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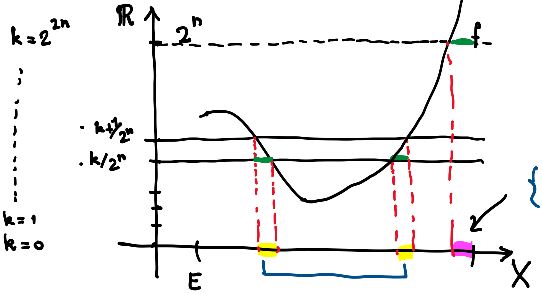
Let $(X, \mathcal{F}, \mu)$ be a meas sp.

$f \in L(E)$ $(E \subset X)$, $f \geq 0$ on E



$f \in [2^n, +\infty[$

$\{f \geq 2^n\} \subset E$

$\shortparallel$

E_n

$\{f \in [\frac{k}{2^n}, \frac{k+1}{2^n}[\}$ $\quad E_{k,n} := \left\{ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \right\} \subset E$

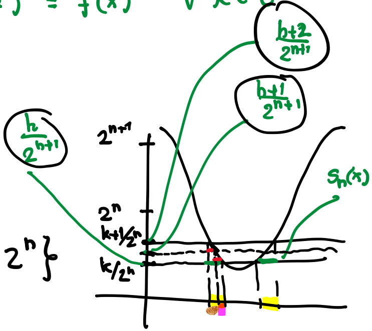
$$s_n(x) := \begin{cases} \boxed{\frac{k}{2^n}}, & x \in E_{k,n} \\ 2^n & x \in E_n \end{cases}$$

$$\boxed{s_n(x) := \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} 1_{E_{k,n}} + 2^n 1_{E_n}}$$

Prop: Let $(X, \mathcal{F}, \mu)$ be a meas sp, $f \in L(E)$, $f \geq 0$. Then the seq (s_n) is made of simple and meas functs s.t.

1) $0 \leq s_n \leq s_{n+1}$ $\quad \forall x \in E \quad \forall n$

2) $\lim\limits_{n\to+\infty} s_n(x) = f(x) \quad \forall x \in E$



Proof:

$$\boxed{s_{n+1}(x) = 2^{n+1}}$$

$\Downarrow$

$x \in \{f \geq 2^{n+1} \geq 2^n\}$

$\Downarrow$

$x \in \underbrace{\{f \geq 2^n\}}_{E_n} \Rightarrow s_n(x) = 2^n \leq \underset{\underset{s_{n+1}(x)}{\parallel}}{2^{n+1}}$

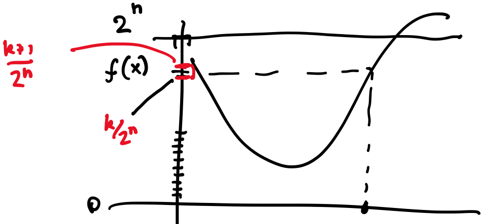
this proves $s_n \leq s_{n+1} \quad \forall x$

2) $s_n(x) \longrightarrow f(x) \quad \forall x \in E$

$\Updownarrow$

$0 \leq f(x) - s_n(x)$

$x \in E \qquad f(x) \in [0, +\infty[$



Let n big enough s.t $2^n > f(x)$

$$\exists\, k : \quad \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}$$

$$(x \in E_{k,n})$$

$$\Rightarrow \quad s_n(x) = \frac{k}{2^n}$$

$$\Rightarrow \quad 0 \leq f(x) - s_n(x) = f(x) - \frac{k}{2^n} \leq \frac{1}{2^n}$$

so : if $x \in E$ is fixed, $\forall n \geq N$:

$$2^n \geq f(x)$$

$$\begin{array}{ccccc} 0 \leq & f(x) & - \; s_n(x) & \leq & \dfrac{1}{2^n} \qquad \forall n \geq N \\ \downarrow & & \downarrow & & \downarrow \; n \to 0 \\ 0 & & 0 & & 0 \end{array}$$

