Tuesday, 3 October 2023   08:40

Rmk: We say that

$$\exists \lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}) = \vec{l}$$

$$\Leftrightarrow \boxed{\begin{array}{l} \forall (\vec{x}_n) \subset D,\ \vec{x}_n \neq \vec{p},\ \vec{x}_n \to \vec{p} \\ \Rightarrow \vec{F}(\vec{x}_n) \longrightarrow \vec{l}. \end{array}}$$

In part we have that

if $\exists (\vec{x}_n), (\vec{y}_n) \subset D$, $\vec{x}_n \neq \vec{p}$, $\vec{y}_n \neq \vec{p}$

$\vec{x}_n \to \vec{p}$, $\vec{y}_n \to \vec{p}$

s.t

$\vec{F}(\vec{x}_n) \longrightarrow \vec{l}_1$, $\vec{F}(\vec{y}_n) \longrightarrow \vec{l}_2$

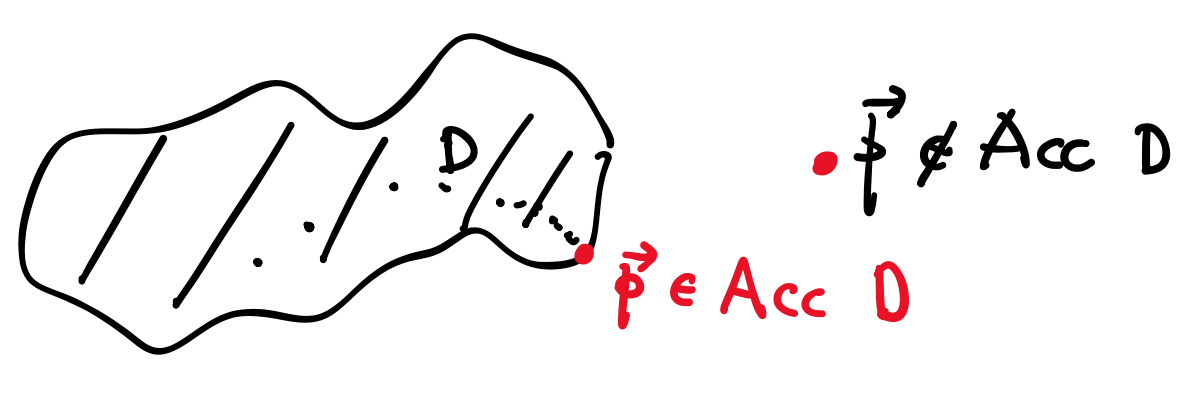
with $\vec{l}_1 \neq \vec{l}_2$

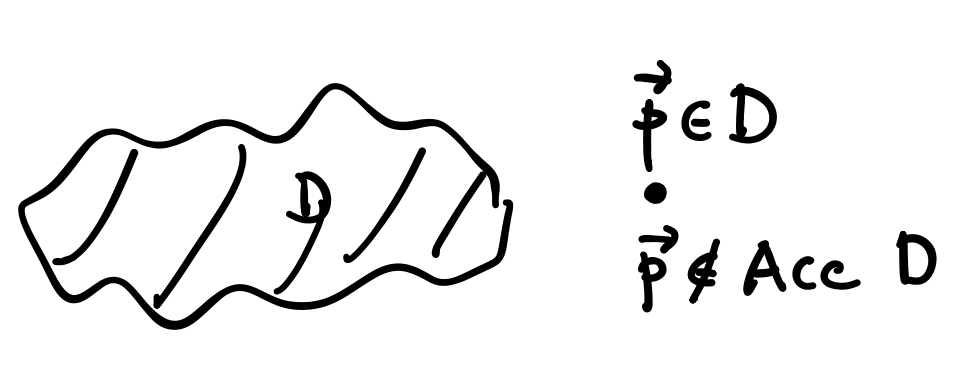
$$\Rightarrow \nexists \lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}).$$

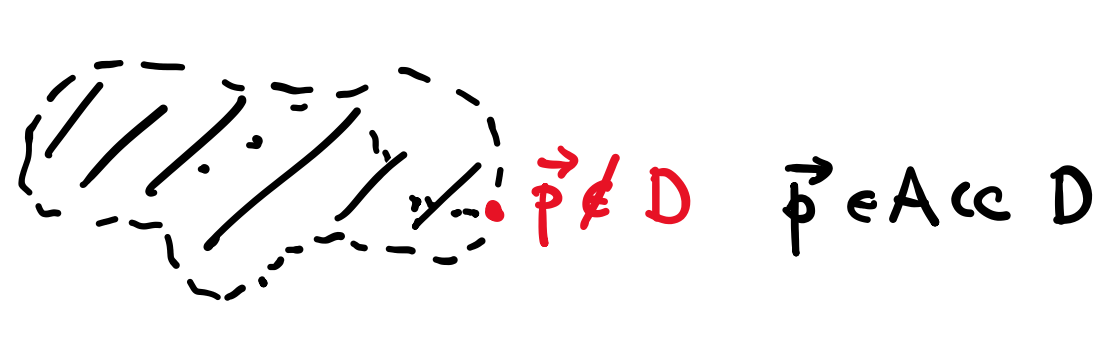
To complete the def of lim. we need to say what does it mean $\vec{p} \in \text{Acc } D$

