Ex. 2.3.1 (*)

$N \subset \mathbb{R}$, $\lambda(N) = 0$

$\Rightarrow$ N^c is dense in $\mathbb{R}$ that is

$\forall\ ]a,b[\subset \mathbb{R}$, $]a,b[\cap N^c \neq \emptyset$.

Sol: If false $\Rightarrow$ $\exists\ \underset{a<b}{]a,b[}$: $\boxed{]a,b[} \cap N^c = \emptyset$

$\Downarrow$

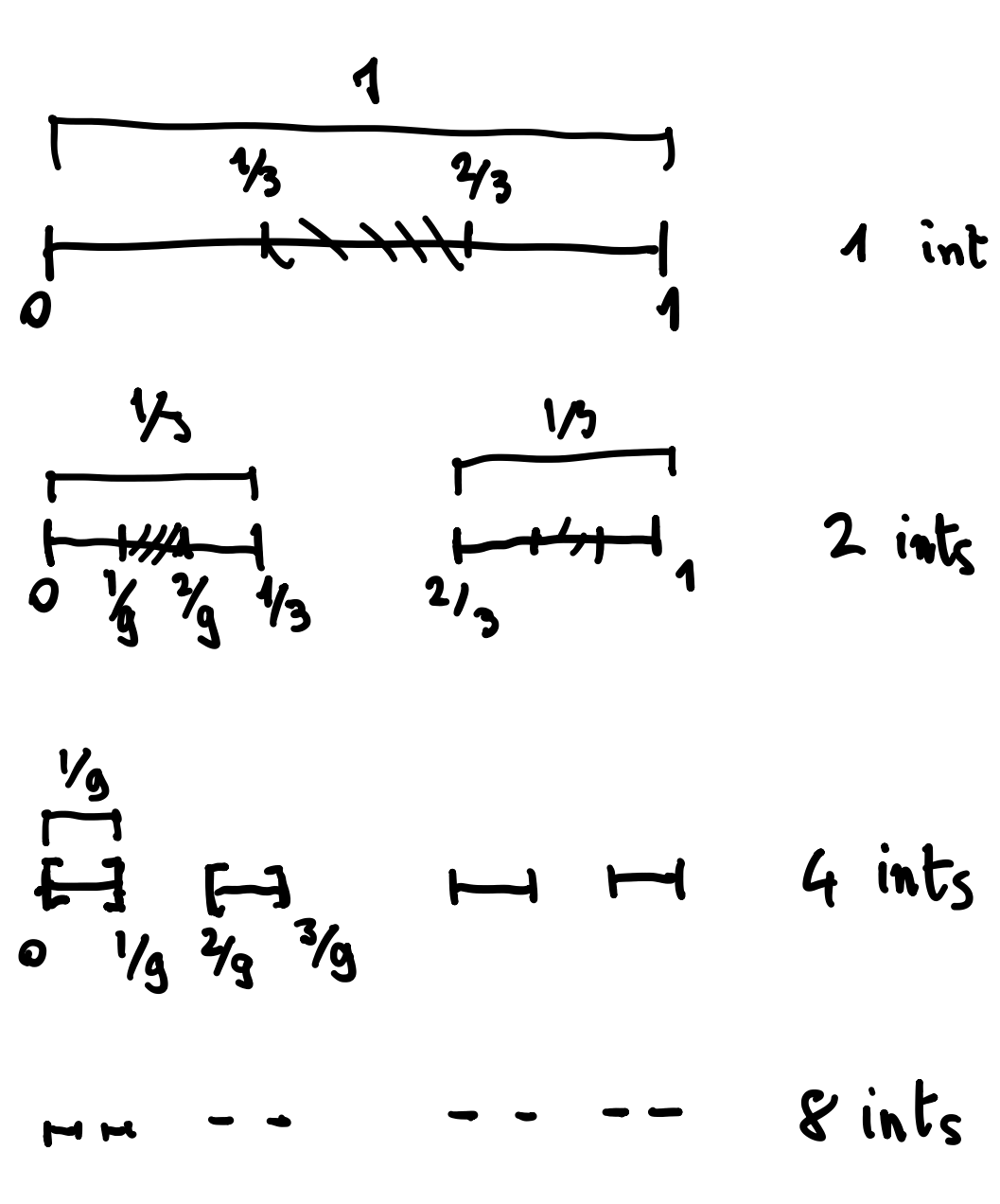
$]a,b[\subset N$

$\Downarrow$

$0 < b-a = \lambda(]a,b[) \leq \underset{=0}{\lambda(N)}$

(imposs.) ∎

Ex 2.3.2. (Cantor set)



