Wednesday, 8 October 2025 12:26

Ex 1.8.7

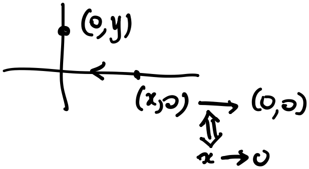
Check that the following lims do not $\exists$

1. $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}.$$

Sol: $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ defd on

$D = \mathbb{R}^2 \setminus \{(0,0)\}$, $(0,0) \in \operatorname{Acc} D$.

$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \xrightarrow[x\to 0]{} \boxed{1}$$



$$(x,0) \longrightarrow (0,0) \Updownarrow x \to 0$$

$$f(0,y) = \frac{-y^2}{y^2} \equiv -1 \xrightarrow[y\to 0]{} \boxed{-1}$$

$$\Rightarrow \nexists \lim_{(x,y)\to\vec{0}} f. \quad \blacksquare$$

2. $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{\underbrace{x^2+y^2}_{}}.$$

f is defd on $f(x,y)$

$D = \mathbb{R}^2 \setminus \{(0,0)\}$

$$f(x,0) = \frac{x^2}{x^2} \equiv 1 \xrightarrow[x \to 0]{} \boxed{1}$$

$$f(0,y) = y^3/y^2 = y \xrightarrow[y \to 0]{} \boxed{0}$$

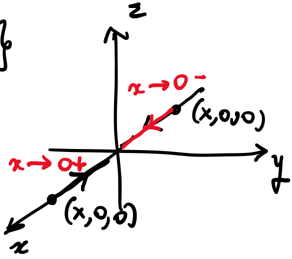
$$\Rightarrow \nexists \lim_{(0,0)} f.$$

4. $$\lim_{(x,y,z) \to \vec{0}} \frac{x+y^2+z^3}{\sqrt{\underbrace{x^2+y^2+z^2}_{\geq 0}}}.$$

Here $f(x,y,z)$ is defd

$$D = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \neq 0\}$$

$$= \mathbb{R}^3 \setminus \{(0,0,0)\}$$



$$f(x,0,0) = \frac{x}{\sqrt{x^2}} = \frac{x}{|x|} = \operatorname{sgn}(x).$$

$$\xrightarrow[x\to 0+]{} \boxed{+1}$$

$$\xrightarrow[x\to 0-]{} \boxed{-1}$$

$$\Rightarrow \nexists \lim_{(0,0)} f .$$

∎

6. (**)

$$\lim_{(x,y,z)\to \vec{0}} \underbrace{\frac{xz}{x^4+y^2+z^2}}_{f}$$

f is defd on $D = \{(x,y,z) : x^4+y^2+z^2 \neq 0\}$

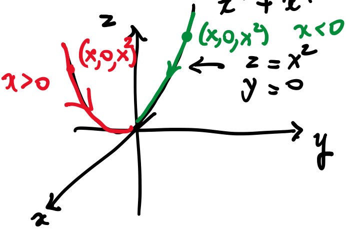
$$= \mathbb{R}^3 \setminus \{(0,0,0)\}$$

$$f(x,0,0) = \frac{0}{x^4} \equiv 0 \xrightarrow[x\to 0]{} \boxed{0}$$

$$f(x,0,z) = \frac{xz}{x^4+z^2}$$

$$f(x,0,x^2) = \frac{x\cdot x^2}{x^4+x^4} = \frac{x^3}{2x^4} = \frac{1}{2}\,\frac{1}{x}$$

$$\frac{1}{x} \xrightarrow[x\to 0+]{} +\infty \qquad \frac{1}{x} \xrightarrow[x\to 0-]{} -\infty$$



$$(x,0,x^2) \to (0,0,0)$$

$$\Updownarrow$$

$$x \to 0$$

$$\Rightarrow \nexists \lim_{(x,y,z)\to\vec{0}} f . \qquad \blacksquare$$

$$\ulcorner \; f(x,0,mx) = \frac{mx^2}{x^4+m^2x^2} \qquad m\neq 0$$

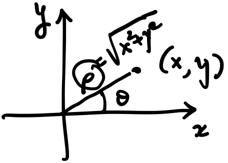
$$= \frac{m\cancel{x^2}}{\cancel{x^2}(m^2+x^2)} \xrightarrow[x\to 0]{} \frac{m}{m^2} = 1/m \; \lrcorner$$

Ex 1.8.8

1. (*) $$\lim_{(x,y)\to\vec{0}} \underbrace{\frac{\overset{\nearrow 0}{xy}}{\sqrt{x^2+y^2}\to 0}}_{f(x,y)} .$$

Here f is defd

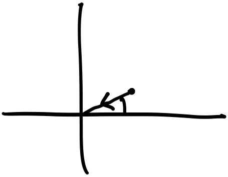
on $D=\{(x,y) : x^2+y^2=0\} = \mathbb{R}^2\setminus\{(0,0)\}$.

