Yesterday COMPACTNESS THEOREMS

$f_n \in L^p(U)$ $\|f_n\|_p \le C$ $\Longrightarrow$ $\exists$ $f \in L^p(U)$ f_{n_j} subsequence

$p \in (1, +\infty)$ $f_{n_j} \longrightarrow f$ in $L^p(U)$

(KAKUTANI) that is $\forall g \in L^q(U)$ $\int_U f_n g\, dx \to \int_U fg\, dx$

$(L^p)^{**} \simeq L^p$, L^p separable $\frac{1}{p} + \frac{1}{q} = 1$

$f_n \in L^\infty(U)$ $\|f_n\|_\infty \le C \Longrightarrow \exists$ $f \in L^\infty(U)$ f_{n_j} subsequence

(BANACH-ALAOGLU) $f_{n_j} \overset{*}{\longrightarrow} f$ in $L^\infty(U)$

$L^\infty = (L^1)^*$, L^1 separable that is $\forall g \in L^1(U)$ $\int_U f_n g\, dx \to \int_U fg\, dx$

in $L^1(U)$ NEVER TRUE :

$$\rho(x) = \begin{cases} C\, e^{\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \ge 1 \end{cases} \qquad c\int_{\mathbb{R}^n} e^{\frac{1}{|x|^2-1}}\, dx = 1$$

$$\rho_k(x) = k^n \rho(kx) \qquad k \to +\infty \quad \left| \begin{array}{l} \rho_k \ge 0 \quad \rho_k \in L^1(\mathbb{R}^n) \\ \|\rho_k\|_{L^1} = \|\rho\|_{L^1} = 1 \end{array} \right.$$

ρ_k is not converging weakly (up to any subseq.)!

Take $g \in C_c(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} g(x)\, \rho_k(x)\, dx = \int_{\mathbb{R}^n} g(x)\, k^n \rho(kx)\, dx \overset{y=kx}{=} \underbrace{\int_{\mathbb{R}^n} g\left(\frac{y}{k}\right) \rho(y)\, dy \xrightarrow{k\to+\infty} g(0)}_{\text{Lebesgue dominated conv.}}$$

Q: $\exists f \in L^1(\mathbb{R}^n)$ such that $g(0) = \int_{\mathbb{R}^n} f(x)\, g(x)\, dx \quad \forall g \in C_c(\mathbb{R}^n)$

A: NO: ASSUME IT IS TRUE for $f \in L^1(\mathbb{R}^n)$:

$g_k(x) = \rho(kx) \qquad g_k \geq 0 \quad g_k(0) = \frac{c}{e} = \|g_k\|_\infty$ indep of k.

$\operatorname{supp} g_k \subseteq B(0, \frac{1}{k}) \subseteq B(0,1)$

$$g_k(0) = \frac{c}{e} = \int_{\mathbb{R}^n} f(x)\, g_k(x)\, dx = \int_{B(0,1)} \underbrace{f(x)\,\rho(kx)}_{\downarrow 0} \, dx \longrightarrow 0$$

+ Lebesgue dominated convergence

Compactness does not hold in L^1. We embed $L^1(U)$ in the space of signed Radon measures!

$f \in L^1(U) \longrightarrow I_f : C_0(U) \longrightarrow \mathbb{R}$ is linear and continuous

$\phi \longrightarrow \int_U \phi(x) f(x) dx$

$$|I_f(\phi)| = \left| \int_U \phi(x) f(x) dx \right| \leq \int_U |\phi(x)||f(x)| dx \leq \|\phi\|_\infty \|f\|_{L^1}.$$

$(\|I_f\| = \|f\|_{L^1})$

$f = f^+ - f^-$ $\quad f^+ = \max(f,0)$ $\quad f^- = \max(-f,0)$

$f^+, f^- \in L^1(U)$ $\quad f^+, f^- \geq 0$ $\quad \mu^+(A) = \int_A f^+(x) dx$ is a Radon finite meas.

