[ I follow mainly Evans - Gariepy (ch 1)
PAY ATTENTION that definitions can be slightly different!
(measure for EG is a "outer measure" for me).
+ Folland ch 1+7 ]

[ALMOST NO PROOFS IN THIS PART
(topics of REAL ANALYSIS / LINEAR FUNCT AN.)]

$\mu : \mathcal{B} \to [0, +\infty]$ be a Radon measure (locally finite)

we defined also $\int_{\mathbb{R}^n} f \, d\mu$.

<u>Def</u> $\operatorname{supp} \mu = \{x \mid \mu(B(x,r)) > 0 \ \forall r > 0\} = \mathbb{R}^n \setminus \bigcup \{A \subseteq \mathbb{R}^n \ A \text{ open}, \mu(A) = 0\}$

If $U \subseteq \mathbb{R}^n$ is an open set we may consider

$\mu : \mathcal{B}_{|U} \to [0, +\infty]$ (measures defined on the $\sigma$-algebra of Borelians $\cap\, U$)

$\mathcal{M}(U) = \{\text{Radon measures on } U\}$ $\qquad \mu \in \mathcal{M}(\mathbb{R}^n) \to \mu_{|U} \in \mathcal{M}(U)$

$\mathcal{M}(\mathbb{R}^n) = \{\text{Radon measures on } \mathbb{R}^n\}$

- **The space $\mathcal{M}(U)$ has the structure of a "dual space".**

**Def** Let $I$ be a POSITIVE LINEAR FUNCTIONAL on $C_c(U)$ ($U \subseteq \mathbb{R}^n$ open, $U$ can also be $\mathbb{R}^n$)

that is $I : C_c(U) \to \mathbb{R}$
① $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$
② if $f \geq 0 \Rightarrow I(f) \geq 0$.
1+2 $f \geq g \Rightarrow I(f) \geq I(g)$

__Proposition__ (POSITIVITY $\Rightarrow$ "CONTINUITY")

$I$ positive linear functional on $C_c(U)$. Then $\forall K \subset\subset U$ $\exists\, C_K > 0$ such that $|I(f)| \leq C_K \|f\|_\infty$ $\forall f \in C_c(U)$ $\operatorname{supp} f \subseteq K$.

__proof__ Take $K \subset\subset U$ and $\phi \in C_c^\infty(U, [0,1])$ $\phi \equiv 1$ on $K$

For $f \in C_c(U)$ with $\operatorname{supp} f \subseteq K \Rightarrow -\|f\|_\infty \phi(x) \leq f(x) \leq \|f\|_\infty \phi(x)$

$\Rightarrow -\|f\|_\infty I(\phi) \leq I(f) \leq \|f\|_\infty I(\phi)$ $\quad C_K = I\phi$ $\quad \square$

**Obs** In $C_c(U)$ we may put the topology related to this convergence

$f_n \to f$ in $C_c(U) \iff \exists K,\ \operatorname{supp} f_n, \operatorname{supp} f \subseteq K$

$\|f_n - f\|_\infty \to 0$.

PREVIOUS PROPOSITION: $I : C_c(U) \to \mathbb{R}$ linear + positive $\Rightarrow$ continuous

that is ($f_n \to f$ in $C_c(U) \Rightarrow I(f_n) \to I(f)$)

**Theorem** (RIESZ 1909, MARKOV 1938, KAKUTANI 1941)

Any linear positive functional on $C_c(U)$ is uniquely associated to a Radon measure $\mu \in \mathcal{M}(U)$ by the representation formula $I(f) = \int_U f\, d\mu$. (and viceversa)

$\mu$ is defined as follows:

$$\mu(A) := \sup\{ I(f),\ f \in C_c(U)\ \operatorname{supp} f \subset\subset A,\ 0 \leq f \leq 1\}$$

$\forall A$ open

$$\mu(K) := \inf\{ I(f),\ f \in C_c(U)\ f \geq 1 \text{ on } K,\ f \geq 0 \text{ on } U \setminus K\}$$

$\forall K$ compact

proof Obviously if $\mu$ is Radon measure then

$$f \longmapsto \int_U f \, d\mu$$ is a positive linear functional!

Viceversa is more involved.

