Prop: Let $(\mathbb{R}^k, \mathcal{M}_k, \lambda_k)$, $f \in \mathcal{C}(\mathbb{R}^k)$ is meas.

Proof:

$\{f > a\} = \{x : f(x) > a\}$

$= \{f(x) \in]a, +\infty[\}$

$= f^{-1}(\overset{\text{open}}{]a, +\infty[})$

Since $f \in \mathcal{C}$ we know that $f^{-1}(A)$ is open $\forall A \subset \mathbb{R}$ is open. So $f^{-1}(]a,+\infty[)$ is open $\Rightarrow f^{-1}(]a,+\infty[) \in \mathcal{M}_k$ ■

Rmk: 1_E is meas $\Leftrightarrow$ E is meas.

So, if $(X, \mathcal{F}, \mu) = (\mathbb{R}, \mathcal{M}_1, \lambda_1)$

1_E is meas $\Leftrightarrow E \in \mathcal{M}_1$

$E = \mathbb{Q} \in \mathcal{M}_1$

$\ulcorner \mathbb{Q} \in \mathcal{M}_1$ because $\lambda^*(\mathbb{Q}) = 0$

Gen fact: any finite/count. set N has $\lambda^*(N) = 0$ $\Rightarrow N \in \mathcal{M}_k$

$\overset{N}{\shortparallel}$ countable

$\{x_n : n \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} \{x_n\}$

$0 \leq \boxed{\lambda^*(N)}$

$= \lambda^*\left(\bigcup_n \{x_n\}\right) \textcircled{\leq} \sum_n \underset{0}{\underset{\shortparallel}{\lambda^*(\{x_n\})}} = \boxed{0}$

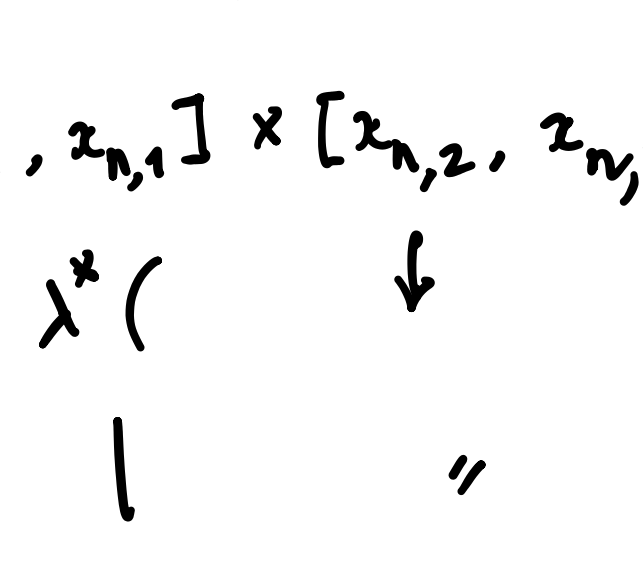
$x_n = (x_{n,1}, \dots, x_{n,k})$

$\{x_n\} = [x_{n,1}, x_{n,1}] \times [x_{n,2}, x_{n,2}] \times \dots$

$\lambda^*(\{x_n\}) = \lambda^*(\quad\downarrow\quad)$

$= | \quad " \quad | = 0$

$\Rightarrow \lambda^*(N) = 0$ $\lrcorner$



From this it follows that

$1_{\mathbb{Q}}$ is Leb-meas on $\mathbb{R}$.

$1_{\mathbb{R} \setminus \mathbb{Q}} = 1_{\mathbb{Q}^c}$ " " "

(Dirichlet funct)

Rmk: $1_{\mathbb{Q}}, 1_{\mathbb{Q}^c}$ are never cont.

Prop: Let $f, g : E \subset \mathbb{R}^k \longrightarrow \mathbb{R}$ be such that the set

$N := \{x \in E : f(x) \neq g(x)\}$

has L-meas $= 0$ $(\lambda_k(N) = 0)$.

Then f is L-meas $\Leftrightarrow$ g is L-meas.

Proof: Assume f meas $(\{f \in I\} \in \mathcal{M}_k \ \forall I \subset \mathbb{R}$ interv$)$

Th: g is meas.