$\mu^-(A) = \int_A f^-(x) dx$ is a finite (Radon) measure

$$\mu^+(U) = \int_U f^+(x) dx, \quad \mu^-(U) = \int_U f^-(x) dx$$

Therefore $f \in L^1(U) \rightsquigarrow \exists\, \mu^+, \mu^- \in \mathcal{M}(U)$ finite

such that $\forall \phi \in C_0(U)$

$$\int_U f \phi \, dx = \int_U f^+ \phi \, dx - \int_U f^- \phi \, dx = \int_U \phi \, d\mu^+(x) - \int_U \phi \, d\mu^-(x)$$

Compactness If $f_n \in L^1(U)$ $\exists\, C$ $\|f_n\|_{L^1} \leq C$

then $\exists\, f_{n_j}$ subsequence and $\mu^+, \mu^- \in \mathcal{M}(U)$ finite

such that $\forall \phi \in C_0(U)$

$$\int_U f_{n_j} \phi \, dx \longrightarrow \int_U \phi \, d\mu^+(x) - \int \phi \, d\mu^-(x)$$

proof $f_n \rightsquigarrow \mu_n^+, \mu_n^-$ (by previous argument)

$$\int_U f_n \phi \, dx = \int_U \phi \, d\mu_n^+ - \int_U \phi \, d\mu_n^- \qquad \forall \phi \in C_0(U)$$

therefore $\mu_n^+(U) \le C \quad \mu_n^-(U) \le C \Rightarrow$ by compactness

in Radon $\exists\, \mu_{n_j}^+, \mu_{n_j}^-$ subsequence, μ^+, μ^-

$$\mu_{n_j}^+ \rightharpoonup \mu^+ \qquad \mu_{n_j}^- \rightharpoonup \mu^-$$

that is $\int_U \phi \, d\mu_{n_j}^+ \to \int_U \phi \, d\mu^+ \quad \forall \phi \in C_0(U)$

$$\int_U \phi \, d\mu_{n_j}^- \to \int_U \phi \, d\mu^- \quad \forall \phi \in C_0(U) \quad \blacksquare$$

Previous example: $\rho_k(x) = k^n \rho(kx) \qquad \rho_k \in L^1(\mathbb{R}^n)$

$\|\rho_k\|_{L^1} = 1 \Rightarrow \rho_k$ is associated to $\mu_k \in \mathcal{M}(\mathbb{R}^n)$ finite

and $\mu_k \rightharpoonup \delta_0 \quad \left[\forall \phi \in C_c(\mathbb{R}^n) \quad \int_{\mathbb{R}^n} \phi \, \rho_k(x) dx = \int_{\mathbb{R}^n} \phi \, d\mu_k(x) \to \phi(0) = \int_{\mathbb{R}^n} \phi \, \delta_0(x) \right.$

Local version.

$f \in L^1_{loc}(U) \Rightarrow f^+ = \max(f, 0) \geq 0$

$f^- = \max(-f, 0) \geq 0 \qquad f = f^+ - f^-$

$f^+, f^- \in L^1_{loc}(U) \qquad f^+, f^- \geq 0$

$\forall \phi \in C_c(U) \qquad \left.\begin{array}{l} T^+(\phi) = \int_U \phi(x) f^+(x)\,dx \\ T^-(\phi) = \int_U \phi(x) f^-(x)\,dx \end{array}\right\}$ positive linear functionals

$\Downarrow$

continuous!

$\to$ RIESZ $\exists \mu^+, \mu^- \in \mathcal{M}(U) \qquad \int_U \phi(x) f^+(x)\,dx = \int_U \phi(x)\,d\mu^+(x)$

$\int_U \phi(x) f^-(x)\,dx = \int_U \phi(x)\,d\mu^-(x)$

$f \rightsquigarrow \mu^+ - \mu^-$

$\forall K \subset\subset U \quad \mu^+(K) = \int_K f^+(x)\,dx \qquad \mu^-(K) = \int_K f^-(x)\,dx$

$\forall \phi \in C_c(U) \qquad \int_U \phi(x) f(x)\,dx = \int_U \phi(x)\,d\mu^+(x) - \int_U \phi(x)\,d\mu^-(x)$

WEAK

Compactness (local version)