C_0
$\cup$
C_1
$\cup$
C_2
$\cup$
C_3
$\vdots$

C_n $\boxed{\text{is made of } 2^n \text{ closed intervals of length } \frac{1}{3^n} \text{ each.}}$

$C = \boxed{\bigcap_{n=0}^{\infty} C_n}$

Show $C \in \mathcal{M}_1$, compute $\lambda(C)$.

1. each $C_n \in \mathcal{M}_1$ $\Rightarrow$ $\bigcap_{n=0}^{\infty} C_n \in \mathcal{M}_1$
($\mathcal{M}_1$ is a σ-alg)

alternative: each C_n is closed

$\Downarrow$

$\bigcap_n C_n$ is closed

$\Downarrow$

$\in \mathcal{M}_1$.

2. Since $\lambda(C_0) = \lambda([0,1]) = 1 < +\infty$

and $C_n \downarrow C$, we apply the cont.

from above

$\lambda(C) = \lim_n \lambda(C_n)$

$= \lim_n 2^n \cdot \frac{1}{3^n} = \lim \left(\frac{2}{3}\right)^n$

$= 0$

(in alt: $0 \leq \underset{C \subset C_n}{\lambda(C) \leq \lambda(C_n)} \longrightarrow 0$)

∎

2.3.5.

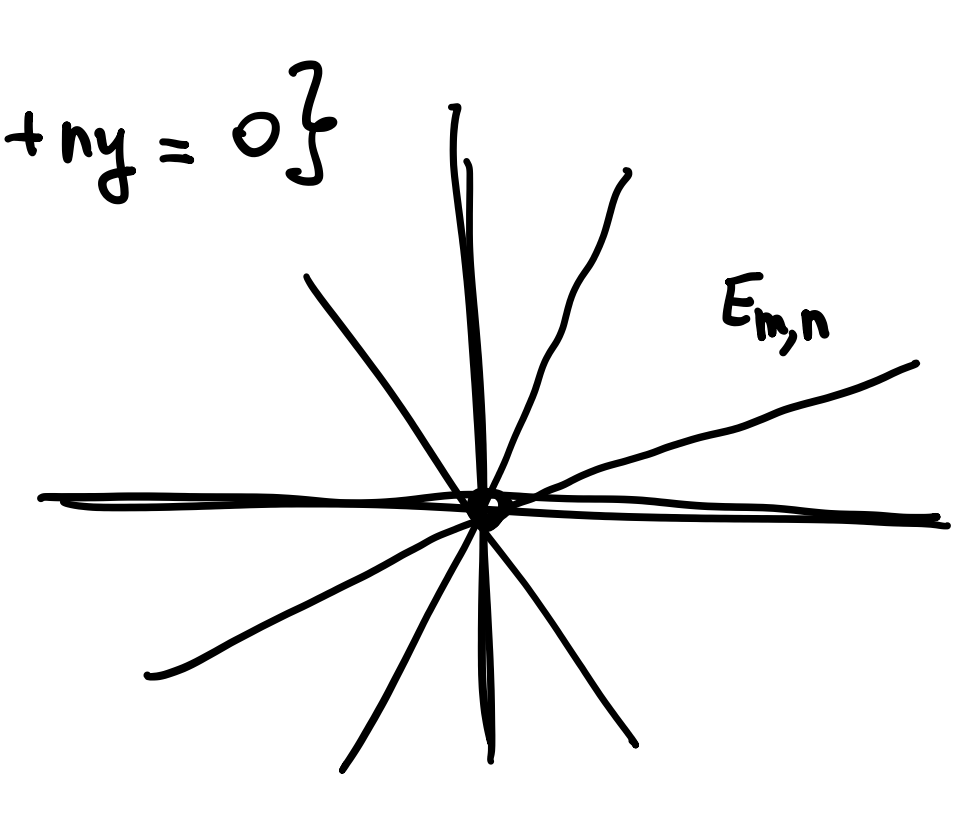
$E = \{ (x,y) \in \mathbb{R}^2 : \exists (m,n) \in \mathbb{N}^2 \setminus \{(0,0)\}$
$mx + ny = 0 \}$

