COMPUTABILITY (13/10/2025)

EXERCISE: URM^s machine : variant of URM

$T(\cancel{m},n)$ $\qquad$ $T_s(m,n)$ $\qquad$ $r_m \longleftrightarrow r_n$

$$\mathcal{C}^s \stackrel{?}{=} \mathcal{C}$$

proof

($\mathcal{C} \subseteq \mathcal{C}^s$) Let $f \in \mathcal{C}$ $\quad f: \mathbb{N}^k \to \mathbb{N}$ $\quad \rightsquigarrow \quad f \in \mathcal{C}^s$

let P be a URM-program such that $f = f_P^{(k)}$.

by a previous exercise ($\mathcal{C}^- = \mathcal{C}$) there is P' URM-program without transfer instructions such that $f_{P'}^{(k)} = f_P^{(k)}$

but P' is also a URM^s program and therefore $f = f_P^{(k)} = f_{P'}^{(k)} \in \mathcal{C}^s$

($\mathcal{C}^s \subseteq \mathcal{C}$) Take $f \in \mathcal{C}^s$ $\quad f: \mathbb{N}^k \to \mathbb{N}$ and let P a URM^s program such that $f = f_P^{(k)}$. We want to "transform" P into a URM program say P', such that

$$f_P^{(k)} = f_{P'}^{(k)}$$

$T_s(m,n)$

$T(m,i)$ $\quad R_i$ not used by P
$T(n,m)$
$T(i,n)$

> A URM^s-program P can be transformed into P' URM-program such that $f_P^{(k)} = f_{P'}^{(k)}$

We proceed by induction on h = number of T_s instructions in P

($h=0$) P is already a URM-program, hence $P' = P$

$(n \to n+1)$ Let P URM^S program with $n+1$ T_S instructions

$$P \begin{cases} I_1 \\ \vdots \\ I_t \quad T_S(m,n) \\ \vdots \\ I_s \end{cases} \quad \rightsquigarrow \quad P'' \begin{cases} I_1 \\ \vdots \\ I_t \quad J(1,1,SUB) \\ \\ I_s \\ I_{s+1} \quad J(1,1,END) \\ SUB: \quad T(m,i) \\ \quad\quad T(n,m) \\ \quad\quad T(i,n) \\ \quad\quad J(1,1,t+1) \end{cases}$$

assuming:

- P assumed to be well-formed (if it halts, it does at $s+1$)
- register i not used by P

$$i = \max\left(\{J \mid R_J \text{ is used by } P\} \cup \{k\}\right) + 1$$

Then P'' is such that

$$f_{P''}^{(k)} = f_P^{(k)}$$

and it contains n swap instructions. Hence by inductive hyp. there is P' URM-program such that $f_{P'}^{(k)} = f_{P''}^{(k)}$

Summing up

$$f = f_P^{(k)} = f_{P''}^{(k)} = f_{P'}^{(k)}$$

and thus $f \in \mathcal{C}$.

The proof is wrong: I am using the inductive hyp. on P'' which is not a URM^S program

Solution: prove a stronger assertion

"Every program P which uses both swap T_S and transfers T can be transformed into a URM-program P' s.t. $f_P^{(k)} = f_{P'}^{(k)}$." $\square$

EXERCISE : Consider URM$^=$ without jump instructions

Show (a) $\mathcal{C}^= \subsetneq \mathcal{C}$ and (b) characterise the shape of functions in $\mathcal{C}^=$

proof

(a) An URM$^=$ program

$$P \begin{cases} I_1 \\ \vdots \\ I_s \end{cases}$$

$\ell(P) = s$ length of program P

P terminates after $\ell(P)$ steps

$\leadsto$ all functions in $\mathcal{C}^=$ are total $\leadsto$ $\mathcal{C}^= \subsetneq \mathcal{C}$

e.g. $f : \mathbb{N} \to \mathbb{N}$

$f(x)\uparrow \quad \forall x \in \mathbb{N}$

$f \in \mathcal{C}$ $\quad J(1,1,1)$

$f \notin \mathcal{C}^=$ because is not total

(saying " the program uses jumps" is not sufficient for $f \notin \mathcal{C}^=$

e.g. $I_1 : J(1,1,2)$ $\quad P$

$f_P^{(1)}(x) = x$

but $f_P^{(1)} \in \mathcal{C}^=$

(b)

shape of functions in $\mathcal{C}^=$?

1			
x	0	0	...

conjecture :

$f(x) = c$ (constants)

or $f(x) = x + c$ (increment by a constant)

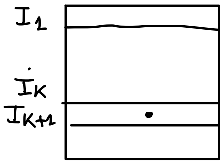
$c \in \mathbb{N}$ suitable constant

denote $r_1(x, k)$ = content of register 1 after k steps starting from $x\,|\,0\,|\,0\,|\,\ldots$

We prove by induction on k that $r_1(x,k) \begin{cases} c \\ x + c \end{cases}$ $\quad c \in \mathbb{N}$

$(k=0)$ $\quad r_1(x,0) = x = x+0 \quad c=0$

$(k \to k+1)$ By induction hyp $\quad r_1(x,k) = \begin{cases} c \\ x+c \end{cases}$



different cases according to the shape of I_{k+1}

* I_{k+1} $\quad Z(n)$ $\quad$ two sub cases

  $(n=1)$ $\quad r_1(x, k+1) = 0$ $\quad$ ok.

  $(n>1)$ $\quad r_1(x, k+1) = r_1(x,k)$ $\quad$ ok by ind. hyp.