Let $f_n \in L^1_{loc}(U)$ s.that $\forall K \subset\subset U$ $\exists C_K > 0$: $\int_K |f_n(x)|\,dx \leq C_K$

then $\exists \mu^+, \mu^- \in \mathcal{M}(U)$, and f_{n_j} subsequence such that

$\forall \phi \in C_c(U)$ $\quad \int_U \phi f_{n_j}(x)\,dx \to \int_U \phi\, d\mu^+(x) - \int_U \phi\, d\mu^-(x)$

proof $\quad f_n \rightsquigarrow \mu_n^+ - \mu_n^- \qquad \mu_n^+(K) = \int_K f_n^+\,dx \leq C_K$

$$\mu_n^-(K) \leq \int_K f_n^-\,dx \leq C_K$$

$\exists \mu^+, \mu^-$ $\mu_{n_j}^+$ $\mu_{n_j}^-$ subsequence such that $\mu_{n_j}^+ \to \mu^+$

$$\mu_{n_j}^- \to \mu^-$$

$$\int_U \phi f_{n_j}(x)\,dx \longrightarrow \int_U \phi\, d\mu^+ - \int \phi\, d\mu^- \quad .. \square$$

VECTOR VALUED VERSION of RIESZ THEOREM

Let $I : C_c(U, \mathbb{R}^k) \to \mathbb{R}$ be a linear operator such that

$\forall K \subset\subset U$

$$\sup \{ I(f) \mid f \in C_c(U; \mathbb{R}^k) \quad \operatorname{supp} f \subseteq K , \quad |f| \leq 1 \} = C_K < +\infty$$

$\exists$ μ Radon measure on U and μ-measurable function $\sigma : U \to \mathbb{R}^k$ such that

1) $|\sigma(x)| = 1 \quad \mu - a.e\ x$

2) $I(f) = \int_U f(x) \cdot \sigma(x) \, d\mu \qquad \forall f \in C_c(U, \mathbb{R}^k)$

μ = VARIATION MEASURE

$A \subseteq U$ open

$$\mu(A) = \sup \left\{ \int_U f(x) \cdot \sigma(x) \, d\mu , \ \begin{array}{l} |f| \leq 1 \\ \operatorname{supp} f \subseteq A \end{array} \right\}$$

$$\mu(U) < +\infty \iff \sup_{\substack{|f| \leq 1 \\ f \in C_c(U)}} I(f) := C < +\infty$$

$\|I\| = C = \mu(U)$ operator norm

Back to some measure theory

COVERING THEOREMS (Vitali & Besikovitch)

VITALI (ACTUALLY a COROLLARY of Vitali theorem)

$\forall\, U \subseteq \mathbb{R}^n$ open, $\forall\, \delta > 0$ $\exists$ a COUNTABLE family $\mathcal{G}$ of closed, DISJOINT balls in U with diam $\leq \delta$ s.t. $\left|U \setminus \bigcup_{B \in \mathcal{G}} \bar{B}\right| = 0$.

(FILL UP AN OPEN SET with countably many disjoint balls)

BESICOVITCH Let μ Borel, $\mathcal{F}$ family of closed balls

$A = \{\text{center of } B,\ B \in \mathcal{F}\}$ If $\mu(A) < +\infty$, $\inf\{r,\ B(a,r) \in \mathcal{F}\} = 0$ $\forall a \in A$

$\rightarrow \forall\, U \subseteq \mathbb{R}^n$ open $\exists\, \mathcal{G}$ countable family of disjoint closed balls in $\mathcal{F}$ s. that $\bigcup_{B \in \mathcal{G}} B \subseteq U$ $\mu\left(U \cap A \setminus \bigcup_{B \in \mathcal{G}} B\right) = 0$

Used to prove DIFFERENTIATION THMS

1) $f \in L^1_{loc}(\mathbb{R}^n)$ $\quad A_r(x) = \dfrac{1}{r^n \omega_n} \displaystyle\int_{B(x,r)} f(y)\,dy$ is jointly continuous in x, r

$$\lim_{r \to 0^+} A_r(x) = f(x) \text{ for a.e } x$$

→ representative of f

$$f(x) = \begin{cases} \lim_{r\to 0^+} \displaystyle\fint_{B(x,r)} f(y)\,dy \\ 0 \text{ elsewhere} \end{cases}$$