$\Rightarrow$

$\lambda_2(E) = 0$

Sol: If $(m,n) \in \mathbb{N}^2 \setminus \{(0,0)\}$ the set

$E_{m,n} = \{ mx + ny = 0 \}$



$E = \bigcup_{(m,n) \in \mathbb{N}^2 \setminus \{\vec{0}\}} E_{m,n}$

$\lambda_2(E) \leq \sum \underbrace{\lambda_2(E_{m,n})}_{0} = 0$.

∎

2.3.7 $A, B, C \subset [0,1]$ s.t

$\boxed{\forall x \in [0,1] \text{ belongs to at least 2 sets among } A,B,C.}$

$\Rightarrow$ one of $\lambda(A), \lambda(B), \lambda(C) \geq \frac{2}{3}$.

Sol: We know

$[0,1] \subset \underset{\subset[0,1]}{A\cap B} \cup \underset{\subset[0,1]}{A \cap C} \cup \underset{\subset[0,1]}{B \cap C}$

$\subset [0,1]$

$\Rightarrow$ $\boxed{[0,1] = A\cap B \cup A \cap C \cup B \cap C}$

$\Downarrow$

Hp: (*) $\boxed{1 = \lambda\left((A\cap B) \cup (A \cap C) \cup (B \cap C)\right)}$

Th: one of $\lambda(A), \lambda(B), \lambda(C) \geq 2/3$.

By contradiction we assume Th false

$\boxed{\lambda(A), \lambda(B), \lambda(C) < 2/3.}$

$1 = \lambda$ $\leq \lambda(\underset{\subset[0,1]}{A \cup B \cup C}) \leq 1$

$\Rightarrow \lambda(A \cup B \cup C) = 1$

$\boxed{\begin{array}{l}\mu(A\cup B) = \mu(A) + \mu(B) \\ \quad - \mu(A \cap B)\end{array}}$
$(A \cup B) \cup C$

$1 = \lambda(\overbrace{A \cup B}\cup \overbrace{C})$

$= \lambda(A) + \lambda(B) + \lambda(C)$

$- \lambda(A\cap B) - \lambda(A \cap C) - \lambda(B \cap C)$

$+ \lambda(A \cap B \cap C)$

$\boxed{\begin{array}{l} 1 = \lambda(A) + \lambda(B) + \lambda(C) \\ - \left(\boxed{\lambda(A\cap B) + \lambda(A\cap C) + \lambda(B \cap C)}\right) \\ + \lambda(A \cap B \cap C)\end{array}}$

$\lambda\left((A \cap B) \cup (A \cap C) \cup (B\cap C)\right)$

$= \lambda(A\cap B) + \lambda(A \cap C) + \lambda(B \cap C)$

$- \lambda((A\cap B) \cap (A \cap C)) - \lambda(A \cap B \cap C)$
$\underbrace{\qquad}_{\lambda(A\cap B\cap C)}$ $- \lambda(A \cap B\cap C)$

$+ \lambda(A \cap B \cap C)$

$\underset{\text{Hp}}{1} = \boxed{\lambda(A\cap B) + \lambda(A\cap C) + \lambda(B\cap C)} - 2\lambda(A\cap B\cap C)$

$\Rightarrow \underline{\lambda(A\cap B) + \lambda(A\cap C) + \lambda(B\cap C)} = \underline{1 + 2\lambda(A\cap B\cap C)}$

So

$1 = \lambda(A) + \lambda(B) + \lambda(C)$

$- \left(1 + 2\lambda(A\cap B\cap C)\right) + \lambda(A\cap B\cap C)$

$= \underline{\lambda(A) + \lambda(B) + \lambda(C)} - 1 - \underline{\lambda(A\cap B\cap C)}$

$\Downarrow$

$\boxed{\lambda(A) + \lambda(B) + \lambda(C) = 2 + \lambda(A\cap B\cap C)} \geq 0$

if these $< \frac{2}{3}$ lhs $< 3 \cdot \frac{2}{3} = 2$

rhs ≥ 2

Contradiction!