$\{g \in I\} = \{x \in E : g(x) \in I\}$

$= \{x \in E \setminus N : f(x) \in I\}$

$\cup \underset{③}{\{x \in E \cap N : g(x) \in I\}}$

$\{f \in I\} = \{x \in E : f(x) \in I\}$

$= \underset{\text{by ass.}}{\underset{\in \mathcal{M}_k}{\underset{①}{\{x \in E : f(x) \in I\}}}} \setminus \underset{\subset N}{\underbrace{\{x \in E \cap N : f(x) \in I\}}_{②}} \in \mathcal{M}_k$

$0 \leq \lambda^*(②) \leq \lambda^*(N) = 0$

$\Downarrow$

$\lambda^*(②) = 0$

$\Downarrow$ key prop. of Leb class

② is L-meas

Similarly ③ $\in \mathcal{M}_k$

$\Rightarrow \{g \in I\} = ① \setminus ② \cup ③ \in \mathcal{M}_k$ ■

From this result we can draw a gen. fact.

Def: Given a meas sp $(X, \mathcal{F}, \mu)$, we say that $\mathcal{F}$ is complete if $\mathcal{F}$ contains all the subsets of any meas 0 set

Formally:

if $E \in \mathcal{F}$, $\mu(E) = 0 \Rightarrow \forall F \subset E$, $F \in \mathcal{F}$.

Rmk: The Leb-class $\mathcal{M}_k$ is complete (w.r.t. Leb meas)

Prop: Let $(X, \mathcal{F}, \mu)$ be a meas sp, $\mathcal{F}$ complete

Then, if $f, g : E \subset X \longrightarrow \mathbb{R}$ s.t.

$N = \{x \in E : f(x) \neq g(x)\}$

has $\mu(N) = 0$,

f is $\mathcal{F}$-meas $\Leftrightarrow$ g is $\mathcal{F}$-meas.

Proof: exercise ■

Rmk: On $\mathbb{R}^k$ an important σ-alg is

$\mathcal{B}_{\mathbb{R}^k} = \sigma(\{\text{open sets}\}) \subset \mathcal{M}_k$

(Borel σ-algebra) (Leb-class)

It can be proved that

i) $\mathcal{B}_{\mathbb{R}^k} \subsetneq \mathcal{M}_k$

ii) $\mathcal{B}_{\mathbb{R}^k}$ is not complete. ■

We write the cond

$\mu(\{x : f(x) \neq g(x)\}) = 0$

as

$f(x) = g(x)$ for almost every $x \in E$

or

$f(x) = g(x)$ a.e. x

or

$f = g$ a.e.

(p.p.) (French)

(almost surely)

a.s.

Prop: Let $(X, \mathcal{F}, \mu)$ be a meas sp.

$f, g : E \subset X \longrightarrow \mathbb{R}$, both meas. $(f, g \in L(E))$

(Notation: we write $f \in L(E)$ to say that f is a meas funct $E \to \mathbb{R}$)

Then:

i) $f \pm g \in L(E)$

ii) $f \cdot g \in L(E) \leftarrow$

iii) $f/g \in L(E)$ if $g \neq 0$.

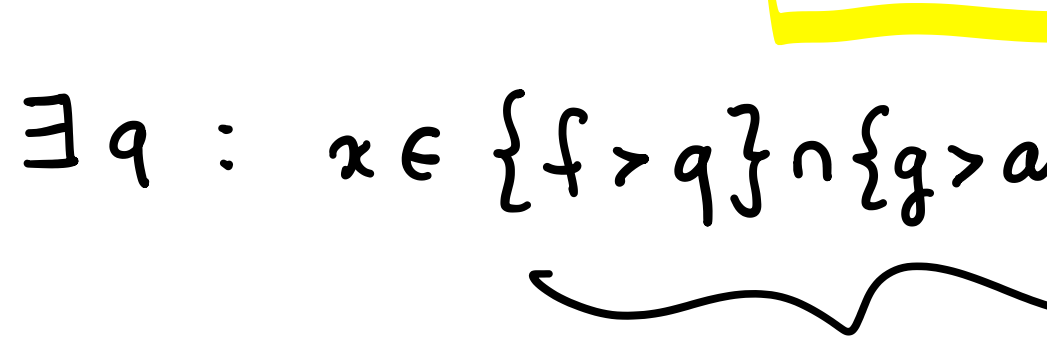
Proof: We prove that

$\{f + g > a\} \in \mathcal{F}$ $\forall a \in \mathbb{R}$.

