$$\begin{cases} 0! = 1 \\ (n+1)! = \underbrace{n!}_{} * (n+1) \end{cases}$$

known

$$\begin{cases} fib(0) = 1 \\ fib(1) = 1 \\ fib(n+2) = \underbrace{fib(n)}_{} + \underbrace{fib(n+1)}_{} \end{cases}$$

known

<u>Def</u>. Given $f : \mathbb{N}^k \to \mathbb{N}$

$g : \mathbb{N}^{k+2} \to \mathbb{N}$

define $h : \mathbb{N}^{k+1} \to \mathbb{N}$

$$\begin{cases} h(\vec{x}, 0) = f(\vec{x}) \\ h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y)) \end{cases}$$

take $x \in \mathbb{R}$

$$x = \frac{\sqrt{x}}{e^x} \cdot \log x$$

- is there a solution?
- is it unique? canonical?

($(\mathcal{D}, \sqsubseteq)$ complete partial order continuous } → existence of a (least) solution

unique solutions - - - - - - - induction)

Examples:

* $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x + y$

$$\begin{cases} x + 0 = x \\ x + (y+1) = (x+y) + 1 \end{cases}$$

$f(x) = x = U_1^1(x)$

$g(x,y,z) = z+1$

* $h' : \mathbb{N}^2 \to \mathbb{N}$

$h'(x,y) = x * y$

$$\begin{cases} x * 0 = 0 \\ x * (y+1) = (x*y) + x \end{cases}$$

$f(x) = 0$

$g(x,y,z) = z + x$

Proposition: The class $\mathcal{C}$ is closed under primitive recursion

proof

Let $f : \mathbb{N}^k \to \mathbb{N}$, $g : \mathbb{N}^{k+2} \to \mathbb{N}$ be in $\mathcal{C}$

and let F, G programs (std form) for f and g

Define $h : \mathbb{N}^{k+1} \to \mathbb{N}$

$h(\vec{x}, 0) = f(\vec{x})$

$h(\vec{x}, y+1) = g(\vec{x}, y, h(\vec{x}, y))$

Idea

$h(\vec{x}, 0) = f(\vec{x})$ use F

$h(\vec{x}, 1) = g(\vec{x}, 0, \underline{h(\vec{x}, 0)})$ use G

$\vdots$

$h(\vec{x}, i) = g(\vec{x}, i-1, h(\vec{x}, i-1))$ " "

$i = y$? if so $\to$ output

otherwise increment $i = i+1$ and continue

1	…	K	K+1	… m	m+1	…	m+K	m+K+1	m+K+2	m+K+3
x_1	..	x_K	y		x_1	. . .	x_K	i	$h(\vec{x},i)$	y

$$m = \max\{\rho(F), \rho(G), K+2\}$$

$T(1, m+1)$
$\vdots$
$T(K, m+K)$

$T(K+1, m+K+3)$

$F[m+1, \dots, m+K \to m+K+2]$ // $h(\vec{x},0) = f(\vec{x})$

LOOP : $J(m+K+1, m+K+3, \text{RES})$ // $i = y$?

$G[m+1 \dots m+K+2 \to m+K+2]$ // $h(\vec{x}, i+1) = g(\vec{x}, i, h(\vec{x},i))$

$S(m+K+1)$ // i++

$J(1,1, \text{LOOP})$

RES : $T(m+K+2, 1)$

$\square$

<u>Examples</u> :

* $h : \mathbb{N}^2 \to \mathbb{N}$

$h(x,y) = x + y$

$$\begin{cases} x + 0 = x \\ x + (y+1) = (x+y) + 1 \end{cases}$$

$f(x) = x = U_1^1(x)$

$g(x, y, z) = z + 1$

* $h' : \mathbb{N}^2 \to \mathbb{N}$

$h'(x,y) = x * y$

$$\begin{cases} x * 0 = 0 \\ x * (y+1) = (x*y) + x \end{cases}$$

$f(x) = 0$
$g(x, y, z) = z + x$

- exponential

$$\begin{cases} x^0 = 1 \\ x^{y+1} = (x^y) * x \end{cases}$$

- predecessor $\quad y \dot- 1 = \begin{cases} 0 & \text{otherwise} \\ y-1 & y>0 \end{cases}$

$$\begin{cases} 0 \dot- 1 = 0 \\ (y+1) \dot- 1 = y \end{cases}$$

- difference $\quad x \dot- y = \begin{cases} 0 & x \le y \\ x-y & x > y \end{cases}$

$$\begin{cases} x \dot- 0 = x \\ x \dot- (y+1) = (x \dot- y) \dot- 1 \end{cases}$$

- sign $\quad sg(y) = \begin{cases} 1 & \text{if } y>0 \\ 0 & \text{if } y=0 \end{cases}$

$$\begin{cases} sg(0) = 0 \\ sg(y+1) = 1 \end{cases}$$

- $\overline{sg}(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{otherwise} \end{cases}$ exercise

(solution
$\overline{sg}(x) = 1 \dot- sg(x)$)

- $\min(x,y) = \overbrace{x \dot- \overbrace{(x \dot- y)}^{0}}^{x}$ $\quad x \le y$

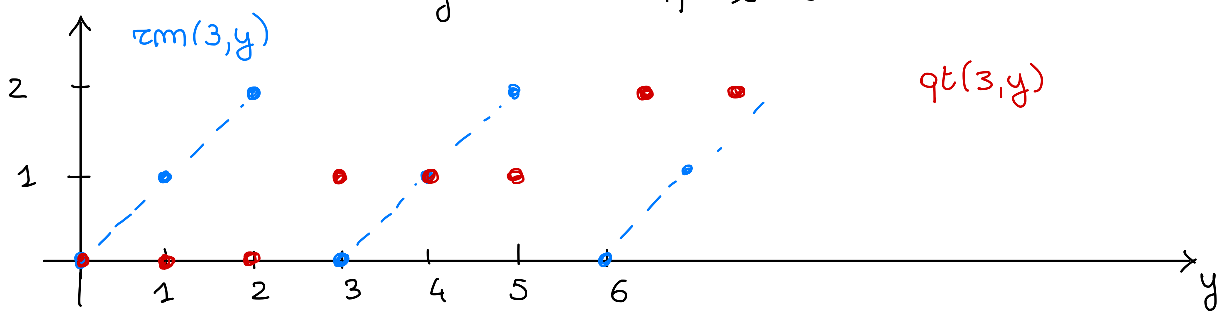
$\underbrace{x \dot- \underbrace{(x \dot- y)}_{x-y}}_{x-(x-y)=y}$ $\quad x > y$

- $\max(x,y)$ exercise

(SOLUTION
$\max(x,y) = x + y \dot- x$)

- $rm(x,y)$ = remainder of x divided by y

$$= \begin{cases} y \bmod x & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}$$



$$\begin{cases} rm(x,0) = 0 \\ rm(x, y+1) = \begin{cases} rm(x,y)+1 & \text{if } rm(x,y)+1 < x \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

$$= (rm(x,y) + 1) * sg(x \dot{-} (rm(x,y)+1))$$

something $\begin{cases} 1 & \text{if } rm(x,y)+1 < x \\ 0 & \text{otherwise} \end{cases}$

* quotient

$qt(x,y)$ (convention $qt(0,y) = 0$)

(exercise) SOLUTION:

↳ $qt(x,0) = 0$

$$qt(x, y+1) = \begin{cases} qt(x,y)+1 & \text{if } rm(x,y+1) = 0 \\ qt(x,y) & \text{otherwise} \end{cases}$$

$$= qt(x,y) + \overline{sg}(rm(x, y+1))$$