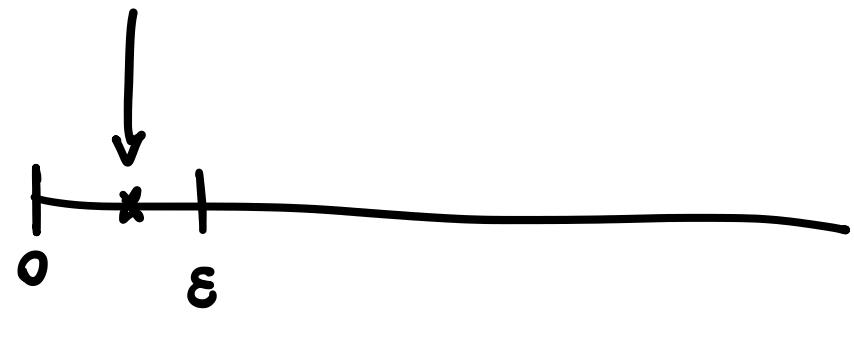
∎

Do 2.3.8

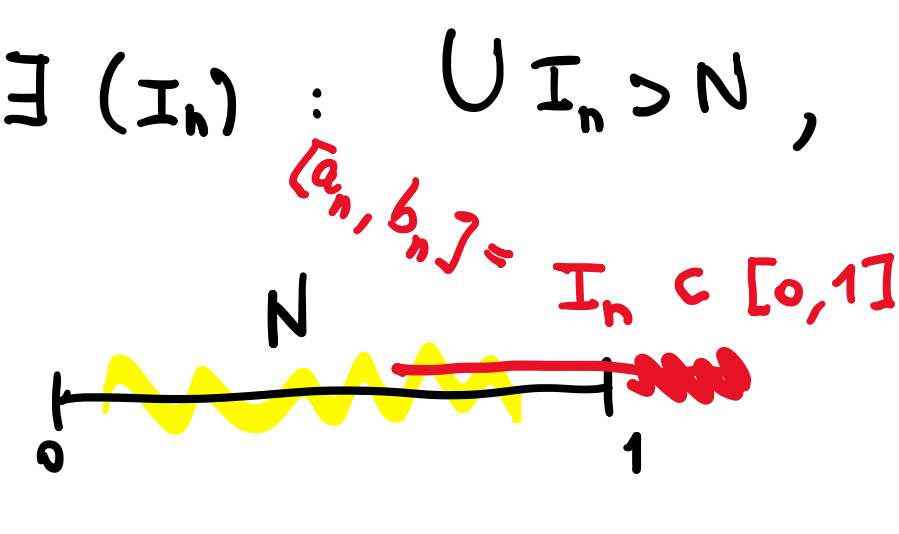
$(N \subset [0,1]$ $\boxed{\lambda(N) = 0}$, $N^2 := \{ x^2 : x \in N \}$)

$\Rightarrow \lambda(N^2) = 0$

$\inf \{ \sum |I_n| : N \subset \bigcup I_n \} = 0$



$\forall \varepsilon > 0$ $\exists (I_n)$: $\bigcup I_n \supset N$, $\sum |I_n| \leq \varepsilon$



Do 3.4.1/2/4/6/7.