→ Uniqueness of $\mu$ derives from the fact that Radon measures are regular. (and we fix the value of $\mu$ on open and compact sets)

→ Proof of the fact that $\mu$ defined on open and compact sets as above is a Radon measure is based on the $\boxed{\text{Carathéodory criterium}}$.

□

# Carathéodory criterium

$X = \mathbb{R}^n$ $\quad X = U \subseteq \mathbb{R}^n$ open.

**How to construct a measure?** Start defining a positive function $\tilde{\mu}$ on a family of "elementary sets" $\mathcal{E}$ (e.g. rectangle $\prod_{i=1}^{n} (a_i, b_i)$, or open sets) $\rightarrow$ **PREMEASURE $\tilde{\mu}$**

$\tilde{\mu}(\emptyset) = 0$, $\emptyset, \mathbb{R}^n \in \mathcal{E}$

then EXTEND IT to all $E \in \mathcal{P}(X)$ $\longrightarrow$ **EXTERIOR MEASURE**

$$\mu^*(E) = \inf \left\{ \sum_{i} \tilde{\mu}(E_i) \quad E_i \in \mathcal{E} \quad E \subseteq \bigcup_i E_i \right\}$$

$$\mu^* : \mathcal{P}(X) \longrightarrow [0, +\infty] \qquad \mu^*(\emptyset) = 0 \qquad \mu^*\left(\bigcup_i A_i\right) \leq \sum_i \mu^*(A_i)$$

$$A_i \cap A_j = \emptyset$$

<u>Def</u> $A \in \mathcal{P}(X)$ is **$\mu^*$ MEASURABLE** if

$\forall B \in \mathcal{P}(X)$ "test set" $\qquad \mu^*(B) = \mu^*(B \setminus A) + \mu^*(B \cap A)$

$$\mathcal{M} = \{\text{all } \mu^* \text{ measurable sets}\} \subseteq \mathcal{P}(X)$$

# CARATHEODORY CRITERIUM:

1) $\mathcal{M}$ is a $\sigma$-algebra and $\mu^*|_{\mathcal{M}}$ is a COMPLETE MEASURE

2) If $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ $\forall A, B \in \mathcal{P}(X)$ $d(A,B)>0$ then $\mu^*$ is a Borel measure (that is $\mathcal{M} \supseteq \mathcal{B}$)

Back to the Riesz theorem.

$$\mathcal{M}(U) \longleftrightarrow \{\text{positive linear functionals on } C_c(U)\}$$

$(C_c(U), \|\ \|_\infty)$ NOT CLOSED $\rightarrow$ $(C_0(U), \|\cdot\|_\infty)$ is the closure (is Banach ch)

$I: C_c(U) \to \mathbb{R}$ positive linear functional can be extended to a CONTINUOUS LINEAR FUNCTIONAL on $C_0(U)$ if it is bounded: $\exists\, C>0$ $|I(f)| \leq C\|f\|_\infty$ $\forall f \in C_c(U)$.

Now that $\mu(U) = \sup\{ \int_U g \, d\mu \quad g \in \mathcal{C}_c(U) \; g \in [0,1] \}$

Observe that:

$$|I(f)| = \left| \int_U f \, d\mu \right| \overset{f \neq 0}{\leq} \|f\|_\infty \int_U \overset{g}{\frac{|f|}{\|f\|_\infty}} d\mu \leq \|f\|_\infty \mu(U).$$

Therefore if $\mu(U) < +\infty \Rightarrow |I(f)| \leq C \|f\|_\infty \rightarrow I$ bounded

If $I$ bounded, $\mu(U) = \sup\{\int_U g \, d\mu \ldots\} \leq \sup\{C\|g\|_\infty\} \leq C$.

$I$ is a linear continuous functional on $(\mathcal{C}_0(U), \|\cdot\|_\infty)$
$\Leftrightarrow \mu(U) < +\infty$. $\xrightarrow{\text{NOTE}}$ $\mu(U) = \|I\|$ (operator norm)

~ . ~

What is the dual of $(\mathcal{C}_0(U), \|\cdot\|_\infty)$ ?

$I : (\mathcal{C}_0(U), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$ linear continuous.

