Def: (Lebesgue class)

$\mathfrak{m}_k = \{E\subset\mathbb{R}^k : \forall\varepsilon>0\ \exists O_\varepsilon \text{ open st } O_\varepsilon\supset E,\ \lambda^*(O_\varepsilon\setminus E)\le\varepsilon\}$

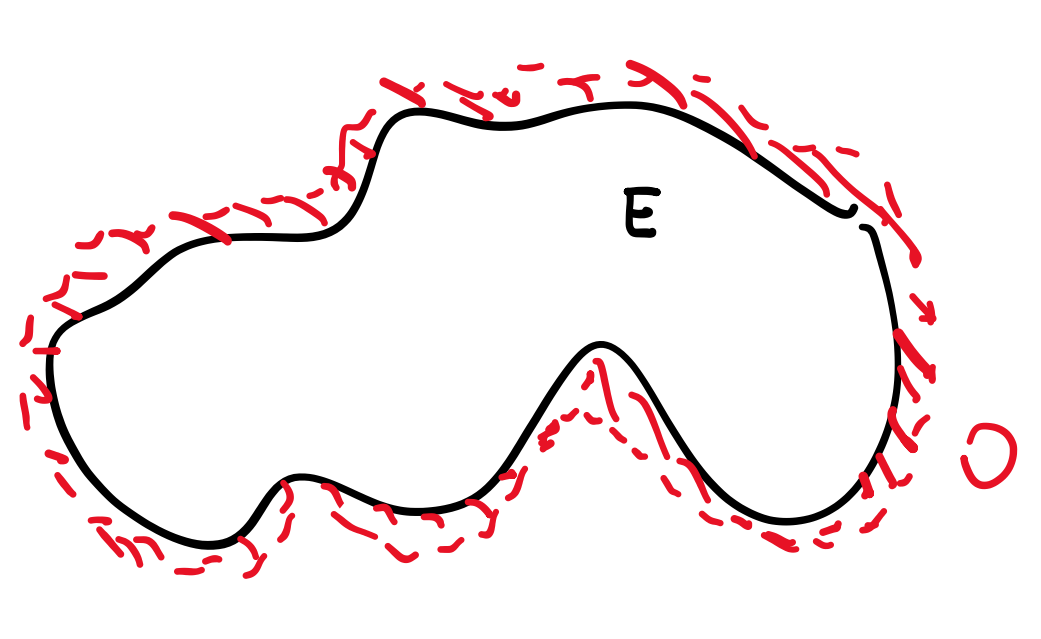
E is "well approx" by an open set

Open set: $O\subset\mathbb{R}^m$ is open ⟺ $\forall x\in O\ \exists B(x,r]$

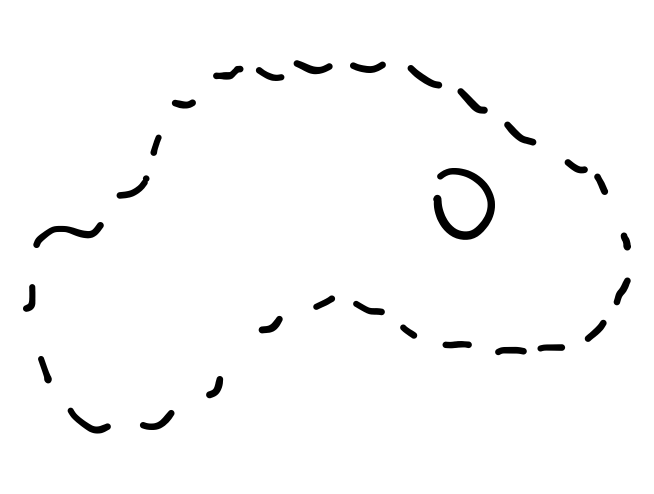
: $B(x,r] = \{y\in\mathbb{R}^k : \|y-x\|\le r\}$

$\emptyset$ is assumed open.

Ex: $O = \{x\in\mathbb{R}^k : g(x)<0\}$ with $g\in C(\mathbb{R}^k)$ is open.



Facts: 1. If O is open ⇒ $O\in\mathfrak{m}_k$.