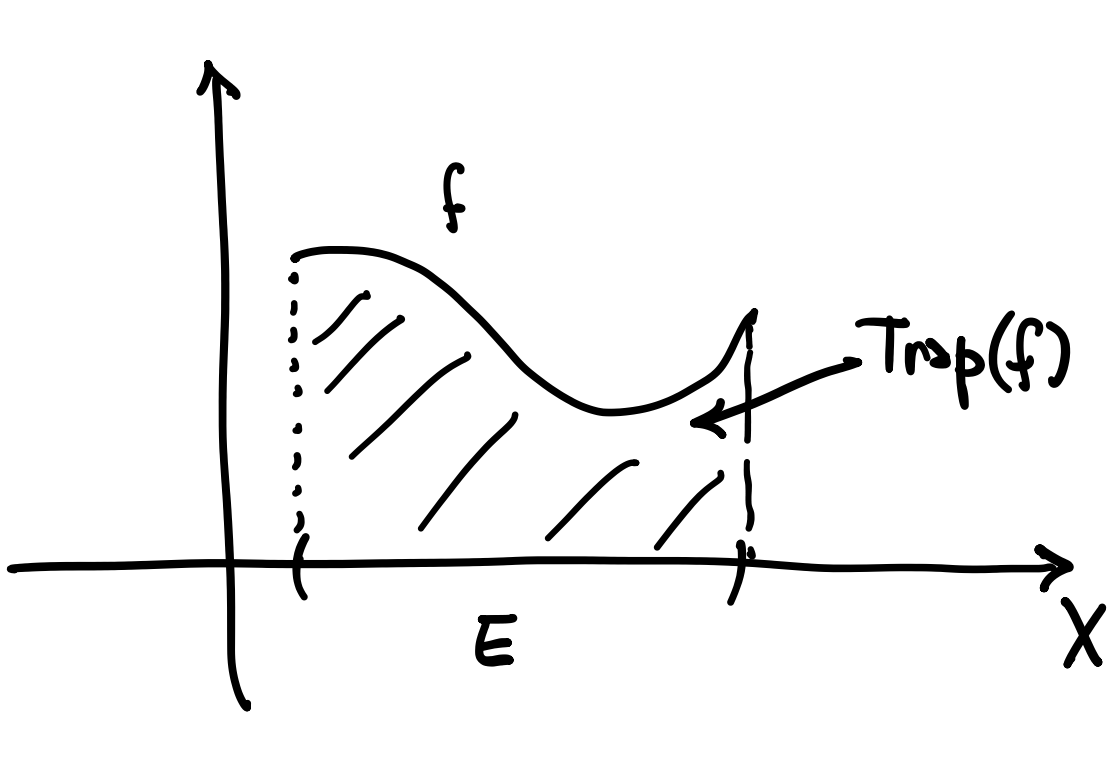
Abstract Integral

Goal: Let $(X, \mathcal{F}, \mu)$ be a meas sp.

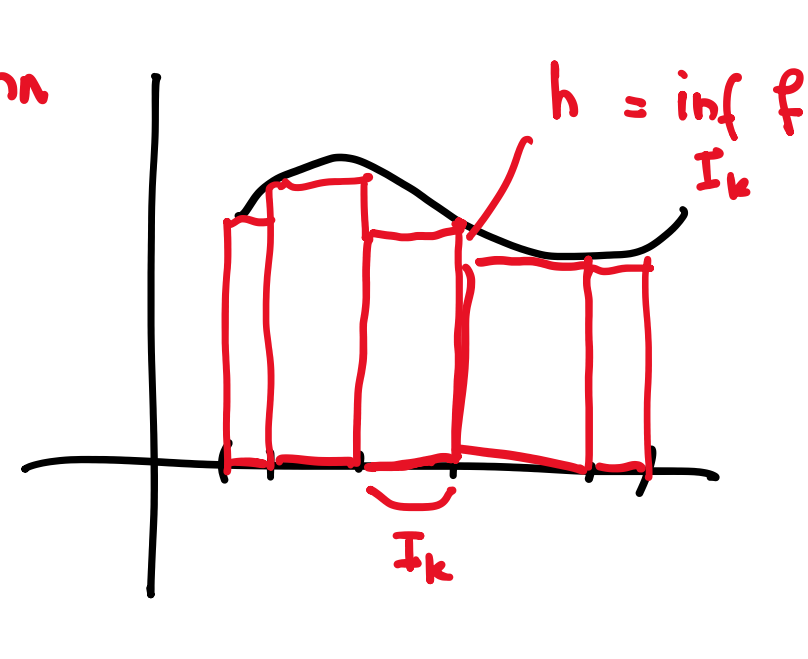
$f : E \subset X \longrightarrow \mathbb{R}$, we aim to def

$\int_E f \, d\mu$. (integral of f on E w.r.t μ)