Def: We say that $\vec{p} \in \mathbb{R}^d \cup \{\infty_d\}$ is an accumulation point for $D \subset \mathbb{R}^d$ if

$$\exists (\vec{x}_n) \subset D : \vec{x}_n \neq \vec{p},\ \vec{x}_n \to \vec{p}$$







Ex 1.4.1 (*) Discuss existence and value (if any) of

$$\lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2}.$$

Sol: Let

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad (x,y) \in D$$

where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\}$$

$x^2+y^2 = 0 \Leftrightarrow x = y = 0$

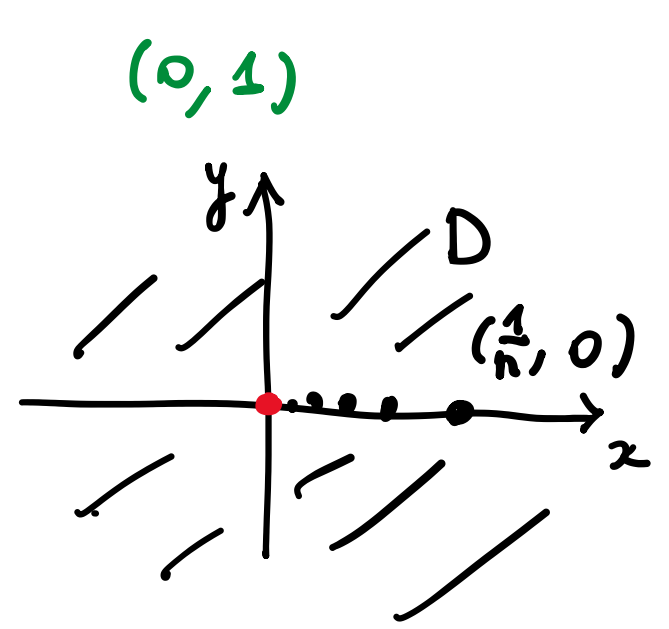
$$\Rightarrow D = \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$$

$\Updownarrow$

(1) $\boxed{x \neq 0 \ \vee \ y \neq 0}$ ←

(2) $\boxed{x \neq 0 \ \wedge \ y \neq 0}$ ←

(1) correct ✓
(2) correct ✗
both wrong ✗
both correct ✗

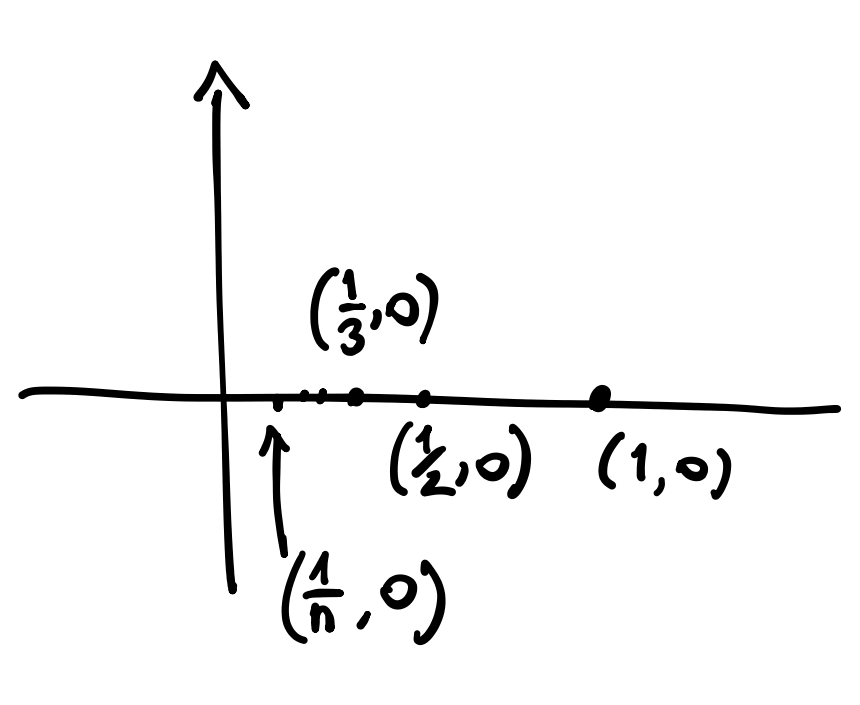


$$D = \mathbb{R}^2 \setminus \{(0,0)\}$$

We have to discuss

$$\lim_{\substack{(x,y) \to (0,0) \\ \| \\ \vec{p}}} f$$

Let's check that $\vec{p} \in \text{Acc } D$.