(We know $\{f > \alpha\}, \{g > \beta\} \in \mathcal{F}$ $\forall \alpha, \beta)$

$\{f + g > a\} = \{f > a - g\}$

$= \{x \in E : f(x) > a - g(x)\}$



$x \in$ $\Rightarrow$ $f(x) > a - g(x)$

$\Rightarrow \exists q :$ $f(x) > q > a - g(x)$

$\boxed{f(x) > q}$, $a - g(x) < q$

$\Updownarrow$

$\boxed{g(x) > a - q}$

$\Rightarrow \exists q : x \in \underbrace{\{f > q\} \cap \{g > a - q\}}_{E_q \in \mathcal{F}}$

$\forall x \in \{f + g > a\} \Leftrightarrow x \in \bigcup_{q \in \mathbb{Q}} E_q \in \mathcal{F}$

$\Longrightarrow \{f + g > a\} = \bigcup_{q \in \mathbb{Q}} E_q \in \mathcal{F}$ $\forall a$.

$\{fg > a\} = \{f(x) g(x) > a\}$

$= \underline{\{g = 0, f \overset{0}{\overset{\shortparallel}{g}} > a\}} \cup \{g > 0 : \overset{f > \frac{a}{g}}{---}\}$

$\{g = 0, 0 > a\}$ $\cup \{g < 0 : --- \}$

■

Prop: Let $(X, \mathcal{F}, \mu)$ meas sp.

$f : E \longrightarrow \mathbb{R}$, $f \in L(E)$, $g : \mathbb{R} \to \mathbb{R}$, $g \in \mathcal{C}(\mathbb{R})$ (continuous)

$\Rightarrow g \circ f \in L(E)$ $(g \circ f(x) = g(f(x)))$

Warning:

So, for ex, if $f \in L(E)$

$f(x)^2$, $f(x)^3$, $e^{f(x)}$, $\sin(f(x)) \in L(E)$

$\shortparallel$ $\shortparallel$ $\shortparallel$ $\shortparallel$

$g(f(x))$ $g(f(x))$ $g(f(x))$

$g(y) = y^2$ $g(y) = y^3$ $g(y) = e^y$

$|f(x)| \in L(E)$

$\shortparallel$

$g(f(x))$ $g(y) = |y|$

but be careful: $|f|$ could be meas with $f \notin L$

$f(x) = 1_E - 1_{E^c}$ E is a non meas set

$= \begin{cases} 1 & x \in E \\ -1 & x \notin E \end{cases}$ $(E \subset X, E \notin \mathcal{F})$

Is $f \in L(X)$? Since f is simple $\Rightarrow$ f is meas $\Leftrightarrow E, E^c \in \mathcal{F}$ false

$\Rightarrow f \notin L(X)$

$|f| \equiv 1 \in L(X)$. ■

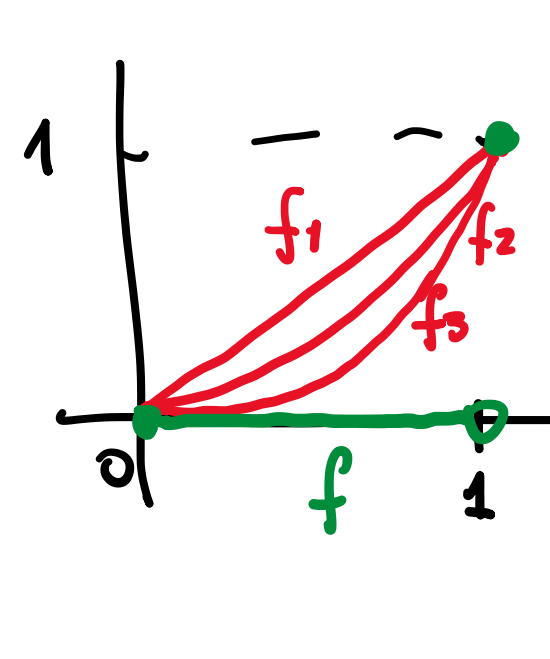
Thm: Let $(X, \mathcal{F}, \mu)$ be a meas sp. $(f_n)_n \subset L(E)$ be such that

$\forall x \in E \quad \exists \lim_{n \to +\infty} f_n(x) =: f(x)$

(point wise limit)

$\Rightarrow f \in L(E)$.

Ex: $f_n(x) = x^n$ $x \in [0,1] =: E$



$f_1(x) = x$

$f_2(x) = x^2$

$f_3(x) = x^3$

$$\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} x^n = \begin{cases} 1 & x = 1 \\ 0 & 0 \leq x < 1 \end{cases}$$

$=: f(x)$