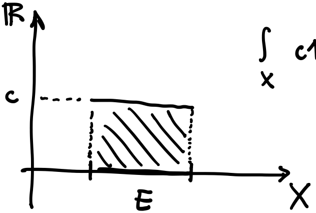
$$\Rightarrow \quad f(x) - s_n(x) \longrightarrow 0 \qquad n \to +\infty$$

(squeeze thm)

$$\Rightarrow \quad s_n(x) \longrightarrow f(x) \qquad \forall x \in E.$$

$$E_{k,n} = \left\{ \frac{k}{2^n} \le f < \frac{k+1}{2^n} \right\}$$

So if $$s_n(x) = \sum_{k=0}^{2^{2n}-1} \underbrace{\frac{k}{2^n} 1_{E_{k,n}}}_{} + \underbrace{2^n 1_{E_n}}_{}$$



$$\int_X c1_E \, d\mu = c\cdot \mu(E)$$

(we expect that the $\int$ does this)

$$I_n = \int_E s_n \, d\mu = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n}\mu(E_{k,n}) + 2^n \mu(E_n)$$

Thm: Let $(X,\mathcal{F},\mu)$ be a meas sp, $f \in L(E)$, $f \ge 0$. Then $I_n \nearrow$

$$\exists \lim_{n\to+\infty} I_n =: \int_E f \, d\mu.$$

Some first properties:

1) $f, g \in L(E)$, $0 \le f \le g \Rightarrow \int_E f \, d\mu \le \int_E g \, d\mu$

$(s_n^f \le s_n^g)$

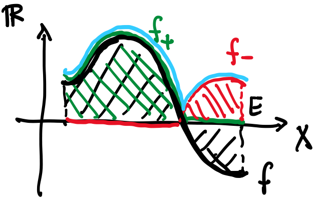
2. $f, g \in L(E)$, $\alpha, \beta \geq 0$ $\Rightarrow$ $\alpha f + \beta g \in L(E)$
$\alpha f + \beta g \geq 0$

and

$$\int_E (\alpha f + \beta g)\, d\mu = \alpha \int_E f\, d\mu + \beta \int_E g\, d\mu.$$

So far we defd $\int_E f\, d\mu$ for
$f \in L(E)$, $f \geq 0$.

We want to extend this to
$f \in L(E)$ $(f : E \longrightarrow \mathbb{R})$



$$f_+ = \begin{cases} f & f \geq 0 \\ 0 & f < 0 \end{cases}$$ (pos. part of f)

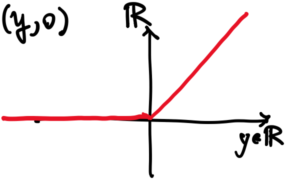
$$f_- = \begin{cases} 0 & f \geq 0 \\ -f & f < 0 \end{cases}$$ (neg. part of f)

Notice that

$$f \in L(E) \iff f_\pm \in L(E) \quad f_\pm \geq 0$$

$$\Rightarrow \quad f_+(x) = \max(f(x),0) = \varphi(f(x)) = \overset{\overset{\mathcal{C}}{\Downarrow}}{\varphi} \circ \overset{\overset{L(E)}{\swarrow}}{f}$$

$$\mathcal{C} \ni \varphi(y) = \max(y,0)$$



$$\Rightarrow \quad f_+ \in L(E)$$

$$f_- = \max(-f,0) \overset{?}{=} -\min(f,0)$$

$$\Leftarrow \quad f_\pm \in L(E) \quad \Rightarrow \quad \boxed{f = f_+ - f_-} \in L(E)$$

So the idea should be

$$\int_E f \, d\mu := \int_E f_+ \, d\mu - \int_E f_- \, d\mu$$

green "area" red "area"

