Ex 1.8.8

Check that the following two do not:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Sol: $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ defn on $D = \mathbb{R}^2 \setminus \{(0,0)\}$, (0,0) not in D.

$f(x,0) = \frac{x^2}{x^2} = 1 \xrightarrow[x\to 0]{} 1$

$f(0,y) = \frac{-y^2}{y^2} = -1 \xrightarrow[y\to 0]{} -1$

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ DNE ■

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$f = \frac{xy}{\sqrt{x^2+y^2}}$ defn on $D = \mathbb{R}^2 \setminus \{(0,0)\}$

$f(x,x) = \frac{x^2}{\sqrt{2x^2}} = \frac{x^2}{\sqrt{2}|x|} = \frac{x}{\sqrt{2}}$

$\Rightarrow \lim_{x\to 0} f(x,x) = 0$

$f(x,-x) = \frac{-x^2}{\sqrt{2x^2}} = -\frac{x}{\sqrt{2}}$

$\Rightarrow \lim_{x\to 0} f(x,-x) = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ DNE ■

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2+1}{x^2+y^2-1}$

Here $f(x,y) = \frac{x^2+y^2+1}{x^2+y^2-1}$ is defn on $D = \mathbb{R}^2 \setminus \{(x,y) : x^2+y^2-1 = 0\}$

$= \mathbb{R}^2 \setminus \{(x,y) : x^2+y^2 = 1\}$

$f(x,0) = \frac{x^2+1}{x^2-1} \xrightarrow[x\to 0]{} -1$

$f(0,y) = \frac{y^2+1}{y^2-1} \xrightarrow[y\to 0]{} -1$

$f(x,x) = \frac{2x^2+1}{2x^2-1} \xrightarrow[x\to 0]{} -1$

$\Rightarrow \lim_{(x,y)\to(0,0)} f = -1$ ■

$f(x,mx) \xrightarrow[x\to 0]{} \frac{1+m^2}{-1} \neq -1$ if $m \neq 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ DNE ■

Ex 1.8.8

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)^2}$

Here $f = \frac{x^2y}{(x^2+y^2)^2}$ is defn on $D = \{(x,y) : x^2+y^2 \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$f(x,0) = \frac{0}{x^4} = 0 \xrightarrow[x\to 0]{} 0$

$f(0,y) = \frac{0}{y^4} = 0 \xrightarrow[y\to 0]{} 0$

$f(x,x) = \frac{x^3}{(2x^2)^2} = \frac{x^3}{4x^4} = \frac{1}{4x} \xrightarrow[x\to 0]{} \pm\infty$

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ DNE ■

$f(x,mx) \xrightarrow[x\to 0]{} \frac{mx^3}{(x^2+m^2x^2)^2} = \frac{mx^3}{x^4(1+m^2)^2} = \frac{m}{x(1+m^2)^2} \xrightarrow[x\to 0]{} \pm\infty$ if $m \neq 0$

$f(x,0) = 0 \xrightarrow[x\to 0]{} 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ DNE ■

(b) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

Here $f = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is defn on $D = \{(x,y) : x^2+y^2 \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$f(x,0) = \frac{\sin x^2}{x^2} \xrightarrow[x\to 0]{} 1$ (by standard limit)

$f(0,y) = \frac{\sin y^2}{y^2} \xrightarrow[y\to 0]{} 1$ (by standard limit)

$f(x,x) = \frac{\sin 2x^2}{2x^2} \xrightarrow[x\to 0]{} 1$ (by standard limit)

$\Rightarrow \lim_{(x,y)\to(0,0)} f = 1$ ■

Ex 1.8.8

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

Here $f = \frac{x^2y}{x^4+y^2}$ is defn on $D = \mathbb{R}^2 \setminus \{(0,0)\}$

$f(x,0) = \frac{0}{x^4} = 0 \xrightarrow[x\to 0]{} 0$

$f(0,y) = \frac{0}{y^2} = 0 \xrightarrow[y\to 0]{} 0$

$f(x,x) = \frac{x^3}{x^4+x^2} = \frac{x}{x^2+1} \xrightarrow[x\to 0]{} 0$

$f(x,mx) \xrightarrow[x\to 0]{} \frac{mx^3}{x^4+m^2x^2} = \frac{mx}{x^2+m^2} \xrightarrow[x\to 0]{} 0$

$f(x,x^2) = \frac{x^4}{x^4+x^4} = \frac{1}{2} \xrightarrow[x\to 0]{} \frac{1}{2}$