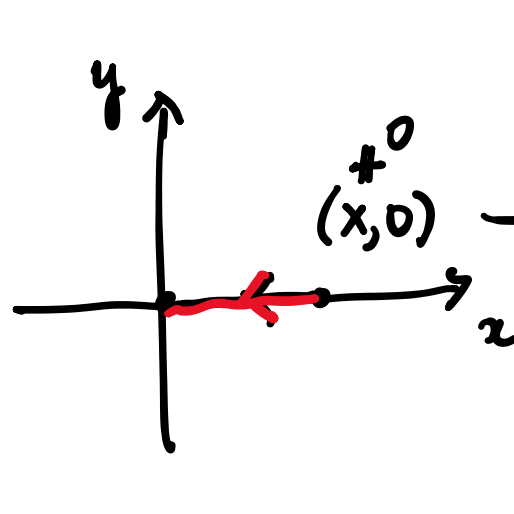


$\vec{x}_n = \left(\frac{1}{n}, 0\right) \in D$, $\vec{x}_n \neq \underset{(\vec{p})}{\vec{0}}$, $\vec{x}_n \longrightarrow \vec{0}$
$\quad\ \ \downarrow \ \ \downarrow$
$\quad\ \ 0 \ \ \ 0$

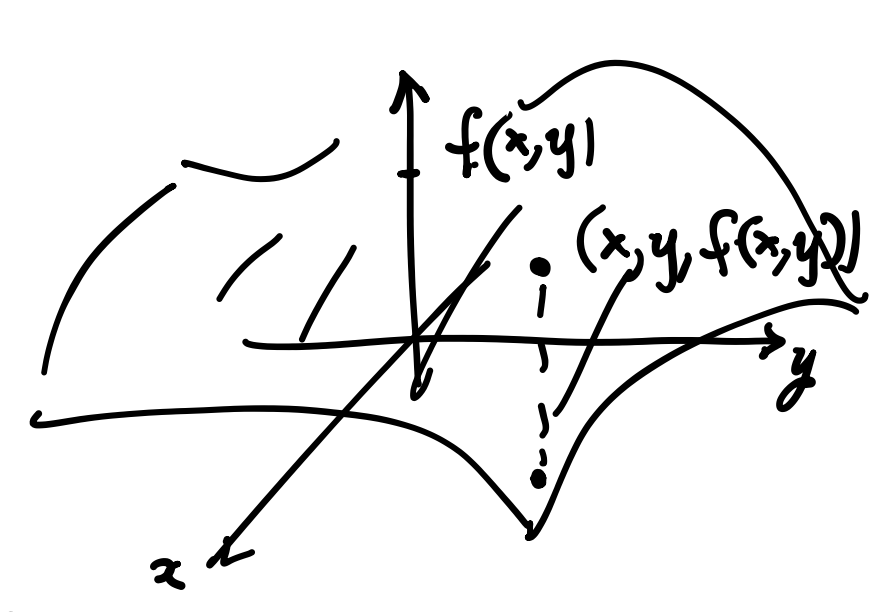
$\Rightarrow \vec{0} \in \text{Acc } D$ and limit op. makes sense.

$$\lim_{(x,y) \to (0,0)} \frac{\overset{\nearrow 0}{\boxed{xy}}}{x^2+y^2 \to 0} \qquad \left(\frac{0}{0}\right) \text{ I.F.}$$

$(x,y) \to (0,0) \Leftrightarrow \begin{array}{l} x \to 0 \\ y \to 0 \end{array}$

