COMPUTABILITY (06/10/2025)

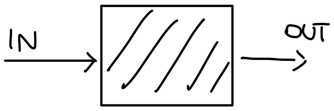
effective procedure (algorithm)
$\rightsquigarrow$
computable function

$\Big\}$ $\rightsquigarrow$ existence of non-computable functions

* <u>Effective procedure</u>

sequence of elementary steps

input data $\rightsquigarrow$ output data



deterministic

function $f : \{\text{inputs}\} \to \{\text{outputs}\}$
(partial)

<u>Def.</u> : A function f is <u>computable</u> if there exists an algorithm such that the induced function is f.

* $GCD(x, y)$ = greatest common divisor (Euclid's algorithm)

* $f(n) = \begin{cases} 1 & n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$

* $h(n) = n^{th}$ digit in π

* $f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are exactly } n \text{ consecutive digits } 5 \\ 0 & \text{otherwise} \end{cases}$

if $\pi = 3{,}145 \; ..... \; 7 \underbrace{5555}_{\text{exactly 4 digits 5}} 4$

$f(4) = 1$

NOT A GOOD ALGORITHM !

<u>idea</u>: - compute all digits of π
- check whether there are exactly n digits 5 in a row

$$g(m) = \begin{cases} 1 & \text{if } \pi \text{ includes at least } m \text{ digits 5 in a row} \\ 0 & \text{otherwise} \end{cases}$$

if $\pi = 3{,}145 \ldots\ldots 7\underbrace{5555}4$

exactly 4 digits 5

(at least 3, 2, 1, 0)

$\downarrow$

$g(4) = 1$

$g(3) = 1$

$g(2) = 1$

$g(1) = 1$

$g(0) = 1$

more generally

$$g(m) = 1 \Rightarrow \forall m' \le m \quad g(m') = 1$$

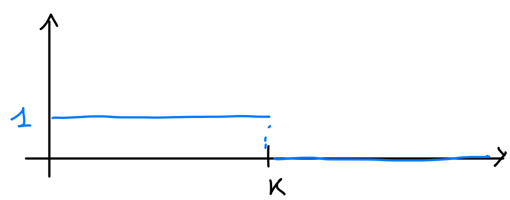
take

$$K = \sup \{ m \mid \text{there are } m \text{ digits 5 in row in } \pi \}$$

two possibilities

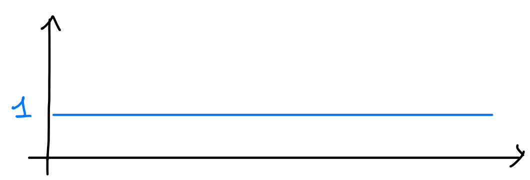
- K finite

$$g(m) = \begin{cases} 1 & \text{if } m \le K \\ 0 & \text{otherwise} \end{cases}$$




```
function g(m):
    if m ≤ K:
        return 1
    else:
        return 0
```

- K is infinite $\quad g(m) = 1 \quad \forall m$




```
function g(m):
    return 1
```

Can't I play the same trick for f?

$$f(n) = \begin{cases} 1 & \text{if in } \pi \text{ there are } \underline{\text{exactly}}\ n \text{ consecutive digits } 5 \\ 0 & \text{otherwise} \end{cases}$$

$A = \{ n \mid \text{there are exactly } n \text{ digits } 5 \text{ in a row in } \pi \}$

and define

```
function f(n):
    if n ∈ A:
        return 1
    else:
        return 0
```

not an algorithm

A might be infinite

* Existence of non-computable functions

* Characteristics of a "good" algorithm

- finite length
- there is a computing agent which executes the algorithm
  - memory (unbounded)
  - discrete steps, deterministic, not probalistic
  - finite limit to the number and power of instructions
  - the computation can
    - → terminate after a finite number of steps → output
    - → diverge (never terminate) → no output

* <u>Math notation</u>

* $\mathbb{N} = \{0, 1, 2, \ldots\}$

* A, B sets $\qquad A \times B = \{(a,b) \mid a \in A, b \in B\}$

$$A^n = A \times A \times \ldots \times A$$
$$\ni$$
$$(a_1, a_2, \ldots, a_n)$$

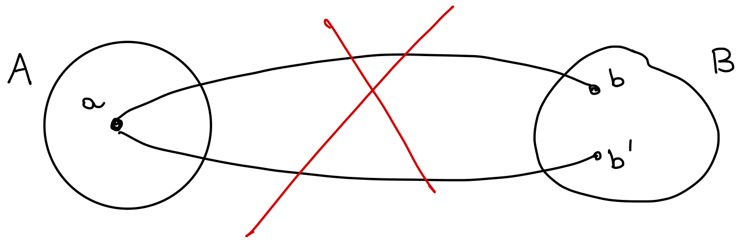
* <u>relations</u> $\qquad r \subseteq A \times B$

n divides m $\qquad$ $\text{divides} = \{(n, n*k) \mid n, k \in \mathbb{N}\}$
$$\cap\!\shortmid$$
$$\mathbb{N} \times \mathbb{N}$$

* <u>functions</u> (partial) $\qquad f : A \to B$

special relation $\qquad f \subseteq A \times B$

$\forall a \in A \ \forall b, b' \in B \quad (a,b), (a,b') \in f \Rightarrow b = b'$



