PARTITIONS of UNITS

Obs

$U \subseteq \mathbb{R}^n$ open. Let $K \subseteq U$ compact

Then $\exists\, \phi \in C_c^\infty(U)$ such that $\phi \equiv 1$ on K

$0 \leq \phi \leq 1 \quad \forall x \in U$

Kind of URYSOHN LEMMA

proof $\delta = \text{dist}(K, \mathbb{R}^n \setminus U) = \inf_{x \in K,\, y \in \mathbb{R}^n \setminus U} |x-y| = \min_{x \in K} d(x, \mathbb{R}^n \setminus U)$

$3\varepsilon < \delta$ $\qquad f(x) = \begin{cases} 1 & x \in U \;\; d(x,K) \leq 2\varepsilon \\ 0 & x \in U \;\; d(x,K) > 2\varepsilon \end{cases}$ $\qquad$ supp $f \subset\subset U$

·) $f * \rho_\varepsilon \in C_c^\infty(U)$ since $\text{supp}(f * \rho_\varepsilon) \subseteq K + \overline{B(0,2\varepsilon)} + \overline{B(0,\varepsilon)} \subseteq U$.

·) $f * \rho_\varepsilon(x) = \int_{B(0,\varepsilon)} \underbrace{f(x-y)}_{\sim} \cdot \underbrace{\rho_\varepsilon(y)\, dy}_{\sim} = 1 \qquad d(x,K) \leq \varepsilon$

PARTITIONS of UNIT SUBORDINATED TO AN OPEN COVER

U measurable set $\qquad U \subseteq \bigcup_{i=1}^\infty A_i \qquad A_i$ open sets then $\exists\ \xi_i \in \mathcal{C}_c^\infty(A_i)$ $\quad 0 \le \xi_i \le 1$

and $\underbrace{\sum_{i=1}^{\infty} \xi_i(x) = 1}$ $\quad \forall x \in U$

and the sum is locally finite that is

$\forall x \in U \quad \exists\ N_x \in \mathbb{N}$ such that $\xi_i(x) \neq 0$ $i \in I_x$ with

$\# I_x = N_x$

(NO PROOF) $\to$ see FOLLAND Prop 4.41.

Partition of units will be used to "localize"

$f : U \to \mathbb{R}$

(extend $f = 0$ in $\mathbb{R}^n \setminus U$)

(ξ_i) partition of unit associated to A_i

$U \subseteq \bigcup_i A_i$

$$f = \sum_i^{\infty} (f \xi_i)$$

$f \xi_i$ has compact support contained in $U \cap A_i$

PRELIMINARIES FROM CALCULUS

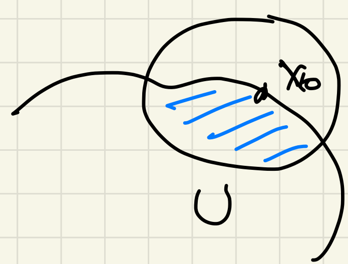
Def $U \subseteq \mathbb{R}^n$ open is of class C^k (or has boundary of class C^k) if

$\forall x_0 \in \partial U \ \exists \ r = r_{x_0} > 0$ and $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ of class C^k s.t.

(up to a rotation and translation of variables)

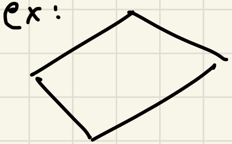
$$B(x_0, r) \cap U = \{ (x_1, \dots x_n) \text{ s.t. } x_n < \gamma(x_1, \dots x_{n-1}) \}$$

this implies $\partial U \cap B(x_0, r) = \{ x_n = \gamma(x_1, \dots x_{n-1}) \}$ but it is not sufficient!