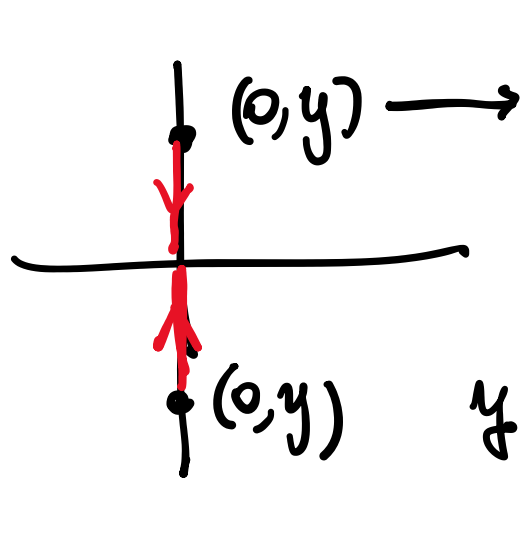


$\longrightarrow (0,0) \Leftrightarrow x \to 0$



$$f(x,0) = \frac{x \cdot 0}{x^2 + 0^2} = \frac{0}{x^2} = 0$$

Since for $x \to 0$, $f(x,0) \equiv 0 \longrightarrow 0$

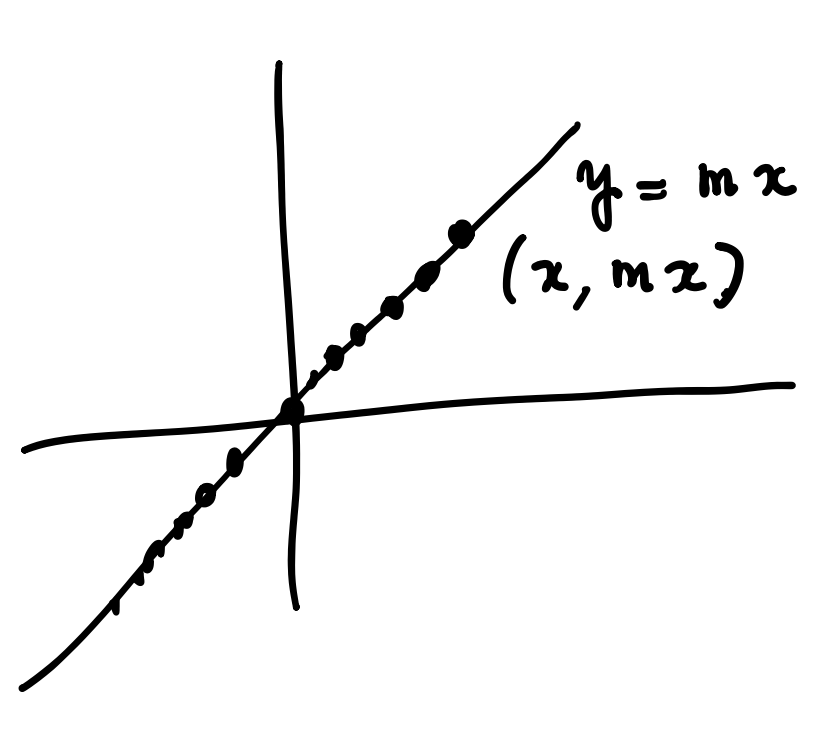


$(0,y) \longrightarrow (0,0) \Leftrightarrow y \to 0$

$y \neq 0$

$$f(0,y) = \frac{0 \cdot y}{0^2 + y^2} = \frac{0}{y^2} = 0$$

$f(0,y) \longrightarrow 0 \qquad y \longrightarrow 0.$



$m \in \mathbb{R}$ fixed
$x \neq 0$

$(x, mx) \longrightarrow (0,0)$ iff $x \longrightarrow 0$

$$f(x,mx) = \frac{x \cdot mx}{x^2 + (mx)^2} = \frac{m\cancel{x^2}}{\cancel{x^2}(1+m^2)}$$