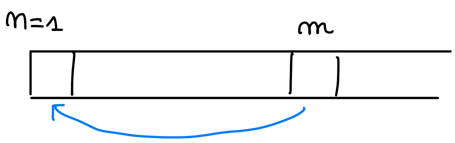
* I_{k+1} $\quad S(n)$ $\quad$ two sub cases

  $(n=1)$ $\quad r_1(x,k+1) = \underbrace{r_1(x,k)}_{\substack{c \\ x+c}} + 1$ ind. hyp. $\quad \begin{cases} c+1 \\ x+(c+1) \end{cases}$ $\quad$ ok!

  $(n>1)$ $\quad r_1(x,k+1) = r_1(x,k)$ $\quad$ ok by ind. hyp.

* I_{k+1} $\quad T(m,n)$ $\quad (n>1$ or $m=1)$ $\quad r_1(x,k+1) = r_1(x,k)$ $\quad$ ok, by ind hyp.

  $(m=1$ and $n>1)$



not working, no info on R_n $\quad n>1$ ...

The key observation is that the same property holds for all registers

$r_j(x,k) =$ content of R_j after k steps of computation starting from $\boxed{x|0|0|\ldots}$

show by induction that $\forall k \ \forall j$ $\quad$ (induction on k)

$$r_j(x,k) = \begin{cases} c \\ x+c \end{cases}$$

The proof now works smoothly. $\quad$ TRY!

EXERCISE

for n-ary functions

$$f(x_1, \dots, x_n) = \begin{cases} c \\ x_j + c \end{cases} \qquad \begin{array}{l} 1 \leq j \leq n \\ c \in \mathbb{N} \text{ constant} \end{array}$$

* Decidable predicates

$div(x, y) \equiv$ " x divides y "

$div \subseteq \mathbb{N} \times \mathbb{N}$

$div = \{ (m, m * k) \mid m, k \in \mathbb{N} \}$

or

$$div : \mathbb{N} \times \mathbb{N} \to \{\underset{1}{true}, \underset{0}{false}\}$$

K-ary predicate $\qquad Q(x_1, \dots, x_k) \subseteq \mathbb{N}^k$

$$Q : \mathbb{N}^k \to \{\underset{\substack{\uparrow \\ false}}{0}, \underset{\substack{\uparrow \\ true}}{1}\}$$

Def. (decidable predicate)

Let $Q(x_1, \dots, x_k) \subseteq \mathbb{N}^k$ a predicate. It is decidable if

$$\chi_Q : \mathbb{N}^k \to \mathbb{N}$$

$$\chi_Q(x_1, \dots, x_k) = \begin{cases} 1 & \text{if } Q(x_1, \dots, x_k) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

is URM - computable

Example : $\qquad Q(x_1, x_2) \subseteq \mathbb{N}^2$

$Q(x_1, x_2) =$ " $x_1 = x_2$ " $\qquad$ decidable

$$\chi_Q : \mathbb{N}^2 \to \mathbb{N}$$

$$\chi_Q(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases} \qquad \text{computable}$$

1	2	3	4	...
x_1	x_2	0	0	
		↑ output		

$J(1,2,\text{YES})$

NO : $J(1,1,\text{RES})$

YES : $S(3)$

RES : $T(3,1)$

<u>Example</u> : $Q(x) \equiv$ "x even" decidable

$\boxed{\text{EXERCISE}}$

1	2	3	
x	0	0	
	↑ K	↑ result	

EVEN : $J(1,2,\text{TRUE})$
$S(2)$

ODD : $J(1,2,\text{FALSE})$
$S(2)$
$J(1,1,\text{EVEN})$

TRUE : $S(3)$

FALSE : $T(3,1)$

* Computability on other domains

D countable

$\alpha : D \to \mathbb{N}$ bijective "effective"

(inverse α^{-1} effective)

$A^*, \mathbb{Q}, \mathbb{Z}, \ldots$ $\cancel{\mathbb{R}}$

Given $f : D \to D$ function is computable if

$$\begin{tikzcd} D \arrow[r, "f"] & D \arrow[d, "\alpha"] \\ \mathbb{N} \arrow[u, "\alpha^{-1}"] \arrow[r, "f^*"] & \mathbb{N} \end{tikzcd}$$

$f^* = \alpha \circ f \circ \alpha^{-1} : \mathbb{N} \to \mathbb{N}$

is URM-computable

Example: Computability on $\mathbb{Z}$

$\alpha : \mathbb{Z} \to \mathbb{N}$

$$\alpha(z) = \begin{cases} 2z & \text{if } z \geq 0 \\ -2z-1 & \text{if } z < 0 \end{cases}$$

$\mathbb{N}$	0	1	2	3	4	5	6	7	..
$\mathbb{Z}$	0	-1	1	-2	2	-3	3	-4	...

$\alpha^{-1} : \mathbb{N} \to \mathbb{Z}$

$$\alpha^{-1}(n) = \begin{cases} n/2 & n \text{ even} \\ -\frac{n+1}{2} & n \text{ odd} \end{cases}$$

$f : \mathbb{Z} \to \mathbb{Z}$

$f(z) = |z|$ computable

$$f^* : \alpha \circ f \circ \alpha^{-1} : \mathbb{N} \to \mathbb{N}$$

$$f^*(n) = \alpha f \alpha^{-1}(n) = \begin{cases} n \text{ even} & \alpha \overbrace{f\left(\frac{n}{2}\right)} = \alpha\left(\frac{n}{2}\right) = 2\frac{n}{2} = n \\ n \text{ odd} & \alpha f\left(-\frac{n+1}{2}\right) = \alpha\left(\frac{n+1}{2}\right) = 2\frac{n+1}{2} = n+1 \end{cases}$$

$$= \begin{cases} n & n \text{ even} \\ n+1 & n \text{ odd} \end{cases}$$

URM- computable

$\Longrightarrow$ f is computable