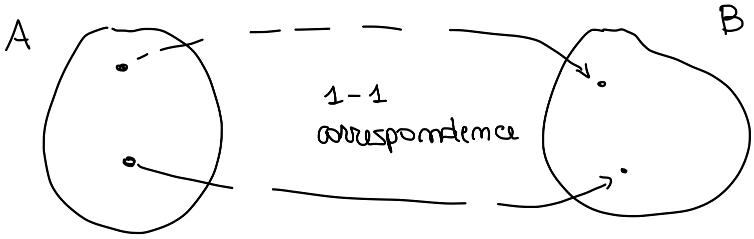
$\text{dom}(f) = \{a \in A \mid \exists b.\ (a,b) \in f\}$

instead of $(a,b) \in f$ we write $f(a) = b$

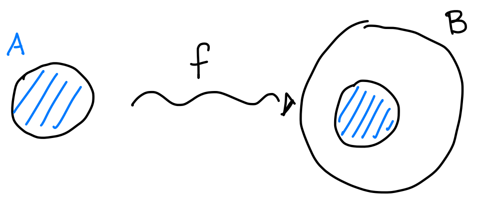
if $a \in \text{dom}(f)$ $\qquad f(a)\downarrow$
$\quad\ a \notin \text{dom}(f)$ $\qquad f(a)\uparrow$

* <u>Cardinality</u>

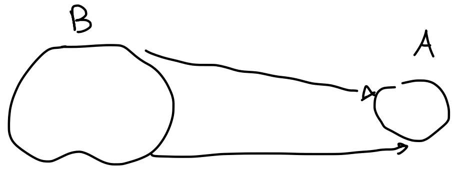
if A, B sets $\qquad |A| = |B|$ if there is $f : A \to B$ bijective



$|A| \leq |B|$ if there is $f: A \to B$ injective



"equivalently" if there is a surjective function $g: B \to A$



* A countable (denumerable)

  $|A| \leq |\mathbb{N}|$

  surjective function $f: \mathbb{N} \to A$

  $$\underbrace{f(0) \quad f(1) \quad f(2) \quad \ldots\ldots}_{\text{list of } \underline{\text{all}} \text{ elements of } A}$$

* A, B countable $\Rightarrow$ $A \times B$ countable

<u>idea</u>:

A	a_0	a_1	a_2	$\ldots$
B	b_0	b_1	b_2	$\ldots$

	a_0	a_1	a_2	$\ldots$
b_0	(a_0, b_0)	(a_1, b_0)	(a_2, b_0)	$\ldots$
b_1	(a_0, b_1)	(a_1, b_1)	(a_2, b_1)	$\ldots$
b_2	(a_0, b_2)	(a_1, b_2)	(a_2, b_2)	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

* $A_0, A_1, \ldots, A_n, \ldots$ countable sets $\Rightarrow$ $\bigcup_{i \in \mathbb{N}} A_i$ is countable

* Existence of non computable function

we restrict to unary functions $f: \mathbb{N} \to \mathbb{N}$ (partial)

$$\mathcal{F} = \{ f \mid f: \mathbb{N} \to \mathbb{N} \} \quad \text{set of all functions}$$

Fix a <u>model of computation</u> $\rightsquigarrow$ set of algorithms $\mathcal{A}$

for each $A \in \mathcal{A}$ $\rightsquigarrow$ $f_A : \mathbb{N} \to \mathbb{N}$

Functions computable by the model

$$\mathcal{F}_{\mathcal{A}} = \{ f \in \mathcal{F} \mid \text{there exists } A \in \mathcal{A} \text{ such that } f_A = f \}$$

$$= \{ f_A \mid A \in \mathcal{A} \}$$

Clearly

$$\mathcal{F}_A \underset{?}{\subsetneq} \mathcal{F}$$

* An algorithm $A \in \mathcal{A}$ is a sequence of instructions from an instruction set I

finite

$$\mathcal{A} = I \cup I \times I \cup I \times I \times I \cup \ldots$$

$$= \bigcup_{i \geq 1} I^i$$

$\uparrow$ countable union of countable sets $\Rightarrow$ countable

$$|\mathcal{A}| \leq |\mathbb{N}| \quad (*)$$

Now

$$\mathcal{A} \to \mathcal{F}_A$$

$$A \mapsto f_A$$

surjective

$$\boxed{|\mathcal{F}_A| \leq |\mathcal{A}| \leq |\mathbb{N}|}$$

* the set $\mathcal{F}$ of all functions is not computable

in fact, assume $|\mathcal{F}| \leq |\mathbb{N}|$

i.e. we can list the elements of $\mathcal{F}$

$$\begin{array}{ccccc}
 & f_0 & f_1 & f_2 & \cdots \\
0 & f_0(0) & f_1(0) & f_2(0) & \cdots \\
1 & f_0(1) & f_1(1) & f_2(1) & \cdots \\
2 & f_0(2) & f_1(2) & f_2(2) & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

define $d: \mathbb{N} \to \mathbb{N}$ by changing systematically the diagonal

$$d(n) = \begin{cases} 0 & \text{if } f_n(n)\uparrow \\ \uparrow & \text{if } f_n(n)\downarrow \end{cases}$$

since $d \in \mathcal{F}$ it must be in the list, there is $n \in \mathbb{N}$ s.t. $d = f_n$

$\to$ if $f_n(n)\downarrow \implies d(n)\uparrow \neq f_n(n)$ contradiction

$\to$ if $f_n(n)\uparrow \implies d(n) = 0 \neq f_n(n)$ //

$\implies$ absurd

$\hookrightarrow$ $\mathcal{F}$ not countable $\implies$ $|\mathcal{F}| > |\mathbb{N}|$

* Summarising

$$\left.\begin{array}{l} \mathcal{F}_A \subseteq \mathcal{F} \\ |\mathcal{F}_A| \leq |\mathbb{N}| < |\mathcal{F}| \end{array}\right\} \implies \boxed{\mathcal{F}_A \subsetneq \mathcal{F}}$$

* How many non-computable functions

$|\mathcal{F} \setminus \mathcal{F}_A| > |\mathbb{N}|$