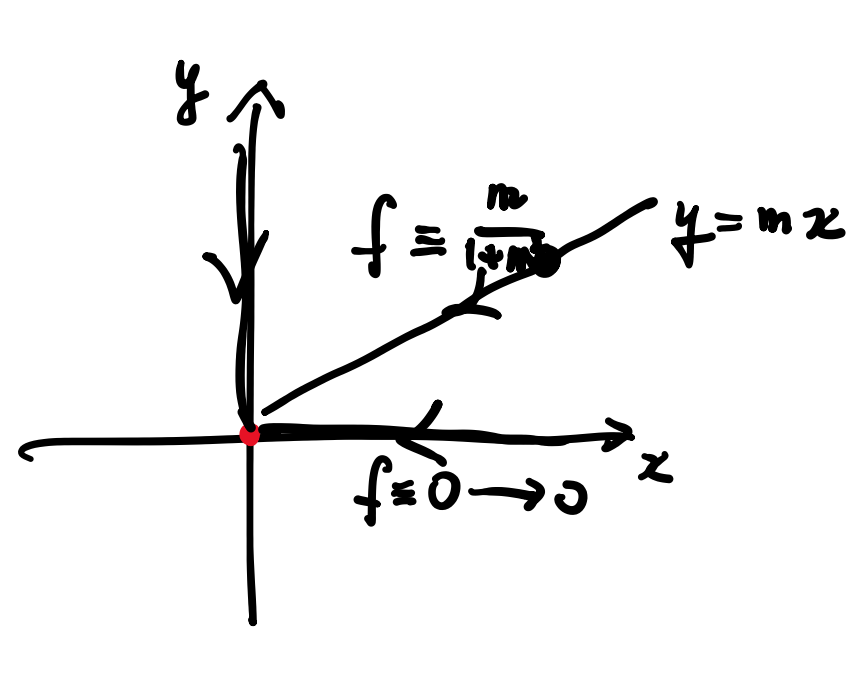
$$= \frac{m}{1+m^2}$$

$$f(x,x) \overset{m=1}{\equiv} \frac{1}{2}$$

when $x \to 0$ $(x,x) \longrightarrow \vec{0}$ and

$f(x,x) \equiv 1/2 \longrightarrow 1/2$

$$f(x,mx) \equiv \frac{m}{1+m^2} \longrightarrow \frac{m}{1+m^2} \neq 0$$



(formally

$\vec{x}_n = \left(\frac{1}{n}, 0\right) \qquad f\left(\frac{1}{n}, 0\right) \equiv 0 \longrightarrow 0$

$\vec{y}_n = \left(\frac{1}{n}, \frac{1}{n}\right) \qquad f\left(\frac{1}{n}, \frac{1}{n}\right) \equiv \frac{1}{2} \longrightarrow \frac{1}{2}$

$$\Rightarrow \nexists \lim_{(x,y) \to \vec{0}} f. \qquad ■$$

Ex 1.4.6 (**)

Q Show that

$$\nexists \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+y^4}.$$

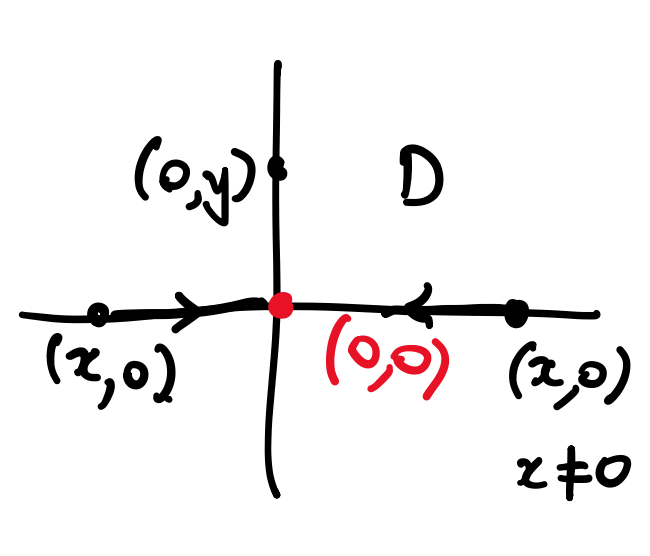
Sol: $f(x,y) = \dfrac{xy^2}{x^2+y^4} \qquad (x,y) \in D \subset \mathbb{R}^2$

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^4 \neq 0\}$$

$\underset{+}{x^2} + \underset{+}{y^4} = 0 \Leftrightarrow x^2 = y^4 = 0 \Leftrightarrow x = y = 0$