2) $f \in L^p_{loc}(\mathbb{R}^n)$

$$\lim_{r \to 0^+} \int_{B(x,r)} \frac{|f(y)-f(x)|^p}{r^n \omega_n}\,dy = 0 \quad \text{a.e } x$$

APPLICATION

E is a measurable set $\quad (f = \chi_E \in L^1_{loc}(\mathbb{R}^n))$

$$\lim_{r \to 0^+} \frac{|B(x,r) \cap E|}{r^n \omega_n} = \underline{\chi_E(x)} \quad \text{a.e } x \in \mathbb{R}^n$$

For a.e $x \in E$ $\quad \displaystyle\lim_{r \to 0^+} \frac{|B(x,r) \cap E|}{\omega_n r^n} = 1$

For a.e. $x \in \mathbb{R}^n \setminus E$ $\quad \displaystyle\lim_{r \to 0^+} \frac{|B(x,r) \cap E|}{\omega_n r^n} = 0$

Measure theoretic interior of $E = \{x \ \lim_{r\to 0^+} \frac{|E\cap B(x,r)|}{\omega_n r^n} = 1\} = \mathrm{Int}_{\mathcal{L}}(E)$

Measure theoretic exterior of $E = \{x \ \lim_{r\to 0^+} \frac{|E\cap B(x,r)|}{\omega_n r^n} = 0\} = \mathrm{Ext}_{\mathcal{L}}(E)$

$\partial_* E$ = measure theoretic boundary of $E = \mathbb{R}^n \setminus (\mathrm{Int}_{\mathcal{L}}(E) \cup \mathrm{Ext}_{\mathcal{L}}(E))$

$= \{x$ such that either $\lim_{r\to 0^+} \frac{|E\cap B(x,r)|}{\omega_n r^n}$ does NOT EXIST or $\lim_{r\to 0^+} \frac{|E\cap B(x,r)|}{\omega_n r^n}$ EXISTS and $\neq 0, 1\}$

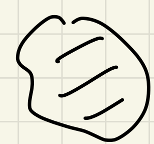
NOTE THAT BY ITS DEFINITION $|\partial_* E| = 0$ (by the Lebesgue theorem)

$\partial_* E \subseteq \partial E$ = topological boundary $= \{x \in \mathbb{R}^n \ \forall U_x$ open neighborhood of x $U_x \cap E \neq \emptyset, U_x \cap (\mathbb{R}^n \setminus E) \neq \emptyset\}$

$= \overline{E} \cap \overline{(\mathbb{R}^n \setminus E)}$

For E with bdry suff. regular $\partial E = \partial_* E$

In general $\partial_* E \subseteq \partial E$ (when $\partial_* E$ is closed $|\partial E \setminus \partial_* E| = 0$)



• isolated point!

Note that it is possible to construct sets in $\mathbb{R}^n$ with $|\partial E| \neq 0$ (topological bdry has NOT 0 measure) but $\partial_* E$, measure theoretic bdry, has ALWAYS measure 0 (due to its definition!)

(so in general $\partial_* E \subsetneq \partial E$).

Ex $\mathbb{Q} = \{q_n\}_{n \in \mathbb{N}}$ q_n is an enumeration of $\mathbb{Q}$

$$E = \bigcup_{n=0}^{+\infty} \underbrace{\left(q_n - \frac{1}{2^{n+1}}, q_n + \frac{1}{2^{n+1}} \right)}_{\text{open interval}} \qquad |E| \leq \sum_{n=0}^{\infty} \frac{2}{2^{n+1}} = 2 \quad \text{but } \overline{E} = \mathbb{R}^n$$

(so $|\partial E| = +\infty$)!

$$\boxed{\begin{array}{l} \text{If } E \text{ is an open set with } C^1 \text{ boundary, } \partial E = \partial_* E \\ \partial^* E = \left\{ x \;\; \lim\limits_{r \to 0^+} \dfrac{|B(x,r) \cap E|}{\omega_n r^n} = \dfrac{1}{2} \right\} \end{array}}$$