Chapter 2

Linear Programming

2.1 Introduction

Linear Programming (LP)[1] is the core model of constrained optimization. Developed by George Dantzig in 1947, the simplex method for solving LP is a cornerstone of applied mathematics, computer science, and operations research. The simplex method has been applied to thousands of applications in a wide variety of fields, such as agriculture, communications, computer science, engineering, finance, manufacturing, transportation, and urban logistics. The majority of the models studied in this course are either LP or Integer Linear Programming (ILP) extensions of LP.

An LP is an optimization problem with (i) q non-negative decision variables, (ii) a linear objective function, (iii) a set of p linear constraints, and (iv) a set of non-negativity restrictions imposed upon the decision variables.

An LP takes the *standard* form

$$\min \sum_{j=1}^{q} c_j x_j \tag{2.1a}$$

$$\text{s.t.} \sum_{j=1}^{q} a_{ij} x_j = b_i \qquad \forall i = 1, \ldots, p \tag{2.1b}$$

$$x_j \in \mathbb{R}_+ \qquad \forall j = 1, \ldots, q \tag{2.1c}$$

Without loss of generality, an LP in the standard form assumes that (i) the objective function is in minimization form, and (ii) the p linear constraints are in the "equal to" form and $b_i \in \mathbb{R}_+$.

In economic planning, each decision variable x_j models the level of the jth activity, each constraint corresponds to a scarce resource available in b_i quantity, and a_{ij} is the amount of the ith resource consumed per unit of the jth activity. The model seeks the best uses of resources to maximize the revenue.

In matrix notation, an LP has the form

$$\min_{x \in \mathbb{R}_+^q} \{ \boldsymbol{cx} \mid \boldsymbol{Ax} = \boldsymbol{b} \} \tag{2.2}$$

[1] LP can refer to linear programming, linear program, or linear problem

where $\boldsymbol{A} = (a_{ij})$ has p rows and q columns, $\boldsymbol{c} \in \mathbb{R}^q$ is a q-dimensional row vector, $\boldsymbol{x} \in \mathbb{R}^q_+$ is a q-dimensional column vector, and $\boldsymbol{b} \in \mathbb{R}^p_+$ is a p-dimensional column vector. We assume that the rows of $\boldsymbol{A}$ are linearly independent, and $\boldsymbol{A}$ has full row rank.

Exercise 2.1.

A firm produces two types of wood desks (rolltop and regular). Wood is cut to a uniform thickness of 1 cm, so wood is measured in units of square meters (m^2). One rolltop desk requires 10 m^2 of pine, 4 m^2 of cedar, and 15 m^2 of maple. One regular desk requires 20 m^2 of pine, 16 m^2 of cedar, and 10 m^2 of maple. The desks yield €115 and €90 profit per sale, respectively.
The firm has currently available 200 m^2 of pine, 128 m^2 of cedar, and 220 m^2 of maple. The firm has backlogged orders for both desks and would like to produce the number of rolltop and regular desks that maximize the profit.
How many of each should it produce?

First, we identify the input data

- D set of desk types
- W set of wood types
- p_d unit profit for desk type $d \in D$
- b_w m^2 of wood type $w \in W$ available
- a_{dw} m^2 of wood type $w \in W$ needed for a single desk of type $d \in D$

Then, we introduce the decision variables

- $x_d \in \mathbb{R}_+$ number of desks of type $d \in D$ to produce

Finally, we write the LP

$$\max \sum_{d \in D} p_d x_d \tag{2.3a}$$

$$\text{s.t.} \sum_{d \in D} a_{dw} x_d \le b_w \quad \forall w \in W \tag{2.3b}$$

$$x_d \in \mathbb{R}_+ \quad \forall d \in D \tag{2.3c}$$

The objective function (2.3a) aims at maximizing the total profit. Constraints (2.3b) ensure that the square meters of each wood type needed to produce the desks do not exceed the square meters available. Constraints (2.3c) are non-negativity constraints.

Notice that by defining the $\boldsymbol{x}$ variables as real, we are not forcing to produce an integer number of desks of each type. Indeed, the problem should have been formulated as an ILP (rather than an LP). However, a feasible non-necessarily optimal solution can be achieved by properly rounding the optimal solution of (2.3).

Exercise 2.2.