$\Rightarrow D = \mathbb{R}^2 \setminus \{(0,0)\}$

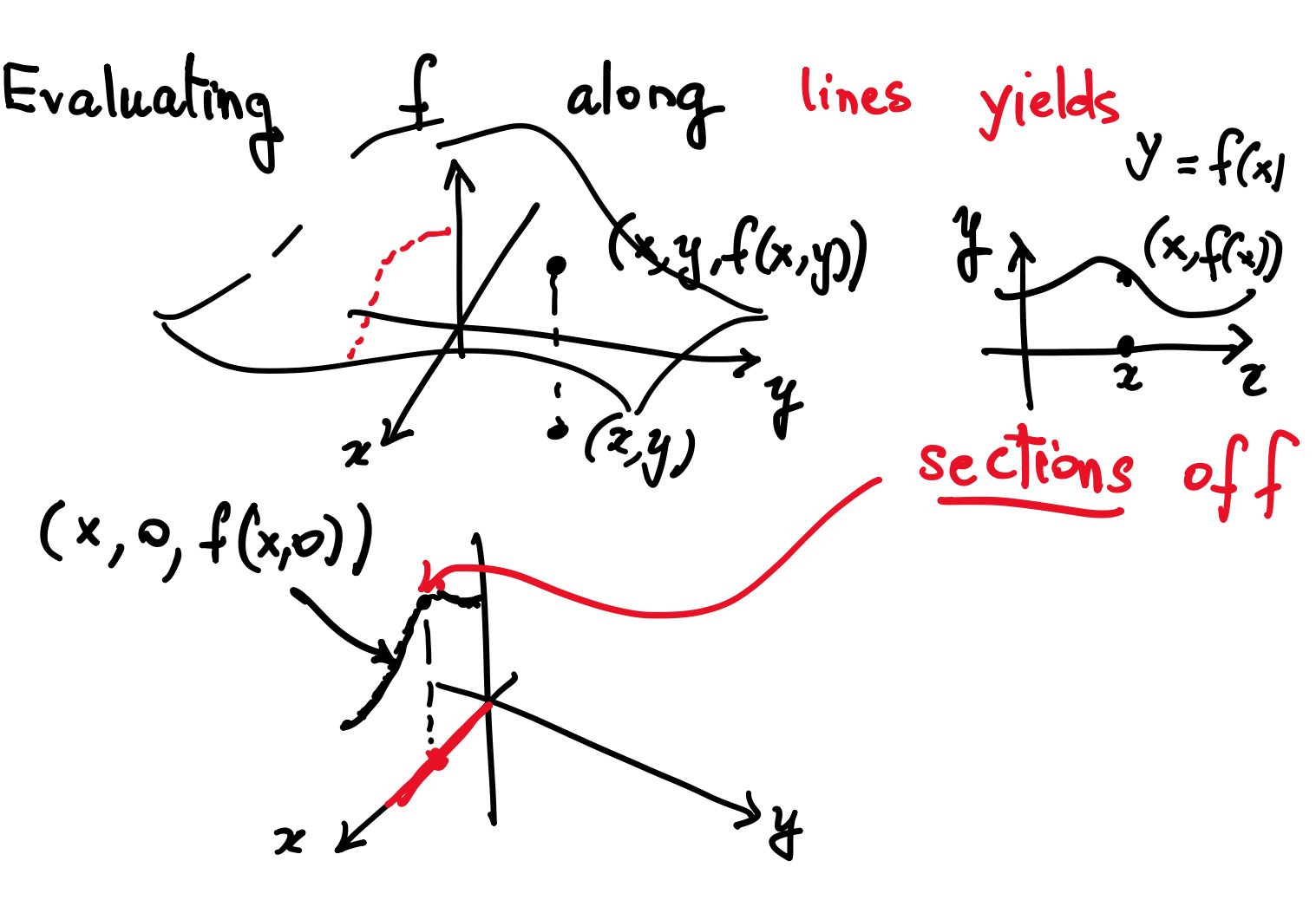
Clearly $(0,0) \in \text{Acc } D$



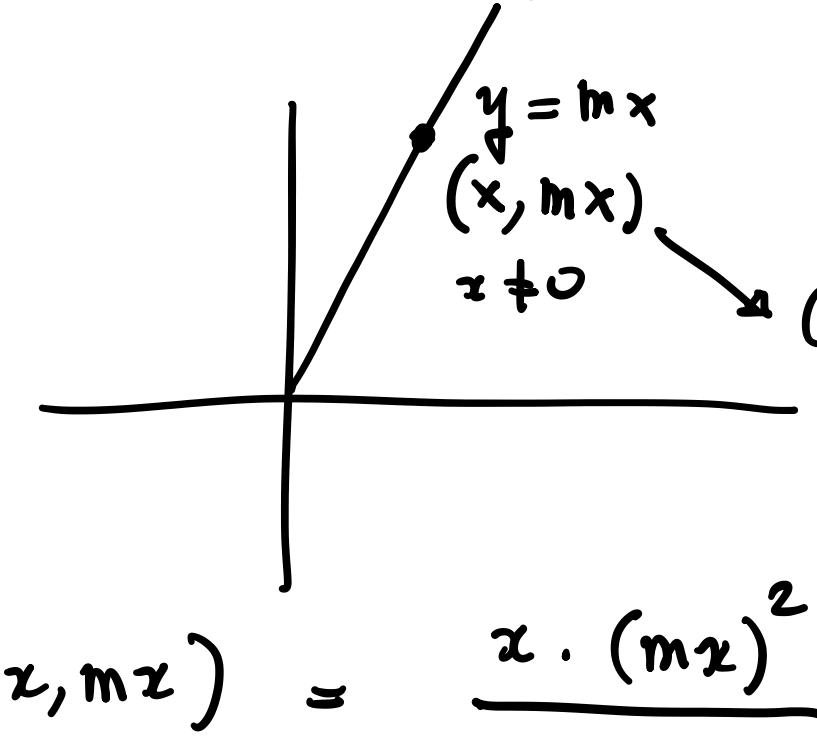
$$f(x,0) = \frac{x \cdot 0^2}{x^2 + 0^4} = \frac{0}{x^2} \equiv 0 \xrightarrow[x \to 0]{} \boxed{0}$$