$O_\varepsilon := O$ is open

$O_\varepsilon\setminus O = \emptyset$

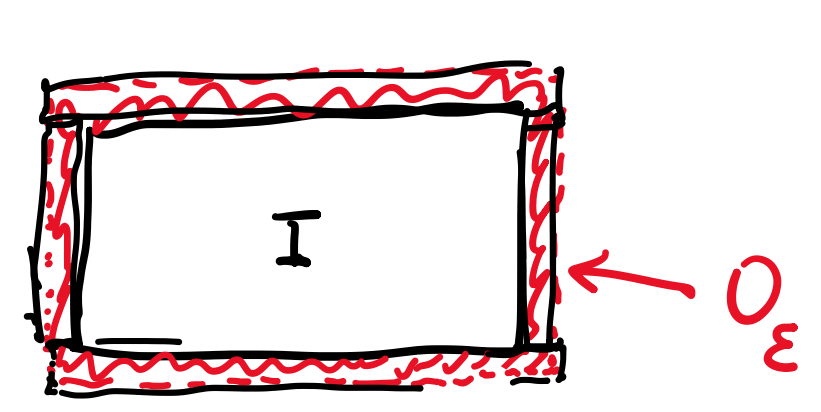
$\Downarrow$

$\lambda^*(O_\varepsilon\setminus O) = \lambda^*(\emptyset) = 0 \le\varepsilon.$

2. $I\subset\mathfrak{m}_k$ $\forall I$ interval.

$k=2$

$I\subset\mathbb{R}^2$ int.



$I = [a,b]\times[c,d]$

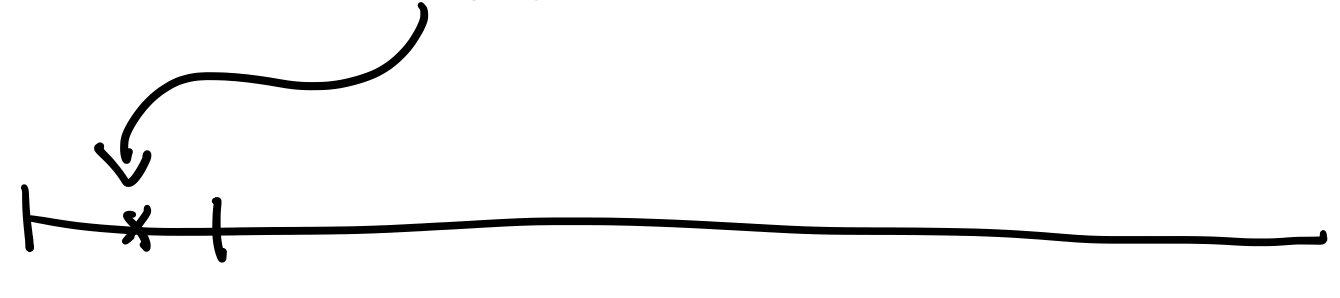
$O_\varepsilon =]a-\varepsilon, b+\varepsilon[\times]c-\varepsilon, d+\varepsilon[$ open

$\supset I$

$\lambda^*(O_\varepsilon\setminus I) = 2(b-a+2\varepsilon)\varepsilon + 2\varepsilon(d-c) = K\varepsilon$ ■

3. If $\lambda^*(N)=0$ ⇒ $N\in\mathfrak{m}_k$.

$\lambda^*(N) = \inf\{\underbrace{\sum|I_n|} : \bigcup I_n\supset N\} = 0$



$\exists (I_n^\varepsilon)$: $\boxed{\bigcup I_n^\varepsilon\supset N}$, $\boxed{\sum|I_n^\varepsilon|\le\varepsilon}$

($k=1$ $I_n^\varepsilon = [a_n,b_n]$ $|I_n^\varepsilon| = b_n-a_n$

$\cap$

$J_n^\varepsilon =]a_n-\varepsilon_n, b_n+\varepsilon_n[$ open

$O_\varepsilon := \underbrace{\bigcup J_n^\varepsilon}_{\text{open}} \supset \bigcup I_n^\varepsilon\supset N$ $\varepsilon_n>0$

$\lambda^*(\underbrace{O_\varepsilon\setminus N}_{\subset O_\varepsilon}) \le \lambda^*(O_\varepsilon) = \lambda^*(\bigcup J_n^\varepsilon)$