The Grand Prix Cars Company manufactures cars in three plants and ships them to four regions of the country. The plants can supply 450, 600, and 500 cars, respectively. The customer demands by region are 450, 200, 300, and 300, respectively. The unit costs of shipping a car from each plant to each region are listed below

	Region 1	Region 2	Region 3	Region 4
Plant 1	131	218	266	120
Plant 2	250	116	263	278
Plant 3	178	132	122	180

Grand Prix wants to find the lowest-cost shipping plan for meeting the demands of the four regions without exceeding the capacities of the plants. Formulate the problem as an LP.

First, we identify the input data

- O set of origins/plants
- D set of destinations/regions
- s_i supply of origin $i \in O$
- d_j demand of destination $j \in D$
- c_{ij} unit shipping cost from origin $i \in O$ to destination $j \in D$

Then, we introduce the decision variables

- $x_{ij} \in \mathbb{R}_+$ number of cars to ship from origin $i \in O$ to destination $j \in D$

Finally, we write the LP

$$\min \sum_{i \in O} \sum_{j \in D} c_{ij} x_{ij} \tag{2.4a}$$

$$\text{s.t.} \sum_{j \in D} x_{ij} \le s_i \qquad \forall i \in O \tag{2.4b}$$

$$\sum_{i \in O} x_{ij} = d_j \qquad \forall j \in D \tag{2.4c}$$

$$x_{ij} \in \mathbb{R}_+ \qquad \forall i \in O \quad \forall j \in D \tag{2.4d}$$

The objective function (2.4a) aims at minimizing the total shipping costs. Constraints (2.4b) ensure that each origin's supply is not exceeded. Constraints (2.4c) guarantee that customer demands are fulfilled. Constraints (2.4d) are non-negativity constraints.

Notice that by defining the $\boldsymbol{x}$ variables as real, we are not forcing the solution to send an integer number of cars from each pair of origin and destination. However, as the matrix $\boldsymbol{A}$ is totally unimodular and we can assume that all s_i and d_j are integer, any feasible solution of (2.4) is integer.[2]

[2] More details about this are beyond the scope of the course

2.2 Graphical Solution Procedure

An LP with two or three variables has a convenient graphical representation that helps understand the nature of LP and the simplex method, which is the most popular method for solving an LP. Consider the following LP

$$\max \quad z = x_1 + x_2 \tag{2.5a}$$
$$\text{s.t.} \quad 2x_1 + 3x_2 \leq 12 \tag{2.5b}$$
$$x_1 \leq 4 \tag{2.5c}$$
$$x_2 \leq 3 \tag{2.5d}$$
$$x_1, x_2 \in \mathbb{R}_+ \tag{2.5e}$$

The shaded region of Figure 2.1 is the set of feasible solutions for (2.5). This set is a *polyhedron*. The points A, B, C, D, and E are the *extreme points* of the polyhedron, where x is an extreme point if it is not a strict convex combination of two distinct points (x^1 and x^2) of the polyhedron, that is, it cannot be represented as $x = \theta x^1 + (1-\theta)x^2$, for $0 < \theta < 1$.

The LP seeks a point (x_1, x_2) in the polyhedron $ABCDEA$ that achieves the maximum possible value of $x_1 + x_2$, i.e., the largest value z for which line $z = x_1 + x_2$ has at least a point in common with $ABCDEA$. The maximum value is $\frac{16}{3}$ achieved at extreme point $D = (4, \frac{4}{3})$. Indeed, every LP always has an extreme point solution as one of its optimal solutions. To solve an LP, we can focus on a finite number of points: the extreme points. The simplex method makes use of this extreme point property: it starts at some extreme point and visits adjacent extreme points to improve the objective function value until it reaches an optimal extreme point.

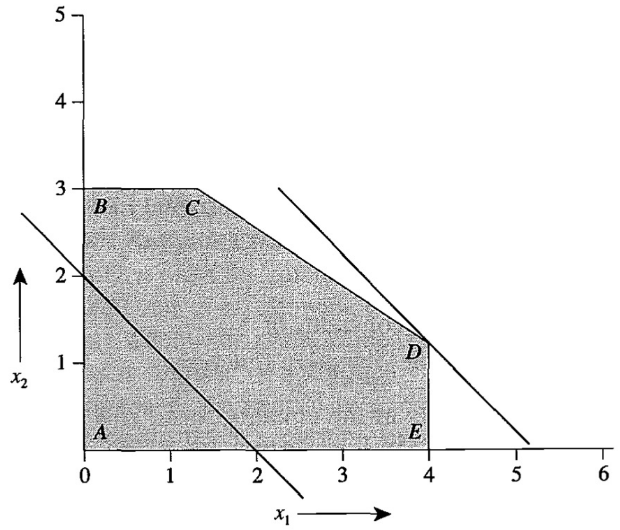

Figure 2.1: Set of feasible solutions for problem (2.5)

2.3 Linear Programming Duality

Each LP, which we call the *primal problem*, has an associated LP, called the *dual problem*. The primal and dual problems define *duality theory*, which we summarize in the following.