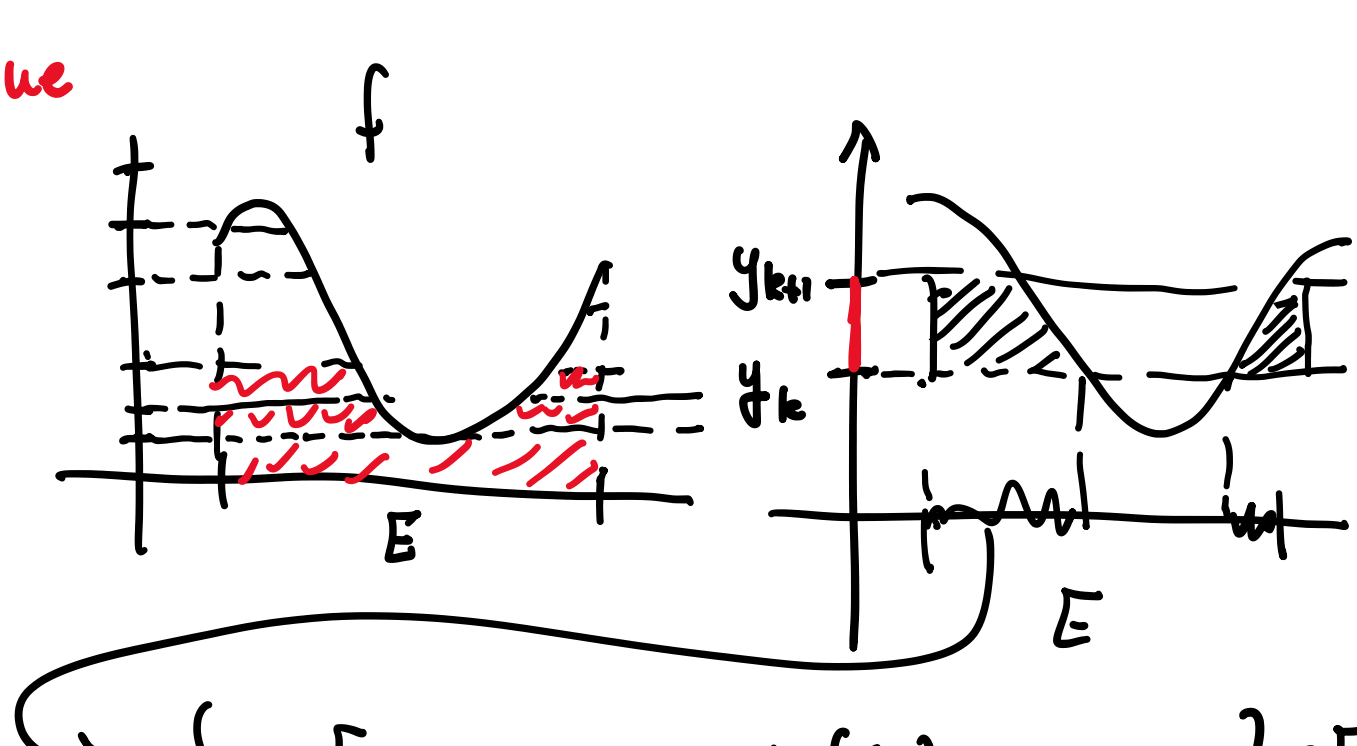
Let $f : E \subset X \longrightarrow [0, +\infty[$



