Preliminaries on CALCULUS

- $f : \mathbb{R}^n \to \mathbb{R}$ $\quad \nabla f = Df = (\partial_{x_1} f, \dots \partial_{x_n} f)$ gradient f

  $D^2 f =$ Hessian of f

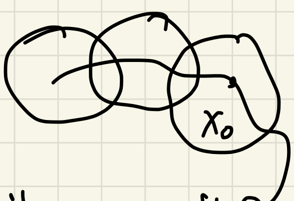
  $\Delta f = \text{trace}(D^2 f) = \sum_{i=1}^{n} \partial_{x_i x_i} f = \text{div}(Df)$

- $F : \mathbb{R}^n \to \mathbb{R}^n$ $\quad \text{div}\, F = \sum_i \partial_{x_i} F_i$

U bounded set of class C^1 $\to$ ∂U is compact $\Rightarrow \exists\, x_0^1 \dots x_0^N \in \partial U$

$\partial U \subseteq \bigcup_{i=1}^{N} B(x_0^i, r_{x_0^i})$ $\quad \forall i = 1 \dots N$

$\partial U \cap B(x_0^i, r_{x_0^i})$ is the graph of γ^i $\to dS = \sqrt{1 + |\nabla \gamma_i|^2}\, dx_1 \dots dx_{n-1}$



$$\int_{\partial U \cap B(x_0^i, r_{x_0^i})} f(x)\, dS = \int_{B' \subseteq \mathbb{R}^{n-1}} f(x_1 \dots x_{n-1}, \gamma_i(x_1 \dots x_{n-1})) \sqrt{1 + |\nabla \gamma_i|^2}\, dx_1 \dots dx_{n-1}$$

it is possible to define integration of $(n-1)$ Hypersurfaces (by PARAMETRIZATION)

- **GAUSS - GREEN THM:** $f \in \mathcal{C}^1(\bar{U})$

$$\int_U \frac{\partial}{\partial x_i} f \, dx = \int_{\partial U} f \cdot \nu_i(x) \, dS$$

DIVERGENCE THM $F \in \mathcal{C}^1(\bar{U}; \mathbb{R}^n)$

$$\int_U \operatorname{div} F \, dx = \int_{\partial U} F(x) \cdot \nu(x) \, dS$$

(related formulas)

$$\int_U \Delta f \, dx = \int_{\partial U} Df \cdot \nu(x) \, dS$$

$$\int_U \Delta f \cdot g - \Delta g \cdot f \, dx = \int_{\partial U} g \, Df \cdot \nu - f \cdot Dg \cdot \nu \, dS$$

INTEGRATION on SPHERES (special case of CO-AREA formula ch. 2.7 Folland)

$S^{n-1} = \{x \in \mathbb{R}^n, \ |x| = 1\}$ sphere $\qquad S^{n-1}_r = \{x \in \mathbb{R}^n, \ |x| = r\}$

$\sigma(S^{n-1})$ = area of the sphere $= \int_{S^{n-1}} dS$

"POLAR COORDINATES"

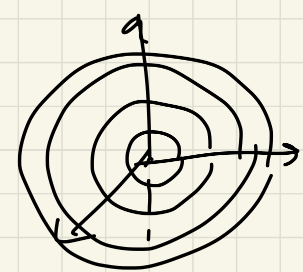
$\phi : x \in \mathbb{R}^n \longrightarrow (r, x') \in [0, +\infty) \times S^{n-1}$ continuous bijection

$r = |x| \qquad x' = \frac{x}{|x|} \quad x \neq 0 \qquad x = r x'$

$f : \mathbb{R}^n \to \mathbb{R}$ measurable ($f \geq 0$ or $f \in L^1(\mathbb{R}^n)$)

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^{+\infty} r^{n-1} \int_{S^{n-1}} f(r x')\,dS(x')\,dr$$

$$\left(= \int_0^{+\infty} \int_{S^{n-1}_r} f(y)\,dS(y)\,dr \right)$$



Obs if $f(x) = \phi(|x|)$ (that is: f has radial symm.)

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^{+\infty} r^{n-1}\phi(r)\,\mathcal{H}(S^{n-1})\,dr$$

Ex $f(x) = \frac{1}{|x|^\alpha}$

$f \in L^1(B(0,1)) \iff \alpha < n$

$$\int_{\mathbb{R}^n} \chi_{B(0,1)} f(x)\,dx = \int_0^1 \mathcal{H}(S^{n-1})\, r^{n-1} \frac{1}{r^\alpha}\,dr < +\infty \iff n-\alpha > 0$$

$\alpha < n$

Ex $f(x) = \begin{cases} \frac{1}{|x|^\alpha} & |x| > 1 \\ 1 & |x| \le 1 \end{cases}$ $f \in L^1(\mathbb{R}^n) \iff \alpha > n$

$$\int_{\mathbb{R}^n} f(x)\,dx = |B(0,1)| + \int_1^{+\infty} \mathcal{H}(S^{n-1}) \frac{r^{n-1}}{r^\alpha}\,dr < +\infty \iff n-\alpha \cancel{>} < 0$$