$\Rightarrow \lim_{(x,y)\to(0,0)} f$ DNE ■

(b) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

Here $f = \frac{x^2y^2}{x^2+y^2}$ is defn on $D = \mathbb{R}^2 \setminus \{(0,0)\}$

$f(x,0) = \frac{0}{x^2} = 0 \xrightarrow[x\to 0]{} 0$

$f(0,y) = \frac{0}{y^2} = 0 \xrightarrow[y\to 0]{} 0$

$f(x,x) = \frac{x^4}{2x^2} = \frac{x^2}{2} \xrightarrow[x\to 0]{} 0$

$f(x,mx) \xrightarrow[x\to 0]{} \frac{m^2x^4}{x^2+m^2x^2} = \frac{m^2x^2}{1+m^2} \xrightarrow[x\to 0]{} 0$

$f(x,x^3) = \frac{x^8}{x^2+x^6} = \frac{x^6}{1+x^4} \xrightarrow[x\to 0]{} 0$

We cannot find a path to prove $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

we have to obtain a bound

$|f(x,y)| \leq g(x,y,z)$

s.t. $g(x) \to 0$ when $x \to 0$

To prove $\lim f = L$ if $\forall \varepsilon \in \mathbb{R}$ we need to bound

$|f(x,y) - L| \leq g(x,y,z)$ with $g(x) \xrightarrow[x\to 0]{} 0$

Thus if

$|f(x) - L| \leq g(x) \xrightarrow[x\to 0]{} 0$

not known

$\Rightarrow |f(x) - L| \xrightarrow[x\to 0]{} 0$

i.e. $f(x) \xrightarrow[x\to 0]{} L$

Ex 1.8.8

(a) $\lim_{(x,y)\to(0,0)} \frac{\ln(1+y^2)}{\ln(1+x^2)}$

Sol: $f(x,y) = \frac{\ln(1+y^2)}{\ln(1+x^2)}$ defn on $D = \{(x,y) : \ln(1+x^2) \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$\frac{\ln(1+y^2)}{\ln(1+x^2)} \to 0$ as $\ln(1+y^2) \to 0$

i.e. as $y \to 0$

$f(x,y) \xrightarrow[y\to 0]{} \frac{y^2 \text{ as } y\to 0}{y^2 \text{ as } y\to 0}$ ($\ln(1+x) \sim x$ as $x \to 0$)

$= 1$ as $y \to 0$ (a)

Now I claim $\forall (x,0) \in D$ (otherwise $\ln(1+x^2) = 0$)

$\ln(1+x^2) = \ln\left(\frac{1+x^2}{1}\right)$

$= \ln\left(\frac{1+x^2}{1+0}\right)$

$= \ln\left(\frac{1+x^2}{1+0^2}\right)$ (b)

$\forall (0,0) \neq (x,0) \in D$ we need to show $\ln\left(\frac{1+x^2}{1+0^2}\right) \neq 0$

$\Rightarrow \frac{1+x^2}{1+0^2} \neq 1$ (because $\ln 1 = 0$)

$\Rightarrow 1+x^2 \neq 1+0^2$

$\Rightarrow x^2 \neq 0$

$\Rightarrow x \neq 0$ (because $\ln x$ is one-one)

$\Rightarrow (x,0) \neq (0,0)$

$\Rightarrow \forall (x,0) \neq (0,0) \in D$ (b) holds

$\forall (0,0) \neq (x,0) \in D$ we have $\ln\left(\frac{1+x^2}{1+0^2}\right) \neq 0$

$\Rightarrow \frac{\ln(1+x^2)}{\ln(1+0^2)} \neq 1$

$\Rightarrow \frac{\ln(1+x^2)}{\ln(1+0^2)} \neq \frac{\ln(1+0^2)}{\ln(1+0^2)}$

$\Rightarrow f(x,0) \neq f(0,0)$ (equal)

This yields to prove $\exists \lim_{(x,y)\to(0,0)} f$ DNE

$\Rightarrow |f(x,y) - 0| = |f(x,y)|$

$= \left| \frac{\ln(1+y^2)}{\ln(1+x^2)} \right|$

$= \left| \frac{\ln(1+y^2)}{y^2} \cdot \frac{x^2}{\ln(1+x^2)} \cdot \frac{y^2}{x^2} \right|$ $\forall (x,y) \neq (0,0) \in D$

$\leq \left| \frac{\ln(1+y^2)}{y^2} \right| \cdot \left| \frac{x^2}{\ln(1+x^2)} \right| \cdot \left| \frac{y^2}{x^2} \right|$