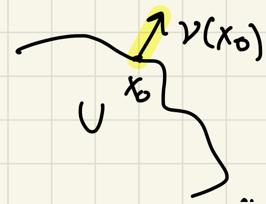
U is analytic if $\gamma \in C^{\omega}(\mathbb{R}^{n-1})$

U is Lipschitz if $\gamma \in C^{0,1}(\mathbb{R}^{n-1})$ ex:



If U is of class C^1, $\forall\, x_0 \in \partial U$ $\exists\, \nu(x_0) \in \mathbb{R}^n$ $|\nu(x_0)| = 1$

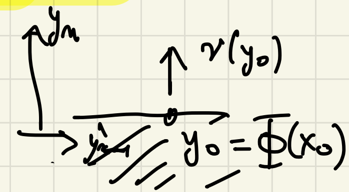
EXTERIOR NORMAL TO U at x_0



$$\nu(x_0) = \left(-\nabla\gamma(x_1^0 \ldots x_{n-1}^0), 1\right)$$

• A typical procedure "STRAIGHTENING THE BOUNDARY"

$$B(x_0, r) \cap U = \{\gamma(x_1 \ldots x_{n-1}) > x_n\} \xrightarrow{\ \Phi\ }$$



$$\Phi : B(x_0, r) \longrightarrow \mathbb{R}^n$$
$$(x_1 \ldots x_n) \longmapsto (x_1, \ldots x_{n-1}, x_n - \gamma(x_1 \ldots x_{n-1})) = (y_1 \ldots y_n)$$

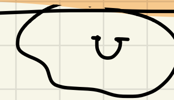
Φ is a C^1 diffeomorphism

$\det D\Phi(x) = 1$

(the body is locally flattened out)

Is it POSSIBLE to extend $\nu(x)$ to a FUNCTION: $(\mathbb{R}^n \to \mathbb{R}^n)$ in a NEIGHBORHOOD of the BOUNDARY?

$U \subseteq \mathbb{R}^n$ open bdd



$d_S(x) =$ signed distance from U $= d(x, \bar{U}) - d(x, \mathbb{R}^n \setminus U)$

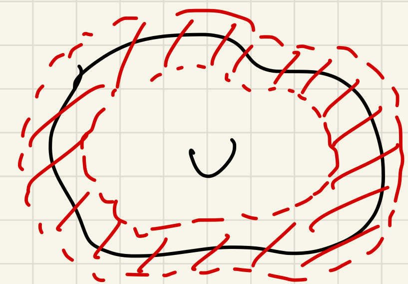
$$d_S(x) = \begin{cases} d(x, \partial U) & x \in \mathbb{R}^n \setminus U \\ -d(x, \partial U) & x \in \bar{U} \end{cases}$$

"$\nabla d_S(x) = \nu(x)$" $x \in \partial U$

d_S is a 1-Lipschitz continuous functions

$\delta > 0$

$$(\partial U)_\delta = \{x,\ d_S(x) \in (-\delta, \delta)\} = \{x,\ |\mathrm{dist}(x, \partial U)| < \delta\}$$



If U is bdd of class C^k, $\exists \delta > 0$ s. that $d_S \in C^k((\partial U)_\delta)$?

$\Rightarrow$ in this case $\nabla d_S \in C^{k-1}((\partial U)_\delta, \mathbb{R}^n)$ is the extension of the exterior normal! $\nabla d_S(x) = \nu(x)$ $x \in \partial U$.

U bdd of class C^k, $k \geq 1$

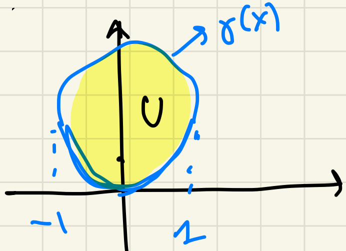
Federer ('50) Possible $\iff$ ∂U has **POSITIVE REACH**

[that is $\exists\, r>0$ such that $\forall\, x \in \mathbb{R}^n$ $|d_S(x)| < r$

$\exists!\ \bar{x} \in \partial U$ such that $d(x, \partial U) = |x - \bar{x}|$]

Note that U of class C^1 $\not\Rightarrow$ ∂U has positive reach!