$\alpha > n$

COMPUTATION of $\mathcal{A}(S^{n-1})$, based $|B(0,1)|$ = volume of $B(0,1)$

$\omega_n = |B(0,1)|$ n-dim volume of the ball $|B(0,r)| = r^n \omega_n$

$\mathcal{A}(S^{n-1})$ = area of the n-dim sphere $\mathcal{A}(r S^{n-1}) = r^{n-1} \mathcal{A}(S^{n-1})$

$$\int_{\mathbb{R}^n} \chi_{B(0,1)}(x)\,dx = \omega_n = \int_0^1 r^{n-1}\left(\int_{S^{n-1}} dS\right) dr = \frac{1}{n}\mathcal{A}(S^{n-1})$$

$$\mathcal{A}(S^{n-1}) = n\,\omega_n$$

integration on spheres

$$\underset{\substack{\downarrow \\ \text{by symmetry \& reduction to } n=2}}{\pi^{n/2}} = \int_{\mathbb{R}^n} e^{-|x|^2} dx \overset{\uparrow}{=} \int_0^{+\infty} e^{-r^2} r^{n-1} \mathcal{A}(S^{n}) \, dr =$$

$$= \underbrace{\mathcal{A}(S^{n-1})}\ \underbrace{\frac{1}{2}\int_0^{+\infty} e^{-s} s^{\frac{n}{2}-1} ds} = n\omega_n \frac{1}{2}\Gamma\left(\frac{n}{2}\right)$$

where Γ is the Gamma-function (by Euler)

$$\Gamma(t) = \int_0^{+\infty} e^{-s} s^{t-1} ds \qquad \left[\begin{array}{l}\Gamma(t+1) = t\,\Gamma(t) \quad t>0 \\ \Gamma(1) = \Gamma(2) = 1\end{array}\right. \qquad \lim_{t\to+\infty}\Gamma(t) = +\infty \qquad \Gamma(n) = (n-1)!$$

$$\mathcal{L}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \qquad \omega_n = \frac{2\pi^{n/2}}{n\,\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$$

Ref:
EVANS - GARIEPY ch 1
FOLLAND ch 1, 2

(GEOMETRIC) MEASURE THEORY

X $(=\mathbb{R}^n)$ (or X locally compact Hausdorff topological space which is $\underbrace{\sigma\text{-compact}}$ $\to X$ countable union of compact sets)

Σ σ-algebra $\subseteq \mathcal{P}(X)$ (contains $\emptyset$, closed by complement & countable union)

$\mathcal{B}$ Borel σ-algebra (smallest σ-algebra which contains all the open sets)

μ Borelian measure $\mu : \mathcal{B} \to [0, +\infty]$

$\mu(\emptyset) = 0$

$\mu(\cup_i A_i) = \Sigma_i \mu(A_i)$
$A_i \cap A_j = \emptyset$.

μ is finite if $\mu(\mathbb{R}^n) < +\infty$

μ is σ-finite if $\mathbb{R}^n = \bigcup_{i=1}^{\infty} X_i \qquad \mu(X_i) < +\infty$

Def: μ is a RADON MEASURE if it is $\boxed{\text{BOREL MEASURE} \\ \text{LOCALLY FINITE}}$, that is $\mu(K) < +\infty \quad \forall K \subseteq \mathbb{R}^n$ compact

Prop: Let μ be a Radon measure on $\mathbb{R}^n$. (Evans-Gariepy Theorem 4 chapter 1)
Then μ is REGULAR:

$$\forall E \in \mathcal{B} \qquad \mu(E) = \inf\{\mu(U) \ U \text{ open} \quad U \supseteq E\}$$
$$\overset{!}{=} \sup\{\mu(K) \ K \text{ compact} \quad K \subseteq E\}$$

$\mathcal{L}$ Lebesgue measure is the unique Radon measure which is INVARIANT by TRANSLATIONS $(|A| = |A+x|)$ and n-HOMOGENEOUS $\quad |\lambda A| = \lambda^n |A| \qquad \lambda > 0$

Let μ be a Borel measure $\mu : \mathcal{B} \to [0, +\infty]$

$\mathcal{N} = \{ A \in \mathcal{B} , \mu(A) = 0 \}$ Borel sets of null measure

(Folland Ch. 1)

$\overline{\mathcal{B}}^{\mu}$ = completion of $\mathcal{B}$ w.r.to μ =

$= \{ E \cup F , E \in \mathcal{B} , \exists N \in \mathcal{N}$ s.that $F \subseteq N \}$

(is a σ-algebra)

$\bar{\mu} : \overline{\mathcal{B}}^{\mu} \to [0, +\infty]$ is the unique extension of μ

($\bar{\mu}$ is called COMPLETE : if $\bar{\mu}(E) = 0$ then $\bar{\mu}(F) = 0 \ \forall F \subset E$)

Let μ be a Borel measure $\mu : \overline{\mathcal{B}}^{\mu} \to [0, +\infty]$

$f : \mathbb{R}^n \to \mathbb{R}$ is μ-measurable if $f^{-1}(a, +\infty) \subseteq \overline{\mathcal{B}}^{\mu}$

$\phi(x) = \sum_{i=1}^{k} c_i \chi_{A_i}(x)$ $\quad c_i \geq 0 \quad A_i \in \overline{\mathcal{B}}^{\mu}$ is a simple function

$$\int_{\mathbb{R}^n} \phi(x) \, d\mu = \sum_{i=1}^{k} c_i \, \mu(A_i)$$

Let $f \geq 0$ $\quad \int_{\mathbb{R}^n} f \, d\mu = \sup \left\{ \int_{\mathbb{R}^n} \phi(x) \, d\mu \quad \phi \text{ simple } 0 \leq \phi \leq f \right\}$

then we extend to f generic $\quad f = f^+ - f^-$

$f^+ = \max(f, 0) \quad f^- = \max(-f, 0) \quad \int_{\mathbb{R}^n} f \, d\mu = \int_{\mathbb{R}^n} f^+ d\mu - \int_{\mathbb{R}^n} f^- d\mu \ldots$

$\Rightarrow$ we may give meaning to

$$\int_{\mathbb{R}^n} f \, d\mu \quad \text{for } f \ \mu\text{-measurable.}$$