$\leq C \cdot \left| \frac{y^2}{x^2} \right|$ $\exists C \in \mathbb{R}$ s.t. $\left| \frac{\ln(1+y^2)}{y^2} \right| \leq C$ (c)

$|f(x,y) - 0| \leq C \left| \frac{y^2}{x^2} \right|$

$\xrightarrow[(x,y)\to(0,0)]{} 0$ $\forall (x,y) \neq (0,0) \in D$

$\Rightarrow \lim_{(x,y)\to(0,0)} f = 0$. ■

Ex 1.8.8 (similar)

Ex 1.8.8

(a) $\lim_{(x,y)\to(0,0)} \frac{\sin x^2y}{x^2+y^2}$

f is defn on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Spherical Coords

$r = \sqrt{x^2+y^2}$

$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= r$

$(x,y,z) \to (0,0,0)$

$\Rightarrow r \to 0$ as $x \to 0, y \to 0, z \to 0$

Let's write f in spherical coords:

$f(x,y,z) = \frac{\sin(r^2\cos^2\theta \cdot r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$

$= \frac{\sin(r^3\cos^2\theta\sin\theta)}{r^2}$

$= \frac{\sin(r^3\cos^2\theta\sin\theta)}{r^3\cos^2\theta\sin\theta} \cdot \cos^2\theta\sin\theta$

$\to 1 \cdot 0 = 0$ as $r \to 0$

$f(x,y,z) = \left(\frac{\sin t}{t}\right)\cdot(\cos^2\theta\sin\theta) \to 0$ as $t \to 0$

$(x,y,z) \to (0,0,0) \Rightarrow f(x,y,z) \to 0$

$|f(x,y,z) - 0| = |f(x,y,z)|$

$= \left|\frac{\sin x^2y}{x^2+y^2}\right| \leq \left|\frac{x^2y}{x^2+y^2}\right| \leq |y|$

$\forall \varepsilon > 0$ $\exists \delta = \varepsilon$ s.t. $|y| < \delta \Rightarrow |f(x,y,z) - 0| < \varepsilon$

$\Rightarrow \lim_{(x,y)\to(0,0)} f = 0$ ■

Ex 1.8.8

(a) $\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)}{\sin(2xy)}$ (1)

Sol:

$D = \{(x,y) : 1+xy > 0\}$, $\sin(2xy) \neq 0$

$1+xy > 0 \Rightarrow xy > -1$

$\sin(2xy) \neq 0 \Rightarrow (xy) \neq k\pi$

$\Rightarrow xy \neq -1, 0, \pi, 2\pi, \ldots$

$\sin(2xy) = 0$ at $(xy) = k\pi$

$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$

$= z\left(1 - \frac{z}{2} + \frac{z^2}{3} - \ldots\right)$

$\Rightarrow \frac{\ln(1+z)}{z} = 1 - \frac{z}{2} + \frac{z^2}{3} - \ldots$

$\Rightarrow \lim_{z\to 0} \frac{\ln(1+z)}{z} = 1$

$\sin(2xy) = 2xy - \frac{(2xy)^3}{3!} + \frac{(2xy)^5}{5!} - \ldots$

$= 2xy\left(1 - \frac{(2xy)^2}{3!} + \frac{(2xy)^4}{5!} - \ldots\right)$

$\sin(2xy)$ is periodic. $\sin t = t + \sin(t)$

Let $t = 2xy$

$\sin(t) = 2xy - \frac{(2xy)^3}{3!} + \frac{(2xy)^5}{5!} - \ldots$

$= t\left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \ldots\right)$

$\Rightarrow \frac{\sin(t)}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \ldots$

$\Rightarrow \lim_{t\to 0} \frac{\sin(t)}{t} = 1$

$\Rightarrow \lim_{2xy\to 0} \frac{\sin(2xy)}{2xy} = 1$

$\Rightarrow \lim_{xy\to 0} \frac{\sin(2xy)}{2xy} = 1$

$\Rightarrow \lim_{xy\to 0} \frac{2xy}{\sin(2xy)} = 1$

$\Rightarrow \lim_{xy\to 0} \frac{\ln(1+xy)}{\sin(2xy)} = \lim_{xy\to 0} \frac{\ln(1+xy)}{xy} \cdot \frac{xy}{\sin(2xy)}$

$= 1 \cdot \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)}{\sin(2xy)} = \frac{1}{2}$ ■