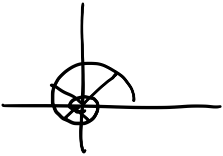


$$\boxed{\begin{array}{c} \underline{(x,y)} \longrightarrow (0,0) \\ \Updownarrow \\ x\to 0 \\ y\to 0 \end{array}}$$

$$\Updownarrow$$

$$\boxed{\rho \to 0 \,,\quad \theta \in [0, 2\pi[}$$





$$f(x,y) = \frac{(\rho\cos\theta)(\rho\sin\theta)}{\rho}$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$= \frac{\rho^{\cancel{2}}}{\cancel{\rho}} \cos\theta \sin\theta$$

$$|f(x,y) - 0| = |f(x,y)|$$

$$= |\rho \cos\theta \sin\theta|$$

$$= \rho \underbrace{|\cos\theta||\sin\theta|}_{\leq 1}$$

$$0 \leq |f(x,y) - 0| \leq \boxed{\rho = \|\underset{\substack{\downarrow \\ \vec{0}}}{(x,y)}\|} \longrightarrow 0$$

$$\Rightarrow \quad |f(x,y) - 0| \longrightarrow 0 \quad \Rightarrow f(x,y) \to 0.$$

■

Do #2

#3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

Here f is defd on $D = \mathbb{R}^2 \setminus \{(0,0)\}$

$$f(x,y) \underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta \\ x^2+y^2=\rho^2}}{=} \frac{\boxed{\rho^3}\cos^3\theta - \boxed{\rho^3}\sin^3\theta}{\rho^2}$$

$$= \underset{\downarrow \atop 0}{\boxed{\rho}} \overset{\text{bounded}}{\underbrace{\left(\cos^3\theta - \sin^3\theta\right)}_{-2 \leq \quad \leq 2}}$$

$$\underset{\downarrow \atop 0}{|f(x,y) - 0|} = |f(x,y)| = \rho \underbrace{|\cos^3\theta - \sin^3\theta|}_{\leq 2}$$

$$\boxed{\leq 2\rho = 2\|(x,y)\|} \xrightarrow[(x,y)\to(0,0)]{} 0$$

$$\Rightarrow \exists \lim_{(x,y)\to\boxed{(0,0)}} f = 0. \qquad \blacksquare$$

We learnt that, to prove

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

we have to obtain a bound

$$|f(x,y)| \leq g(\|(x,y)\|)$$

s.t. $g(\rho) \to 0$ when $\rho \to 0$

To prove $\lim f = \ell \in \mathbb{R}$ we need to bound

$$|f(x,y) - \ell| \leq g(\|(x,y)\|) \quad \text{s.t } \underset{\rho\to 0}{g(\rho)\to}$$

$$\Updownarrow$$

$$|f(\vec{x}) - \ell| \leq g(\|\vec{x}\|) \quad \text{with } g \text{ s.t } \underset{\rho\to 0}{g(\rho) \to 0}$$

Prop: If

$$\underbrace{|f(\vec{x}) - \ell|}_{\text{dist between } f(\vec{x}) \text{ and } \ell} \leq \underbrace{g(\|\vec{x}\|)}_{\text{a funct of } \|\vec{x}\|}$$

s.t $g(\rho) \to 0$ when $\rho \to 0$

$$\Rightarrow \quad \exists \lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \ell .$$

Ex 1.8.8.

#5. $\displaystyle \lim_{(x,y)\to\vec{0}} \frac{xy}{|x|+|y|}$.

