L^p SPACES

$\mathcal{L}$ be the Lebesgue measure on $\mathbb{R}^n$. A measurable

U measurable set

$|A|$ = Lebesgue measure of A.

$$L^p(U) = \{ f : U \to \mathbb{R} \text{ measurable } \int_U |f(x)|^p \, dx < +\infty \}$$

$1 \leq p < +\infty$

<u>NB</u> We identify $f, g \in L^p(U)$ if $|\{x, f(x) \neq g(x)\}| = 0$.

Elements in $L^p(U)$ are class of equivalence.

$(L^p(U), \|\cdot\|_p)$ is a Banach space $\quad \|f\|_p = \left[\int_U |f(x)|^p dx \right]^{1/p}$

$L^\infty(U) = \{ f : U \to \mathbb{R}$ measurable, $\exists\, C_f > 0$ such that $|\{x \in U \ |f(x)| > C\}| = 0 \}$

Obs: $f \in L^\infty(U) \iff$ up to set of measure 0 $|f(x)| \leq C$.

$(L^\infty(U), \|\cdot\|_\infty)$ is a Banach space

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in U} |f(x)| = \inf \{ C > 0 ,\ |\{x \mid |f(x)| \geq C\}| = 0 \}$$

Observation

U bdd $\mathcal{C}(\bar{U}) = \{ f : U \to \mathbb{R}$ continuous, & such that it admits a continuous extension to $\partial U . \}$

$(\mathcal{C}(\bar{U}), \|\cdot\|_\infty)$ is a Banach space (A.A. theorem)

$$f \in \mathcal{C}(\bar{U}) \Rightarrow \|f\|_\infty = \sup_{x \in U} |f(x)| = \max_{x \in \bar{U}} |f(x)| = \operatorname{ess-sup}_x |f(x)|$$

$$(\mathcal{C}(\bar{U}), \|\cdot\|_\infty) \hookrightarrow (L^\infty(U), \|\cdot\|_\infty)$$

$$f \longmapsto f$$

CONTINUOUS EMBEDDING $\quad f \in \mathcal{C}(\bar{U}) \Rightarrow f \in L^\infty(U)$

$(\mathcal{C}(\bar{U}), \|\cdot\|_\infty)$ is a CLOSED SUBSPACE of $(L^\infty(U), \|\cdot\|_\infty)$

$(L^p(U))^* = $ dual of $L^p(U) = \{ T : L^p(U) \to \mathbb{R} \ \underbrace{\text{linear and bounded}} \}$

- $T(\alpha f + \beta g) = \alpha T f + \beta T g$
- $|Tf| \le C \|f\|_{L^p}$

For $p \in [1, +\infty)$ $\quad (L^p(U))^* \simeq L^q(U) \quad \frac{1}{p} + \frac{1}{q} = 1$ (q conjugate exponent of p)

$g \in L^q(U) \Rightarrow T_g f = \int_U f g \, dx \quad T_g \in (L^p(U))^*$

$T \in (L^p(U))^* \Rightarrow \exists! g \in L^q(U) \quad T = T_g$

$(L^\infty(U))^* \supsetneq L^1(U)$

For $p \in (1, +\infty)$ $L^p(U)$ are reflexive $\quad (L^p)^{**} \simeq L^p$

For $p \in [1, +\infty)$ $L^p(U)$ are separable ($\exists$ countable dense set)

For $p = 2$ $(L^2(U), \|\cdot\|_2)$ is Hilbert $\quad (f, g) = \int_U f g \, dx$

$C_c^\infty(U) \subseteq L^p(U) \quad \forall p \in [1, +\infty]$

THEOREM

$$\overline{C_c^\infty(U)}^{\|\cdot\|_p} = L^p(U) \qquad p \in [1, +\infty)$$

that is

(let $f \in L^p(U)$, $p \in [1, +\infty)$, $\varepsilon > 0$ $\exists\, \phi_\varepsilon \in C_c^\infty(U)$ s. that

$$\|f - \phi_\varepsilon\|_p \leq \varepsilon \iff \int_U |f(x) - \phi_\varepsilon(x)|^p dx \leq \varepsilon^p$$

$$\overline{C_c^\infty(U)}^{\|\cdot\|_\infty} = C_0(U)$$

how to prove? Regularization by convolution

CONVOLUTION

$f, g \in L^1_{loc}(\mathbb{R}^n)$ ($\Longleftrightarrow f, g \in L^1(K)$ $\forall K \subseteq \mathbb{R}^n$ compact)

$$\underbrace{f * g(x)}_{} = \int_{\mathbb{R}^n} f(y)\, g(x-y)\, dy = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy$$

if $f(\cdot)\, g(x-\cdot) \in L^1(\mathbb{R}^n)$

① YOUNG INEQUALITY. $p, q, r \in [1, +\infty]$ $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$

If $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ $\Longrightarrow$ $f * g \in L^r(\mathbb{R}^n)$

$\|f * g\|_r \leq \|f\|_p \|g\|_q$

N.B.
$p = q = r = 1$ Fubini Tonelli $(L^1(\mathbb{R}^n), *)$ is an ALGEBRA.