$\underset{\text{sub add}}{\le} \sum\lambda^*(J_n^\varepsilon)$

$\le \sum\lambda^*(\overset{J_n\ \cap}{[a_n-\varepsilon_n, b_n+\varepsilon_n]})$

$= \sum(\underbrace{b_n-a_n}_{|I_n^\varepsilon|} + 2\varepsilon_n)$

$= \underbrace{\sum|I_n^\varepsilon|}_{\le\varepsilon} + 2\underbrace{\sum\varepsilon_n}_{\le\varepsilon}$

$\le 2\varepsilon$

$\boxed{\varepsilon_n = \frac{1}{2^n}\varepsilon}$ $\sum\varepsilon_n = \overbrace{\sum\frac{1}{2^n}}^{1}\cdot\varepsilon$

⇒ $N\in\mathfrak{m}_k$. ■

4. If $E\in\mathfrak{m}_k$ and $\lambda^*(N)=0$ ⇒ $E\cup N\in\mathfrak{m}_k$.
$E\setminus N\in\mathfrak{m}_k$

sketch of the proof:

$\forall\varepsilon>0$ $\exists O_\varepsilon$ open : $\lambda^*(O_\varepsilon\setminus E)\le\varepsilon$, $O_\varepsilon\supset E$

$\exists\tilde{O}_\varepsilon \supset N$, $\lambda^*(\tilde{O}_\varepsilon\setminus N)\le\varepsilon$

Goal: $E\cup N\in\mathfrak{m}_k$

$O_\varepsilon\cup\tilde{O}_\varepsilon$ is open and $\supset E\cup N$.

Prove that $\lambda^*((O_\varepsilon\cup\tilde{O}_\varepsilon)\setminus(E\cup N)) \le K\varepsilon$

Thm: $\mathfrak{m}_k$ is a σ-alg of subsets of $\mathbb{R}^k$, ($\mathfrak{m}_k\supset$ intervals, open, closed sets), and

$\lambda^* : \mathfrak{m}_k \longrightarrow [0,+\infty]$

is a meas.

$\mathfrak{m}_k =:$ Leb. class.

$\lambda^*_k|_{\mathfrak{m}_k} = \lambda_k \equiv \lambda$ Leb. meas.

Do Ex 2.3.1/2/5/6/7/8

Measurable Functions

Def: $(X,\mathcal{F},\mu)$ be a meas sp.
A funct. $f: E\subset X\longrightarrow\mathbb{R}$ $([-\infty,+\infty])$

is called measurable funct. if

$\boxed{\{f\in I\} = \underbrace{\{x\in E : f(x)\in I\}}_{\in\mathcal{F}}\ \forall I\subset\mathbb{R} \text{ interval.}}$

Ex (*)

$f = 1_E(x) = \begin{cases} 1 & x\in E\\ 0 & x\notin E\end{cases}$ (indicator funct of E, unit funct.)

~~$\delta_x(E)$~~ $1_E : X\longrightarrow\mathbb{R}$

1_E is meas ⟺ $E\in\mathcal{F}$.

Sol: We have to check

$\{f\in I\}\in\mathcal{F}$

$\|$

$\{x\in X : 1_E(x)\in I\}$

$\{f\in I\} = \begin{cases}\emptyset & 0,1\notin I\\ E^c & 0\in I,\ 1\notin I\\ E & 0\notin I,\ 1\in I\\ X & 0,1\in I\end{cases} \in\mathcal{F}$ $\forall I$ int

⟺ $\underset{\in\mathcal{F}}{\emptyset}$, E^c, E, $X\in\mathcal{F}$

⟺ $E, E^c\in\mathcal{F}$

⟺ $E\in\mathcal{F}$. ■

Def: We call simple funct. a funct $s: X\longrightarrow\mathbb{R}$ that takes only a finite number of values:

$s(X) = \{c_1, c_2, \dots, c_N\}$

Prop: s is meas ⟺ $\{s=c_j\}\in\mathcal{F}$ $\forall j=1,\dots,N$

($E_j := \{s=c_j\} = \{x\in X : s(x)=c_j\}\subset X$)

$s(x) = \sum_{j=1}^N c_j 1_{E_j}(x)$.