Riemann

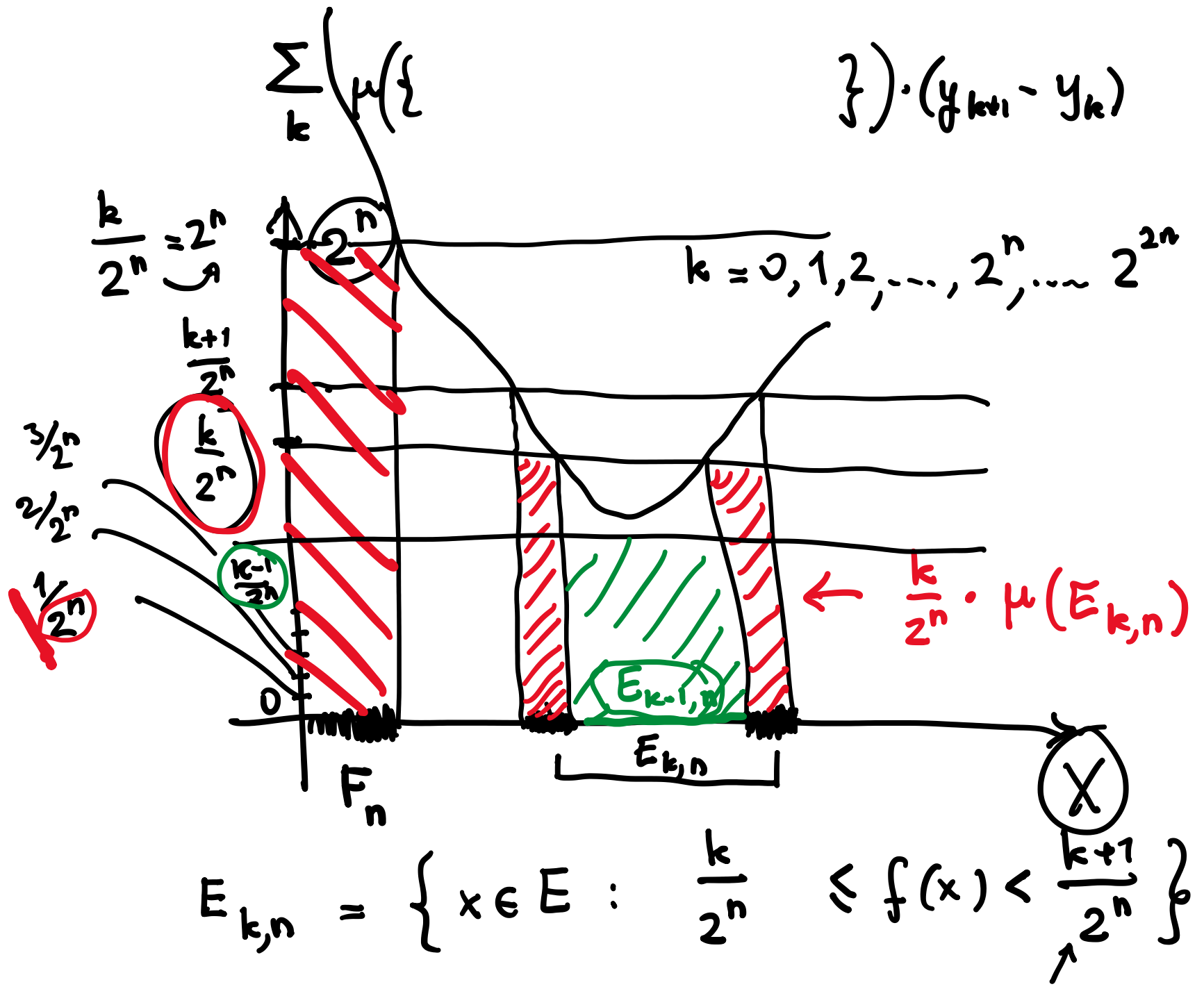


Lebesgue



$\{ x \in E : y_k \leq f(x) < y_{k+1} \} \subset E$

$\sum_k \mu(\{ \qquad \}) \cdot (y_{k+1} - y_k)$



$E_{k,n} = \left\{ x \in E : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\}$

$k = 0, \dots, 2^{2n} - 1$

$= \left\{ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \right\} \in \mathcal{F}$ if $f \in L(E)$