To make this a proper def we require

$$(0 \leq) \int_E f_+ \, d\mu, \ \int_E f_- \, d\mu \ \underline{< +\infty}$$

$$\Updownarrow$$

$$\int_E f_+ \, d\mu + \int_E f_- \, d\mu < +\infty$$

$$\Updownarrow$$

$$\int_E (\underbrace{f_+ + f_-}_{|f|}) \, d\mu < +\infty$$

$$\Updownarrow$$

$$\boxed{\int_E |f| \, d\mu < +\infty}$$

(__Rmk:__ If $f \in L(E) \underset{\not\Leftarrow}{\Rightarrow} |f| \in L(E)$

__Def:__ Let $(X, \mathcal{F}, \mu)$ be a meas. sp. $f \in L(E)$. We say that f is μ-integrable on E if

$$\int_E |f| \, d\mu < +\infty$$

In this case, we call μ-int. of f on E

$$\int_E f \, d\mu = \int_E f_+ \, d\mu - \int_E f_- \, d\mu.$$

The class of int functs is denoted by

$$L^1(X, \mathcal{F}, \mu) = L^1(X, \mu) \equiv \underline{L^1(X)}$$

Last step: extension to $\mathbb{C}$-valued funct.

$f : E \to \mathbb{C}$ $\qquad$ $\underset{\in\,\mathbb{C}}{f(x)} = u(x) + i v(x)$

$u, v : E \to \mathbb{R}$

Real part of f

Imaginary part of f

$$\boxed{f \in L(E) \overset{\text{def}}{\Longleftrightarrow} u, v \in L(E).}$$

We expect that

$$\int_E f \, d\mu = \int_E u \, d\mu + i \int_E v \, d\mu.$$

provided $u, v \in L^1(E) \underset{?}{\Longleftrightarrow} |f| \in L^1(E)$

$$|f| = \sqrt{u^2 + v^2}$$

So if $u, v \in L^1(E) \Rightarrow \underbrace{\int_E |f| \, d\mu}$

$$\Downarrow$$

$$\underbrace{\int_E |u| \, d\mu, \int_E |v| \, d\mu < +\infty}$$

$$|f| = \sqrt{u^2 + v^2} \overset{\cancel{?}}{\leq} \cancel{C}(|u| + |v|)$$

$$\Updownarrow$$

$$u^2 + v^2 \leq \overset{1}{C}(|u| + |v|)^2$$

$$\overset{\|}{\left(u^2+v^2+2|u||v|\right)}$$

$$|f| = \sqrt{u^2+v^2} \leq |u| + |v| \Rightarrow$$

$$\int_E |f|\,d\mu \leq \int_E |u| + |v|\,d\mu = \int_E |u|\,d\mu + \int_E |v|\,d\mu$$

$$< +\infty$$

because $u, v \in L^1$

$$\Leftarrow \quad |f| \in L^1 \Rightarrow \int_E |f|\,d\mu < +\infty$$

$$|f| = \sqrt{u^2 + \underbrace{v^2}_{\geq 0}} \geq \sqrt{u^2} = |u|$$

$$\geq \sqrt{v^2} = |v|$$

$$\Rightarrow |u|, |v| \leq |f| \Rightarrow$$

$$\int_E |u|\,d\mu, \int_E |v|\,d\mu \leq \int_E |f|\,d\mu < +\infty$$

$$\Rightarrow u, v \in L^1$$

Def: $(X, \mathcal{F}, \mu)$ meas sp, $f \in L(E)$, $f : E \longrightarrow \mathbb{C}$

$f = u + iv$. We say that $f \in L^1_{\mathbb{C}}(E)$

if

$$\int_E |f|\, d\mu < +\infty$$

We set

$$\int_E f\, d\mu := \int_E u\, d\mu + i \int_E v\, d\mu.$$

$\blacksquare$

General properties.