Let the primal problem be defined as the following LP in the *canonical form*

$$\min \sum_{j=1}^{q} c_j x_j \tag{2.6a}$$

$$\text{s.t.} \sum_{j=1}^{q} a_{ij} x_j \geq b_i \qquad \forall i = 1, \dots, p \tag{2.6b}$$

$$x_j \in \mathbb{R}_+ \qquad \forall j = 1, \dots, q \tag{2.6c}$$

Let $\pi_i \in \mathbb{R}_+$ be a dual variable associated with the ith constraint (2.6b). The dual of (2.6) is

$$\max \sum_{i=1}^{p} b_i \pi_i \tag{2.7a}$$

$$\text{s.t.} \sum_{i=1}^{p} a_{ij} \pi_i \leq c_j \qquad \forall j = 1, \dots, q \tag{2.7b}$$

$$\pi_i \in \mathbb{R}_+ \qquad \forall i = 1, \dots, p \tag{2.7c}$$

Each constraint in the primal has an associated variable in the dual (and vice versa). The right-hand side of each constraint of the primal (dual, respectively) becomes the cost coefficient of the associated dual (primal, respectively) variable. The dual of the dual is the primal.

The following table summarizes the relationships between the primal and the dual ($\boldsymbol{A}_i$ represents the ith row of matrix $\boldsymbol{A}$, and $\boldsymbol{a}_j^T$ the transpose of the jth column of matrix $\boldsymbol{A}$)

Primal	Dual
min	max
q variables	q constraints
p constraints	p variables
constraint $\boldsymbol{A}_i \boldsymbol{x} \leq b_i$	variable $\pi_i \leq 0$
constraint $\boldsymbol{A}_i \boldsymbol{x} \geq b_i$	variable $\pi_i \geq 0$
constraint $\boldsymbol{A}_i \boldsymbol{x} = b_i$	variable $\pi_i \in \mathbb{R}$
variable $x_j \geq 0$	constraint $\boldsymbol{a}_j^T \boldsymbol{\pi} \leq c_j$
variable $x_j \leq 0$	constraint $\boldsymbol{a}_j^T \boldsymbol{\pi} \geq c_j$
variable $x_j \in \mathbb{R}$	constraint $\boldsymbol{a}_j^T \boldsymbol{\pi} = c_j$

Exercise 2.3.

Write the dual of problem (2.3), which is

$$\max \sum_{d \in D} p_d x_d$$

$$\text{s.t.} \sum_{d \in D} a_{dw} x_d \leq b_w \qquad \forall w \in W$$

$$x_d \in \mathbb{R}_+ \qquad \forall d \in D$$

Let $\boldsymbol{\pi} \in \mathbb{R}_+^{|W|}$ be the dual variables associated with constraints (2.3b). The dual of (2.3) is

$$\begin{aligned} \min \sum_{w \in W} b_w \pi_w & \\ \text{s.t.} \sum_{w \in W} a_{dw} \pi_w \geq p_d & \qquad \forall d \in D \\ \pi_w \in \mathbb{R}_+ & \qquad \forall w \in W \end{aligned}$$

Exercise 2.4.

Write the dual of problem (2.4), which is

$$\begin{aligned} \min \sum_{i \in O} \sum_{j \in D} c_{ij} x_{ij} & \\ \text{s.t.} \sum_{j \in D} x_{ij} \leq s_i & \qquad \forall i \in O \\ \sum_{i \in O} x_{ij} = d_j & \qquad \forall j \in D \\ x_{ij} \in \mathbb{R}_+ \quad \forall i \in O & \quad \forall j \in D \end{aligned}$$