$F_n = \{ x \in E : f(x) \geq 2^n \}$

$= \{ f \geq 2^n \} \in \mathcal{F}$ $f \in L(E)$

Let $f \in L(E)$

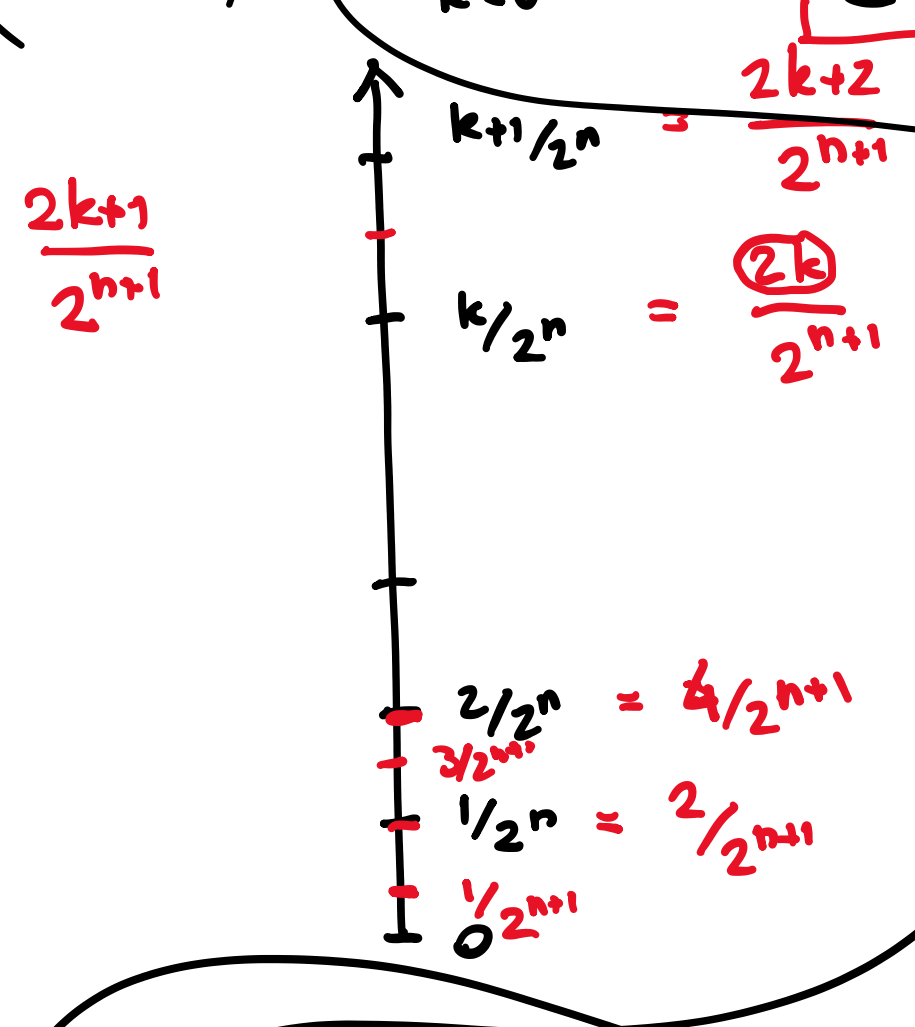
$I_n := \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \underline{\mu(E_{k,n})} + 2^n \mu(F_n)$

Thm: $I_n \nearrow$ $(I_n \leq I_{n+1} \ \forall n)$

so

$\exists \lim_{n \to +\infty} I_n =: \int_E f \, d\mu$.

$I_{n+1} = \sum_{k=0}^{2^{2(n+1)}-1} \frac{k}{2^{n+1}} \mu(E_{k,n+1}) + 2^{n+1} \mu(F_{n+1})$



$\frac{2k}{2^{n+1}} \mu(E_{2k,n+1}) +$

$\frac{2k+1}{2^{n+1}} \mu(E_{2k+1,n+1})$

$= \frac{k}{2^n} \mu\left(\frac{2k}{2^{n+1}} \leq f < \frac{2k+1}{2^{n+1}} \right)$

$\boxed{\begin{array}{l} \frac{k}{2^n} \mu\left(\frac{k}{2^n} \leq f < \frac{2k+1}{2^{n+1}} \right) + \\ = \frac{2k+1}{2^{n+1}} \mu\left(\frac{2k+1}{2^{n+1}} \leq f < \frac{2k+2}{2^{n+1}} \right) \end{array}}$

$\frac{2k}{2^{n+1}} = \frac{k}{2^n}$

$\geq \frac{k}{2^n} \mu\left(\frac{k}{2^n} \leq f < \frac{2k+1}{2^{n+1}} \right) +$
$\frac{k}{2^n} \mu\left(\frac{2k+1}{2^{n+1}} \leq f < \frac{k+1}{2^n} \right)$

$= \frac{k}{2^n} \mu\left(\qquad \cup \qquad \right)$

$= \frac{k}{2^n} \mu\Big(\underbrace{\frac{k}{2^n} \leq f < \frac{k+1}{2^n}}_{E_{k,n}} \Big)$

$\underbrace{\sum_{k=0}^{2^{2(n+1)}-1} \frac{k}{2^{n+1}} \mu(E_{k,n+1})}_{I_{n+1}} \geq \underbrace{\sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \mu(E_{k,n})}_{I_n}$

$2^{n+1} \underset{f \geq 2^{n+1}}{\mu(F_{n+1})} \geq 2^n \mu(f \geq 2^n)$

$\Rightarrow I_{n+1} \geq I_n \Rightarrow \exists \lim I_n$ ∎