② $\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$

3. $f \in \mathcal{C}_c(\mathbb{R}^n)$ $g \in L^1_{loc} \Rightarrow f*g \in \mathcal{C}(\mathbb{R}^n)$

$f \in \mathcal{C}^k_c(\mathbb{R}^n)$ $g \in L^1_{loc} \Rightarrow f*g \in \mathcal{C}^k(\mathbb{R}^n)$

$D^\alpha(f*g) = (D^\alpha f)*g$

Regularization by convolution.

$$\rho(x) := \begin{cases} c\, e^{\frac{1}{|x|^2-1}} & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$$

$c>0$ such that

$$\int_{\mathbb{R}^n} \rho(x)\,dx = c\int_{\{|x|<1\}} e^{\frac{1}{|x|^2-1}}\,dx = 1$$

$\rho \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ $\operatorname{supp}\rho = \overline{B(0,1)}$

(FREDERICK) MOLLIFIERS

$\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$ $\varepsilon>0$ $\varepsilon\in\mathbb{R}$

$\rho_\varepsilon(x) \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ $\operatorname{supp}\rho_\varepsilon = \overline{B(0,\varepsilon)}$ $\int_{\mathbb{R}^n}\rho_\varepsilon(x)\,dx = 1$ $\forall\varepsilon$.

Prop Let $f \in L^p(\mathbb{R}^n)$ $p \in [1, +\infty)$ $p \neq +\infty$!

1) $f * \rho_\varepsilon \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ $\|f * \rho_\varepsilon\|_p \leq \|f\|_{L^p}$ (Young $p = r$, $q = 1$)

2) $\|f * \rho_\varepsilon - f\|_{L^p} \to 0$ as $\varepsilon \to 0$

$\overline{C_c^\infty(\mathbb{R}^n)}^{L^p} = L^p(\mathbb{R}^n)$

$U \subseteq \mathbb{R}^n$ open $\overline{C_c^\infty(U)}^{L^p} = L^p(U)$

Prop ("case $p = +\infty$")

1) Let $f \in C_c(\mathbb{R}^n)$ then $f * \rho_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ $f * \rho_\varepsilon \to f$ UNIFORMLY (in $\|\cdot\|_\infty$)

2) Let $f \in C(\mathbb{R}^n)$ then $f * \rho_\varepsilon \in C^\infty(\mathbb{R}^n)$ $\to$ $f * \rho_\varepsilon \to f$ LOCALLY UNIFORMLY

for both proposition

idea: $f * \rho_\varepsilon(x) - f(x) = \int_{\mathbb{R}^n} f(x-y)\rho_\varepsilon(y)\,dy - \int_{\mathbb{R}^n} f(x)\rho_\varepsilon(y)\,dy =$

$= \int_{\mathbb{R}^n} [f(x-y) - f(x)]\rho_\varepsilon(y)\,dy$

THM:

$\boxed{\overline{C_c^\infty(U)}^{\|\cdot\|_p} = L^p(U)}$

$C_c^\infty(U)$ is DENSE (w.r.t. L^p topology) in $L^p(U)$! $p < +\infty$

(NOT TRUE for L^∞! $\overline{C_c^\infty(U)}^{\|\cdot\|_\infty} = C_0(U)$

IMPORTANT LEMMA

LEMMA DU BOIS– REYMOND (sometimes "fundamental lemma of the calculus of variations").

$U \subseteq \mathbb{R}^n$ open $f \in L^1_{loc}(U)$

$$\int_U f(x)\phi(x)\,dx = 0 \quad \forall \phi \in C^\infty_c(U) \Longleftrightarrow f = 0 \text{ (almost everywhere)}$$

proof $\Leftarrow$ obvious

$\Rightarrow$ Let $K \subset U$ compact $f \in L^1(K)$, fix a representative f

$$g_K(x) := \begin{cases} \operatorname{sign} f(x) & x \in K \\ 0 & x \in U \setminus K \end{cases} \qquad \operatorname{sign} f(x) = \begin{cases} := \dfrac{f(x)}{|f(x)|} & \text{if } f(x) \neq 0 \\ := 0 & \text{if } f(x) = 0 \end{cases}$$