Let $\boldsymbol{u} \in \mathbb{R}_-^{|O|}$ and $\boldsymbol{v} \in \mathbb{R}^{|D|}$ be the dual variables associated with constraints (2.4b) and (2.4c), respectively. The dual of (2.4) is

$$\begin{aligned} \max \sum_{i \in O} s_i u_i + \sum_{j \in D} d_j v_j & \\ \text{s.t.} \quad u_i + v_j \leq c_{ij} & \qquad \forall i \in O \quad \forall j \in D \\ u_i \in \mathbb{R}_- & \qquad \forall i \in O \\ v_j \in \mathbb{R} & \qquad \forall j \in D \end{aligned}$$

Theorem 2.1. ***(Weak Duality)*** *If $\boldsymbol{x}$ is any feasible solution of the primal and $\boldsymbol{\pi}$ is any feasible solution of the dual problem, then*

$$\sum_{i=1}^{p} b_i \pi_i \leq \sum_{j=1}^{q} c_j x_j$$

Proof. Multiplying the ith constraint (2.6b) by π_i and adding them all yields

$$\sum_{i=1}^{p} b_i \pi_i \leq \sum_{i=1}^{p} \pi_i \Big(\sum_{j=1}^{q} a_{ij} x_j \Big), \tag{2.9}$$

while multiplying the jth constraint (2.7b) by x_j and adding them all yields

$$\sum_{j=1}^{q} x_j \Big(\sum_{i=1}^{p} a_{ij} \pi_i \Big) \leq \sum_{j=1}^{q} c_j x_j \tag{2.10}$$

Therefore

$$\sum_{i=1}^{p} b_i \pi_i \leq \sum_{i=1}^{p} \pi_i \Big(\sum_{j=1}^{q} a_{ij} x_j \Big) = \sum_{j=1}^{q} x_j \Big(\sum_{i=1}^{p} a_{ij} \pi_i \Big) \leq \sum_{j=1}^{q} c_j x_j \tag{2.11}$$

which proves the theorem. ∎

The weak duality theorem has a number of consequences.

Lemma 2.1. *The objective function value of any feasible dual solution is a Lower Bound (LB) on the objective function value of every feasible primal solution.*

Lemma 2.2. *The objective function value of any feasible primal solution is an Upper Bound (UB) on the objective function value of every feasible dual solution.*

Lemma 2.3. *If the primal (dual, respectively) has an unbounded solution, the dual (primal, respectively) is infeasible.*

Lemma 2.4. *If the primal has a feasible solution $\boldsymbol{x}$ and the dual has a feasible solution $\boldsymbol{\pi}$ such that $\sum_{i=1}^{p} b_i \pi_i = \sum_{j=1}^{q} c_j x_j$, then $\boldsymbol{x}$ is an optimal primal solution, and $\boldsymbol{\pi}$ is an optimal dual solution.*

Theorem 2.2. ***(Strong Duality)*** *If the primal and dual problems are feasible, there exists an optimal primal solution $\boldsymbol{x}^*$ and an optimal dual solution $\boldsymbol{\pi}^*$ and $\sum_{i=1}^{p} b_i \pi_i^* = \sum_{j=1}^{q} c_j x_j^*$.*

Property 2.1. ***(Complementary Slackness)*** *A pair $(\boldsymbol{x}, \boldsymbol{\pi})$ of the primal and dual feasible solutions satisfies the complementary slackness property if*

$$\pi_i \Big(\sum_{j=1}^{q} a_{ij} x_j - b_i \Big) = 0 \quad \forall i = 1, \ldots, p \tag{2.12}$$

and

$$x_j \Big(c_j - \sum_{i=1}^{p} a_{ij} \pi_i \Big) = 0 \quad \forall j = 1, \ldots, q \tag{2.13}$$

Observe that $\sum_{j=1}^{q} a_{ij} x_j - b_i$ is the amount of slack in the ith primal constraint (2.6b), and π_i is the corresponding dual variable. Similarly, $c_j - \sum_{i=1}^{p} a_{ij} \pi_i$ is the amount of slack in the jth dual constraint (2.7b), and x_j is the corresponding primal variable. The complementary slackness property states that, for each primal and dual solution that fulfill the complementary slackness property, the product of the slack in the constraint and its associated variable is zero.

Theorem 2.3. ***(Complementary Slackness Optimality Conditions)*** *A feasible primal solution $\boldsymbol{x}$ and a feasible dual solution $\boldsymbol{\pi}$ are optimal solutions of the primal and dual problems if and only if they satisfy the complementary slackness property.*