$$U = \left\{ (x,y) \in \mathbb{R}^2 \quad \begin{array}{l} y > |x|^{2-\varepsilon} \quad x \in [-1,1] \\ y \leq \gamma(x) \quad \dots \end{array} \right\} \qquad \varepsilon \in (0,1)$$



U bdd of class C^1

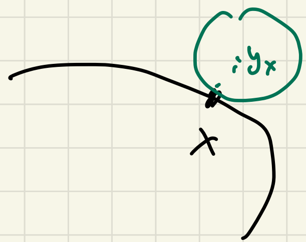
For all $a > 0$ sufficiently small $\to \exists\, x_1 = x_1(a) > 0$

$$d((0,a), \partial U)^2 = |(0,a) - (x_1, x_1^{2-\varepsilon})|^2 = |(0,a) - (-x_1, x_1^{2-\varepsilon})|^2 < |a|^2 = d((0,a),(0,0))^2$$

∂U HAS NOT POSITIVE REACH

A SUFFICIENT (NOT NECESSARY!) CONDITION FOR POSITIVE REACH IS THE UNIFORM EXTERIOR AND INTERIOR SPHERE CONDITION

Def: U has the (UNIFORM) EXTERIOR SPHERE CONDITION if $\exists\, r>0$ such that $\forall x \in \partial U \ \exists\, y_x \in \mathbb{R}^n \setminus U$

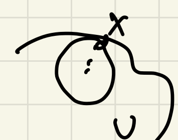


$$B(y_x, r) \cap U = \emptyset$$
$$\overline{B(y_x, r)} \cap \overline{U} = \{x\}$$

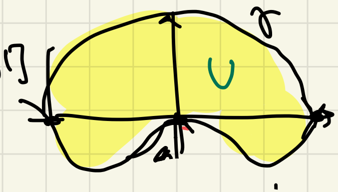
(ex: U convex set has the ext ball condition with $r \in (0, +\infty)$!)

Def U has the (UNIFORM) INTERIOR SPHERE CONDITION if $\exists\, r>0$ such that $\forall\, x \in \partial U \ \exists\, y_x \in U$

$B(y_x, r) \subseteq U \qquad \overline{B(y_x, r)} \cap \mathbb{R}^n \setminus U = \{x\}$



U has the UNIFORM EXTERIOR & INTERIOR SPHERE CONDITION $\iff$ ∂U is of CLASS $C^{1,1}$ (C^1 functions with Lipschitz derivatives)



__Ex__: $U = \{ (x,y) \in \mathbb{R}^2 \quad y > x^2 \lg|x| \text{ for } x \in [-1,1] \ + \text{curve } \gamma \quad y < \gamma(x) \}$

U of class C^1 but ∂U in $(0,0)$ has NOT THE EXTERIOR SPHERE CONDITION

(it is not possible to find $\delta > 0$ such that $\exists\, (0,y)$ $y<0$ with $B((0,y),\delta) \cap U = \emptyset$ $\quad \overline{B((0,y),\delta)} \cap \overline{U} = \{(0,0)\}$)

In $(0,0)$ the "curvature" explodes
(the 2nd derivative of $x^2 \lg|x|$ ..)

U Ext. ball condition $\iff$ $\exists\, C>0, R>0 \quad \forall x \in \partial U$

$$\nu(x)\cdot(y-x) \le C|y-x|^2$$

$$\forall y \in \overline{U} \cap \overline{B(x,R)}$$

U Int ball. condition $\iff$ $\exists\, C>0 \; R>0 \quad \forall x \in \partial U$

$$\nu(x)\cdot(y-x) \ge -C|y-x|^2$$

$$\forall y \in \overline{B(x,R)} \cap (\mathbb{R}^n \setminus U)$$

EXT / INT BALL CONDITION $\approx$ bounds on the curvature of ∂U

Corollary: If U is of class C^k $k \ge 2$, then $\exists\, \delta > 0$ such that $d_s \in C^k((\partial U)_\delta)$ and so $\nabla d_s \in C^{k-1}((\partial U)_\delta, \mathbb{R}^n)$ is an extension of $\nu(x)$.