Prop: The following properties are equiv.

i) f is $\mathcal{F}$-meas

ii) $\{f\ge a\}\in\mathcal{F}$ $\forall a\in\mathbb{R}$

iii) $\{f>a\}\in\mathcal{F}$ $\forall a\in\mathbb{R}$.

Proof:

i) ⇒ ii) trivial

Hp: f is meas $\overset{\text{by def}}{\iff}$ $\{f\in I\}\in\mathcal{F}$ $\forall I$ int

Th: $\{f\ge a\}\in\mathcal{F}$

$\|$

$\{f\in\underbrace{[a,+\infty[}_{I}\}\in\mathcal{F}$

ii) ⇒ i)

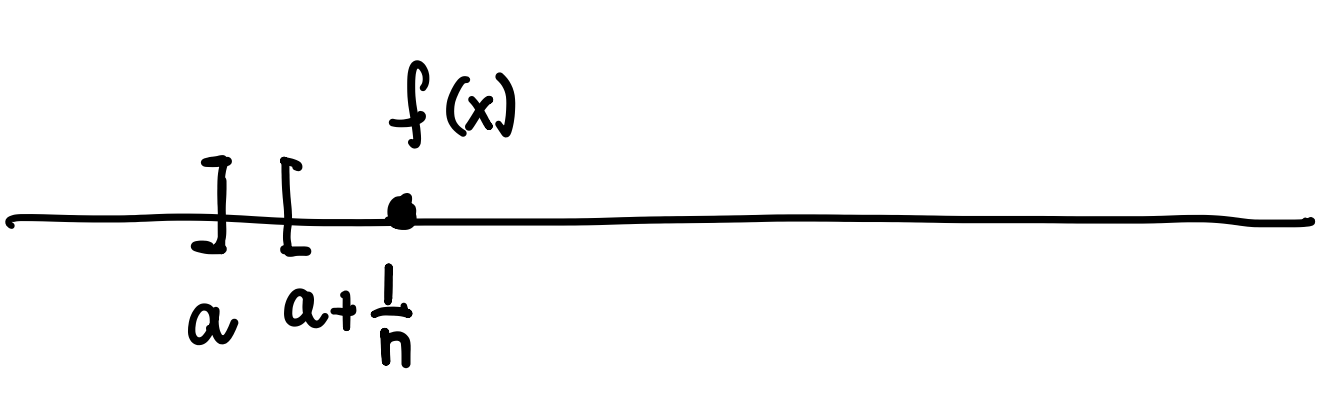
Hp: $\boxed{\{f\ge b\}\in\mathcal{F}\ \forall b}$

Th: $\{f\in I\}\in\mathcal{F}$ $\forall I$ int.

true (by Hp) when $I=[a,+\infty[$.

What if $I =]a,+\infty[$

$\{f\in I\} = \{f\in\underline{]a,+\infty[}\} = \{f>a\}$



$f(x)>a \iff \exists n : f(x)\ge a+1/n > a$

$\Updownarrow$

$x\in\{f>a\} \iff \exists n : x\in\{f\ge a+1/n\}$

$\Updownarrow$

$x\in\bigcup_n\{f\ge a+1/n\}$

⇒

$\{f>a\} = \bigcup_n\underbrace{\{f\ge a+1/n\}}_{\in\mathcal{F}\ (Hp)}\in\mathcal{F}$ ($\mathcal{F}$ σ-alg)

⇒ $\{f>a\}\in\mathcal{F}$ $\forall a$.

$\{f\le b\} = \underset{\in\mathcal{F}}{\{f>b\}^c}\in\mathcal{F}$ ($\mathcal{F}$ σ-alg)

$\{f<b\} = \underset{\in\mathcal{F}}{\{f\ge b\}^c}\in\mathcal{F}$ $\forall b$

$\{a\le f\le b\} = \overset{\in\mathcal{F}}{\{f\le b\}}\cap\overset{\in\mathcal{F}}{\{f\ge a\}}\in\mathcal{F}$

$\|$

$\{f\in[a,b]\}$

and so on.

ii) ⇒ iii) proved through the rel

$\{f>a\} = \bigcup_n\{f\ge a+1/n\}$

iii) ⇒ ii) you have to prove

$\{f\ge a\} = \dots\dots\{\dots>\dots\dots\}$.

(complete the proof.) ■