$\Updownarrow$
$(x,0) \longrightarrow (0,0)$

⌈ Evaluating f along lines yields sections of f



$$\underset{\substack{* \\ 0}}{f(0,y)} = \frac{0 \cdot y^2}{0^2 + y^4} = \frac{0}{y^4} \equiv 0 \longrightarrow 0$$



$m \in \mathbb{R}$ fixed

$\to (0,0) \Leftrightarrow x \to 0$

$$f(x,mx) = \frac{x \cdot (mx)^2}{x^2 + (mx)^4} =$$

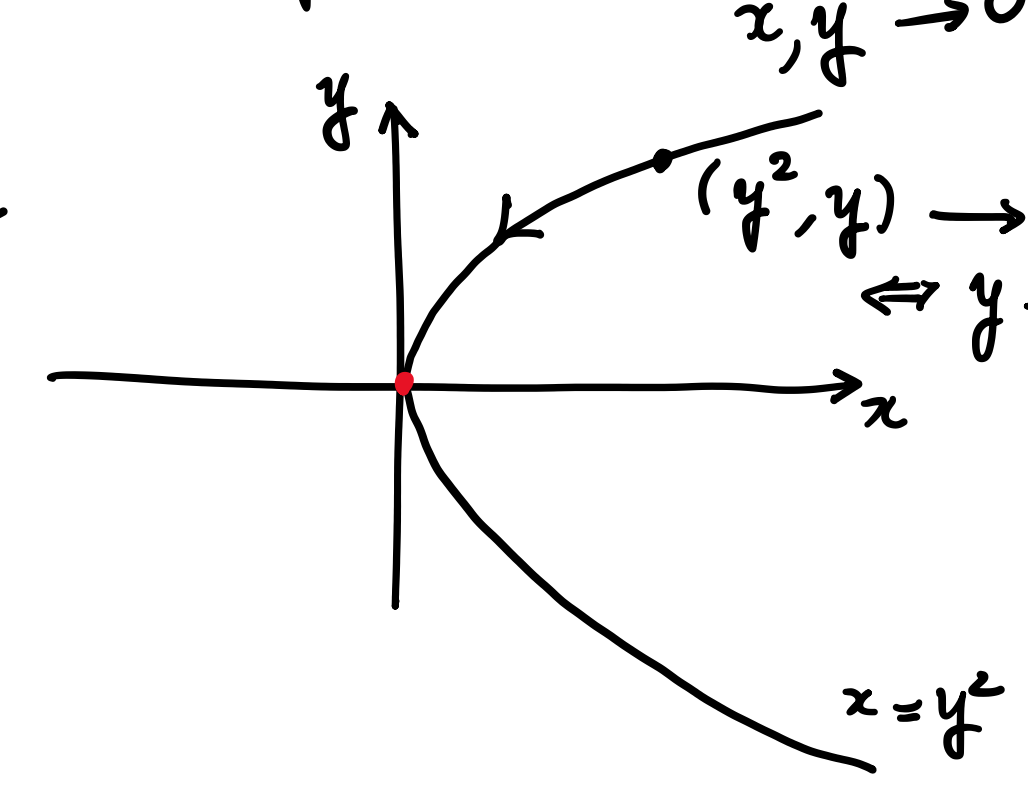
$$= \frac{m^2x^3}{x^2 + m^4x^4} = \frac{m^2\cancel{x^3}}{\cancel{x^2}(1+m^4x^2)}$$

$$= \frac{m^2x}{1+m^4x^2} \xrightarrow[x \to 0]{} \frac{0}{1} = 0$$

So along each straight line going to $\vec{0}$ $f \to 0$.

$$f(x,y) = \frac{xy^2}{x^2+y^4} \qquad \begin{array}{c} (x,y) \to \vec{0} \\ \Updownarrow \\ x, y \to 0 \end{array}$$