Sol: $f(x,y)$ defd on

$$D = \{(x,y) : |x|+|y| \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}.$$

$$\underbrace{|x|}_{\geq 0} + \underbrace{|y|}_{\geq 0} = 0 \iff |x| = |y| = 0$$

$$\iff x = y = 0$$

$$f(x,y) \underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \frac{\rho^{\cancel{2}}\cos\theta\sin\theta}{\cancel{\rho}\left(|\cancel{\rho}\cos\theta| + |\cancel{\rho}\sin\theta|\right)}$$

$$= \rho \frac{\cos\theta\sin\theta}{\boxed{|\cos\theta| + |\sin\theta|}} . \quad (*)$$

$$\varphi(\theta) = |\cos\theta| + |\sin\theta| \geq K > 0$$

$$\forall \theta \in \boxed{[0,2\pi]}$$

φ is a cont funct of θ

$\Rightarrow$ (Weierstrass thm) $\exists \min \varphi$

$$\exists\, \theta_{min} \in [0, 2\pi] \,:\, \varphi(\theta) \geq \underbrace{\varphi(\theta_{min})}_{K}$$

Now I claim $\varphi(\theta_{min}) > 0$

(otherwise, if $\boxed{\varphi(\theta_{min}) = 0}$

$$\Downarrow$$

$$|\cos\theta_{min}| + |\sin\theta_{min}| = 0$$

$$\Downarrow$$

$$\begin{cases} \sin\theta_{min} = 0 \\ \cos\theta_{min} = 0 \end{cases} \quad \text{imposs!})$$

This yields to guess $\exists \lim\limits_{(x,y)\to\vec{0}} f = 0$

$$|f(x,y) - 0| = |f(x,y)|$$

$$= \left| \rho \, \frac{\cos\theta \sin\theta}{|\cos\theta| + |\sin\theta|} \right|$$

$$\overset{\rho \geq 0}{=} \rho \cdot \frac{\overset{\leq 1}{|\cos\theta|}\,\overset{\leq 1}{|\sin\theta|} \leq 1}{|\cos\theta| + |\sin\theta| \geq K}$$

$$\le \boxed{\rho \cdot 1/k} =: C$$

$$|f(x,y) - 0| \le \underbrace{\boxed{C\rho}} = g(\|(x,y)\|)$$

$$g(\rho) \longrightarrow 0 \qquad \rho \to 0$$

$$\Downarrow$$

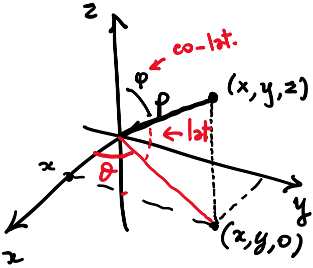
$$\exists \lim_{(x,y)\to\vec{0}} f = 0. \qquad \blacksquare$$

Do #4 (similar)

Ex 1.8.9

#2 $$\lim_{(x,y,z)\to\vec{0}} \underbrace{\frac{xyz}{x^2+y^2+z^2}}_{f(x,y,z)}.$$

f is defd on $\mathbb{R}^3 \setminus \{(0,0,0)\}$.



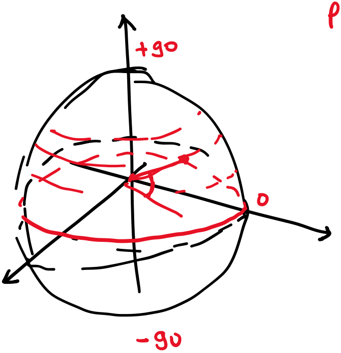
Spherical Coords.

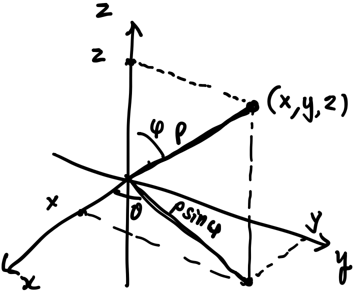
$$\rho = \sqrt{x^2+y^2+z^2}$$
$$= \|(x,y,z)\|$$

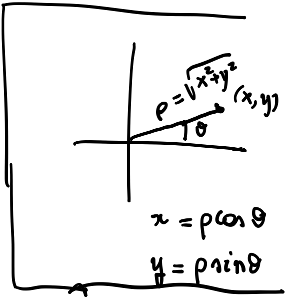
(ρ, φ, θ)