$\operatorname{supp} g_K \subseteq K \qquad g_K \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

$\operatorname{supp}(g_K * \rho_\varepsilon) \subseteq K + \overline{B(0,\varepsilon)} \subseteq U$ if ε is sufficiently small $\boxed{\varepsilon \leq \varepsilon_0}$ $\varepsilon_0 < \operatorname{dist}(K, \mathbb{R}^n \setminus U)$

$\Rightarrow g_K * \rho_\varepsilon \in C^\infty_c(U)$

$\left(K + \overline{B(0,\varepsilon)} = \overline{\bigcup_{x \in K} B(x,\varepsilon)} \right.$ COMPACT $\subset\subset U$ if $d(x, \mathbb{R}^n \setminus U) > \varepsilon_0 > \varepsilon \;\; \forall x \in K$

$\Rightarrow$ by assumption $\int_U f(x)\, g_k * \rho_\varepsilon(x)\, dx = 0$

Let ε_0 such that $\overline{B(0,\varepsilon_0)} + K \subseteq U$. take $\varepsilon \leq \varepsilon_0$

$$0 = \int_U f(x) g_k * \rho_\varepsilon(x)\, dx = \int_{K+B(0,\varepsilon_0)} f(x)\, g_k * \rho_\varepsilon(x)\, dx$$

Recall
$f \in L^1(K+B(0,\varepsilon_0))$

$f(x) \in L^1(K+B(0,\varepsilon_0))$

$\|g_k * \rho_\varepsilon\|_\infty \leq \|g_k\|_\infty$ (Young $p=1$ $q=r=+\infty$)

$|f(x)\, g_k * \rho_\varepsilon(x)| \leq |f(x)| \in L^1(K+B(0,\varepsilon_0))$

$g_k * \rho_\varepsilon \to g_k$ in L^1 and thus a.e (up to a subsequence ε_n)

$\hookrightarrow f(x) g_k * \rho_{\varepsilon_n}(x) \to f(x) g_k(x)$ a.e

by Lebesgue dominated conv.

$$\lim_{\varepsilon \to 0} \int_{K+B(0,\varepsilon_0)} f(x)\, g_k * \rho_\varepsilon(x)\, dx = \int_{K+B(0,\varepsilon)} f(x) g_k(x) = \int_{K+B(0,\varepsilon_0)} |f(x)|\, dx = 0$$

$f(x) = 0$ a.e.

Lemma 2 (1 dimensional)
(it can be generalized to several dim, U connected)

Let $f \in L^1_{loc}(a,b)$.

$\forall \phi \in C^\infty_c(a,b) \quad \int_a^b f(x)\phi'(x)\,dx = 0 \iff \exists c \in \mathbb{R} \quad f(x) = c \text{ a.e. } x$

proof $\quad f = c \Rightarrow \int_a^b c \cdot \phi'(x)\,dx = c\,\phi(b) - c\,\phi(a) = 0 \quad$ since $\phi \in C^\infty_c(a,b)$

$\Leftarrow$ let $\phi \in C^\infty_c(a,b)$ Fix $h(x) \in C^\infty_c(a,b) \quad \int_a^b h(x)\,dx = 1$

$$\psi(x) := \int_a^x \phi(y)\,dy - \int_a^b \phi(z)\,dz \int_a^x h(y)\,dy \qquad ① \; \psi(a) = 0 = \psi(b)$$

② $\psi'(x) = \phi(x) - \left(\int_a^b \phi(z)\,dz\right) h(x) \in C^\infty_c(a,b) \qquad ① + ② \Rightarrow \psi \in C^\infty_c(a,b)$

By assumption

$$\int_a^b f(x)\psi'(x)\,dx = 0 = \int_a^b f(x)\,\phi(x)\,dx - \left[\int_a^b \phi(z)\,dz\right] \int_a^b f(x)\,h(x)\,dx =$$

$$= \int_a^b f(x)\phi(x)\,dx - \int_a^b \phi(x)\,dx \int_a^b f(y)h(y)\,dy = \int_a^b \phi(x)\left[f(x) - \int_a^b f(y)\,h(y)\,dy\right]dx$$

$$\Rightarrow \int_a^b \phi(x) \left[f(x) - \int_a^b f(z) h(z)\, dz \right] dx = 0 \qquad \forall \phi \in C_c^\infty(a,b)$$

By DeBois-Reymond $\quad f(x) - \int_a^b f(z) h(z)\, dz = 0 \quad$ a.e. x

$$\Rightarrow f(x) = \underbrace{\int_a^b f(z)\, h(z)\, dz} := C \quad \text{a.e. } x.$$