$x = y^2$



$$f(y^2,y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} \equiv 1/2$$

$\downarrow \qquad\qquad\qquad\qquad \downarrow y \to 0$

$(0,0) \qquad\qquad\qquad\qquad \boxed{1/2}$

$$\Rightarrow \nexists \lim_{(x,y) \to (0,0)} f. \qquad ■$$

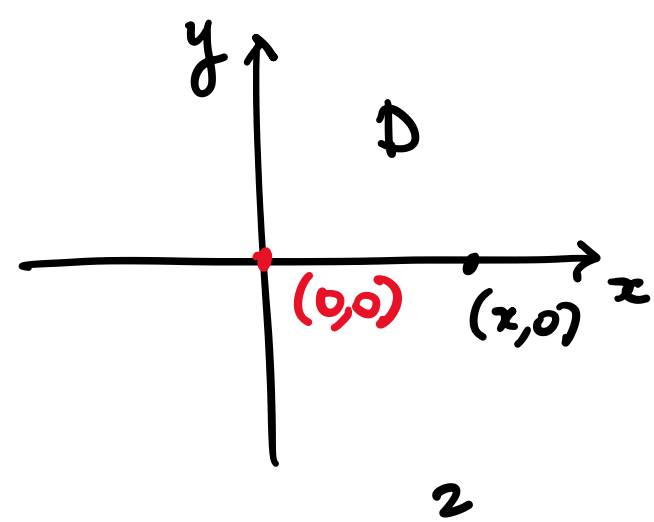
Do Ex 1.8.7.

Ex. 1.4.7 (**)

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+y^2}.$$

Sol: $f(x,y) = \dfrac{xy^2}{x^2+y^2}$ defd on

$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$



Clearly $(0,0) \in \text{Acc} D$.

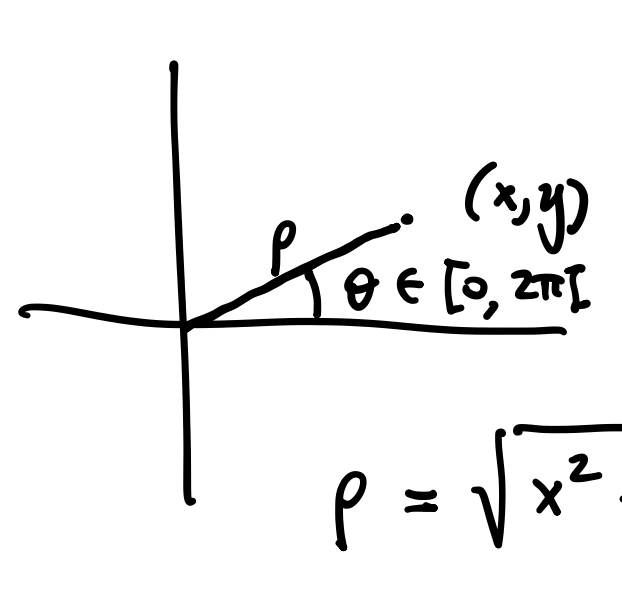
$$\underset{x \neq 0}{f(x,0)} = \frac{x \cdot 0^2}{x^2 + 0^2} = \frac{0}{x^2} \equiv 0 \xrightarrow[x \to 0]{} \boxed{0}$$

$$f(0,y) \equiv 0 \xrightarrow[y \to 0]{} \boxed{0}.$$

At this stage I could say that

if a $\lim_{(x,y) \to \vec{0}} f$ exists $\Rightarrow$ it must be 0.

Guess: $\exists \lim_{\boxed{(x,y) \to \vec{0}}} f = 0$



$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$$

$$\rho = \sqrt{x^2+y^2} = \|(x,y)\|$$

$\boxed{(x,y) \to \vec{0}} \Leftrightarrow \|(x,y) - \vec{0}\| \longrightarrow 0$

$\Leftrightarrow \|(x,y)\| \longrightarrow 0$

$\Leftrightarrow \boxed{\rho \longrightarrow 0}$

$$f(x,y) = \frac{xy^2}{x^2+y^2} \overset{\text{pol. coords}}{=} \frac{\rho\cos\theta\,\rho^2\sin^2\theta}{\rho^2}$$

$$= \frac{\rho^{\cancel{3}}}{\cancel{\rho^2}} \cdot \boxed{\cos\theta \sin^2\theta}$$

$$f(x,y) = \rho \cos\theta \sin^2\theta$$

$\Downarrow \qquad\qquad \downarrow$

$$|f(x,y)| = |\rho| \, |\cos\theta| \, |\sin^2\theta|$$

$$= \rho \, \underbrace{|\cos\theta| \, |\sin^2\theta|}_{\leq 1}$$

$$|f(x,y)| \leq \rho \xrightarrow[\substack{\rho \to 0 \\ \Updownarrow \\ (x,y) \to \vec{0}}]{} 0$$

$\Downarrow$

$f \to 0$ by squeeze thm. ■