$\Rightarrow I = I^+ - I^-$ $\quad I^+, I^- : (\mathcal{C}_0(U), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$ linear positive.

for $f \geq 0$ $I^+(f) := \sup\{ I(g),\ 0 \leq g \leq f\ \ g \in C_c(U)\}$

Note that $I^+(f) \geq 0\ \forall f \geq 0$, $I^+(f) \geq I(f)\ \forall f \geq 0$

$I^- = I^+ - I$. Note that $\forall f \geq 0$ $I^-(f) = I^+(f) - I(f) \geq 0$

$\Rightarrow I^+, I^-$ are positive linear functionals

$\rightarrow$ RIESZ $\exists\, \mu^+, \mu^-$ Radon measures $I^+ f = \int_U f\, d\mu^+$ $I^- f = \int_U f\, d\mu^-$

$I : (C_0(U), \|\cdot\|_\infty) \rightarrow \mathbb{R}$ linear continuous functional

Then $\exists!\, \mu^+, \mu^-$ Radon measures with $\mu^+(U) < \infty$, $\mu^-(U) < +\infty$

such that $I(f) = \int f\, d\mu^+ - \int f\, d\mu^-$.

$\mu = \mu^+ - \mu^-$ is a FINITE SIGNED MEASURE

More generally $\mu$ is a Radon signed measure if $\exists\, \mu^+, \mu^-$ Radon measures such that $\mu = \mu^+ - \mu^-$

## NOTION OF WEAK* CONVERGENCE on $\mathcal{M}(U)$.

$\mu_n, \mu \in \mathcal{M}(U)$ $\quad \mu_n \xrightarrow{*} \mu$ if $\forall f \in C_c(U)$ $\quad \int_U f \, d\mu_n \to \int_U f \, d\mu$

Equivalently: $\quad \lim_n \mu_n(B) = \mu(B) \quad \forall B \in \mathcal{B}_{U}$

B bounded and $\mu(\partial B) = 0$

Equivalently: $\quad \limsup_n \mu_n(K) \leq \mu(K) \quad \forall K \subset\subset U$

$\liminf_n \mu_n(A) \geq \mu(A) \quad \forall A$ open $\subset U$

## WEAK COMPACTNESS THEOREM (derives from Banach Alaoglu)

$\mu_n \in \mathcal{M}(U)$ and $\sup_n \mu_n(K) < C_K \quad \forall K \subset\subset U$

$\exists \, \mu_{n_i}$ subsequence and $\mu \in \mathcal{M}(U)$ such that $\mu_{n_i} \to \mu$

Recall.

## Riesz theorem for $L^p$ spaces, $1 \le p < +\infty$

( $L^q(U)$ is the dual of $L^p(U)$ )

$I : L^p(U) \to \mathbb{R}$ linear bold operator $\Rightarrow \exists!\, g \in L^q(U)$
$f \longmapsto I(f)$

$\frac{1}{p}+\frac{1}{q}=1$ conjugate of $p$
$(p=1 \to q=+\infty)$

such that $I = I_g$ $\quad I_g(f)=\int_U f(x)\,g(x)\,dx.$

$|I_g(f)| \le \|f\|_p \|g\|_p$ Hölder inequality
$(p=1 \to q=+\infty)$

## WEAK CONVERGENCE

$f_n \rightharpoonup f$ in $L^p(U)$ $\quad p\in[1,+\infty)$ $\quad$ if $\forall g \in L^q(U)$ $\quad \frac{1}{p}+\frac{1}{q}=1$

$$\int_U f_n(x)g(x)\,dx \longrightarrow \int_U f(x)g(x)\,dx$$

$f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(U)$ $\quad$ if $\forall g\in L^1(U)$ $\quad \int_U f_n g\,dx \to \int_U f g\,dx$

( $L^\infty(U)$ is the dual of $L^1(U)$ )

# COMPACTNESS THEOREM

1) $p \in (1, +\infty)$ (NEEED REFLEXIVE!)

$f_n \in L^p(U)$ $\|f_n\|_{L^p} \leq C \Rightarrow \exists f_{n_j}$ subsequence, $f \in L^p(U)$

$f_{n_j} \rightharpoonup f$ in $L^p(U)$.

2) $p = +\infty$ $f_n \in L^\infty(U)$ $\|f_n\|_{L^\infty} \leq C \Rightarrow \exists f_{n_j}$ subsequence, $f \in L^\infty(U)$

such that $f_{n_j} \overset{*}{\rightharpoonup} f$ in $L^\infty(U)$.

3) $p = 1$ NOT TRUE!