1) (linearity) $f, g \in L^1(E) \Rightarrow \alpha f + \beta g \in L^1(E)$ $\forall \alpha, \beta \in \mathbb{R}$ $(\mathbb{C})$

and

$$\int_E (\alpha f + \beta g)\, d\mu = \alpha \int_E f\, d\mu + \beta \int_E g\, d\mu$$

2) (monot.) $f, g \in L^1_{\mathbb{R}}(E)$ $f \leq g$ on E

$$\Rightarrow \quad \int_E f\, d\mu \leq \int_E g\, d\mu$$

3) (triang ineq) $f \in L^1(E)$

$$\left| \int_E f\, d\mu \right| \leq \int_E |f|\, d\mu$$

4) $f \in L^1(E), L^1(F)$ $E \cap F = \emptyset \Rightarrow$
$f \in L^1(E \sqcup F)$

$$\int_{E\cup F} f\,d\mu = \int_E f\,d\mu + \int_F f\,d\mu$$

5) If $\mu(E)=0 \Rightarrow \int_E f\,d\mu = 0$

In part $f = g$ a.e. $(\mu(f\neq g)=0)$

$$\Downarrow$$

$$\int_E f\,d\mu = \int_E g\,d\mu.$$

Prop: (Čhebyshev ineq.)

$(X,\mathcal{F},\mu)$ meas sp., $f\in L(E)$, $f\geq 0$.

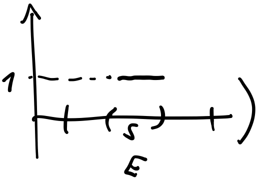
$$\mu(f\geq\alpha)\leq \frac{1}{\alpha}\int_{\boxed{f\geq\alpha}} f\,d\mu \qquad \alpha>0$$

$$\leq \frac{1}{\alpha}\int_E f\,d\mu.$$

Proof:

$$\mu(f\geq\alpha) = \int_E \overset{f/\alpha}{1}\cdot\boxed{1_{\{f\geq\alpha\}}}\,d\mu$$

$\Big(\mu(S) = \int_E 1_S\,d\mu$



$\Big)$

$S\subset E$

$$f \geqslant \alpha \Leftrightarrow \boxed{\frac{f}{\alpha} \geqslant 1} \qquad (\alpha > 0$$

$$\leqslant \int_E \frac{f}{\alpha} \, 1_{\underbrace{\{f \geqslant \alpha\}}} {}_{\subset E} \, d\mu$$

$$\mu(f \geqslant \alpha) \leqslant \frac{1}{\alpha} \int_{\underbrace{\{f \geqslant \alpha\}}_{\subset E}} f \, d\mu$$

$$\leqslant \frac{1}{\alpha} \int_E f \, d\mu. \qquad \blacksquare$$

Rmk: (Variants)

$$\mu(f \geqslant \alpha) \leqslant \frac{1}{\alpha} \int_E f \, d\mu$$

So if $f \in L^1(E)$ $\left(\int_E f \, d\mu < +\infty \right)$

$$\Rightarrow \quad \mu(\{f \geqslant \alpha\}) \leqslant \frac{C}{\alpha} \longrightarrow 0$$

Other bounds: if $p \geqslant 1$

$$\mu(f \geq \alpha) = \int_E 1 \cdot 1_{\underbrace{f \geq \alpha}} \, d\mu$$

$$f \geq \alpha \iff \underset{\overset{\shortparallel}{1}}{1^p} \leq \left(\frac{f}{\alpha}\right)^p = \frac{f^p}{\alpha^p}$$

$$\leq \int_E \frac{f^p}{\alpha^p} 1_{f \geq \alpha} \, d\mu$$

$$\boxed{\mu(f \geq \alpha) \leq \frac{1}{\alpha^p} \int_E f^p \, d\mu}$$

So $(p=2)$ if $\int_E f^2 \, d\mu < +\infty$

$$\Rightarrow \mu(f \geq \alpha) \leq \frac{1}{\alpha^2} \int_E f^2 \, d\mu .$$

Do Ex 4.3.4/5/6/7.