$\rho \geq 0, \quad 0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$







$x = \rho \cos\theta$

$y = \rho \sin\theta$

$$\begin{cases} x = \rho(\sin\varphi)(\cos\theta) \\ y = \rho(\sin\varphi)(\sin\theta) \\ z = \rho\cos\varphi \end{cases}$$

$$\boxed{x^2+y^2+z^2 = \rho^2}$$

Let's write f in spher. coords:

$$f(x,y,z) = \frac{\overset{x}{\textcircled{\rho}\sin\varphi\cos\theta} \cdot \overset{y}{\textcircled{\rho}\sin\varphi\sin\theta} \; \overset{z}{\textcircled{\rho}\cos\varphi}}{\rho^2}$$

$$= \frac{\rho^{\cancel{3}}}{\cancel{\rho^{2}}} \left(\sin^2\varphi \, \tfrac{1}{2} \, 2\underbrace{\cos\theta \, \sin\theta}_{\frac{1}{2}\sin(2\theta)} \cos\varphi \right)$$

$$-1 \leq \qquad\qquad\qquad\qquad \leq 1$$

$$f(x,y,z) = \rho \cdot \left(\sin^2\varphi \, \cos\theta \, \sin\theta \, \cos\varphi \right)$$

$$(x,y,z) \longrightarrow \vec{0} \iff \underbrace{\|(x,y,z)\|}_{\rho} \longrightarrow 0$$

$$|f(x,y,z) - 0| = |f(x,y,z)|$$

$$= \rho \, \underbrace{|\sin^2\varphi| \, |\cos\theta| \, |\sin\theta| \, |\cos\varphi|}_{\leq 1}$$

$$\leq \underset{\substack{\| \\ g(\rho)}}{\rho} = \|(x,y,z)\|$$

$$g(\rho) \longrightarrow 0 \qquad \rho \longrightarrow 0$$

$$\Rightarrow \exists \lim_{(x,y,z) \to \vec{0}} f = 0 \qquad \blacksquare$$

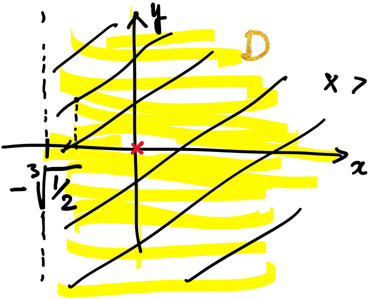
Do #3

$$\# 4 \quad \lim_{(x,y)\to\vec{0}} \underbrace{\frac{\log(1+2x^3)}{Sh(x^2+y^2)}}_{f(x,y)} \quad \left(\frac{0}{0}\right)$$

Sol :

$$D = \{\boxed{(x,y)} : 1+2x^3>0, \quad Sh(x^2+y^2)\neq 0\}$$

$$1+2x^3>0 \iff \underbrace{x^{\cancel{3}} > -\sqrt[3]{1/2}}$$



$$x > -\sqrt[3]{1/2} .$$

$$Sh(x^2+y^2) = 0 \iff x^2+y^2=0 \iff \boxed{x=y=0}$$

$$Sh(x^2+y^2) \neq 0 \iff (x,y)\neq(0,0)$$

$$\log(1+2x^3) \underset{\substack{x=\rho\cos\theta \\ y=\rho\sin\theta}}{=} \log(1+\underbrace{2\rho^3\cos^3\theta}_{\downarrow \atop 0})$$

$$\log(1+t) = t + o(t) \qquad t\to 0$$

$$= \boxed{2}\rho^3\cos^3\theta + o(\rho^3 \cancel{\cos^3\theta})$$

$$= 2\rho^3\cos^3\theta + o(\rho^3)$$

$$\mathrm{Sh}(x^2+y^2) \underset{\text{pol coords.}}{=} \mathrm{Sh}\,\rho^2 = \rho^2 + o(\rho^2)$$

$$\mathrm{Sh}\,t = t + o(t)$$

$$f(x,y) = \frac{2\rho^3\cos^3\theta + o(\rho^3)}{\rho^2 + o(\rho^2)}$$

$$= \frac{\rho^{\cancel{3}}\left(2\cos^3\theta + \overset{0}{\uparrow}\frac{o(\rho^3)}{\rho^3}\right)}{\cancel{\rho^2}\left(1 + \underset{0}{\downarrow}\frac{o(\rho^2)}{\rho^2}\right)}$$

$$= \rho\,\frac{2\cos^3\theta + \varepsilon_\rho^{\nearrow 0}}{1 + \underset{0}{\downarrow}\varepsilon_\rho} \xrightarrow{?} 0$$

$$|f - 0| \le$$