UNIVERSITÀ DEGLI STUDI DI PADOVA

# Network Science

A.Y. 23/24

ICT for Internet & multimedia, Data science, Physics of data

# Centrality

Importance of nodes in a network

# Centrality

From Wikipedia, the free encyclopedia

*For the statistical concept, see Central tendency.*

In graph theory and network analysis, indicators of **centrality** identify the most important vertices within a graph. Applications include identifying the most influential person(s) in a social network, key infrastructure nodes in the Internet or urban networks, and super-spreaders of disease. Centrality concepts were first developed in social network analysis, and many of the terms used to measure centrality reflect their sociological origin.[1] They should not be confused with node influence metrics, which seek to quantify the influence of every node in the network.

## Degree centrality [ edit ]

*Main article: Degree (graph theory)*

## PageRank centrality [ edit ]

*Main article: PageRank*

## Betweenness centrality [ edit ]

*Main article: Betweenness centrality*

## Eigenvector centrality [ edit ]

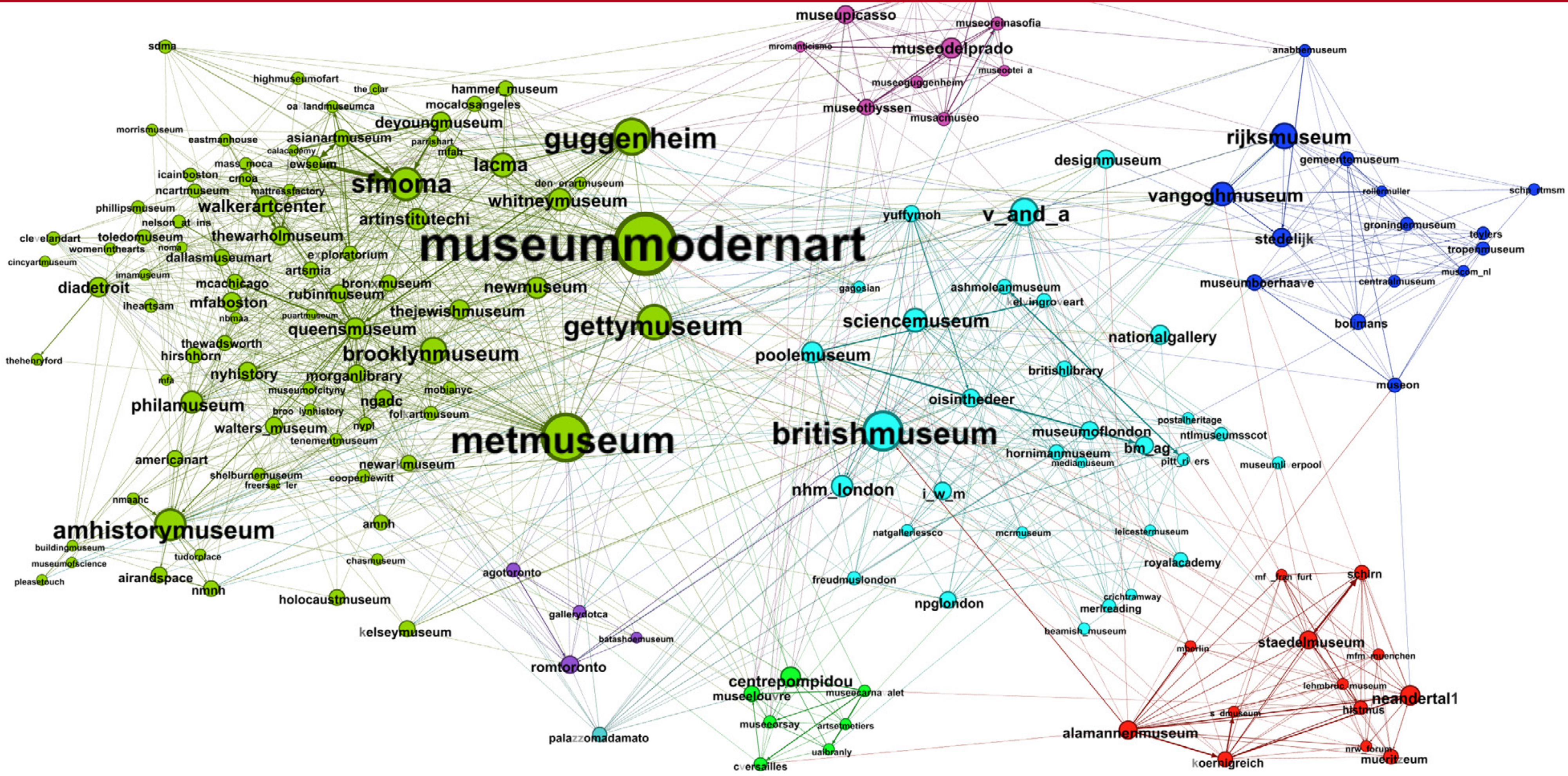
*Main article: Eigenvector centrality*

## Closeness centrality [ edit ]

*Main article: Closeness centrality*

# An example of node centrality

## museums network




Can we do this efficiently, i.e., by using automatic, reliable, and fast methods?

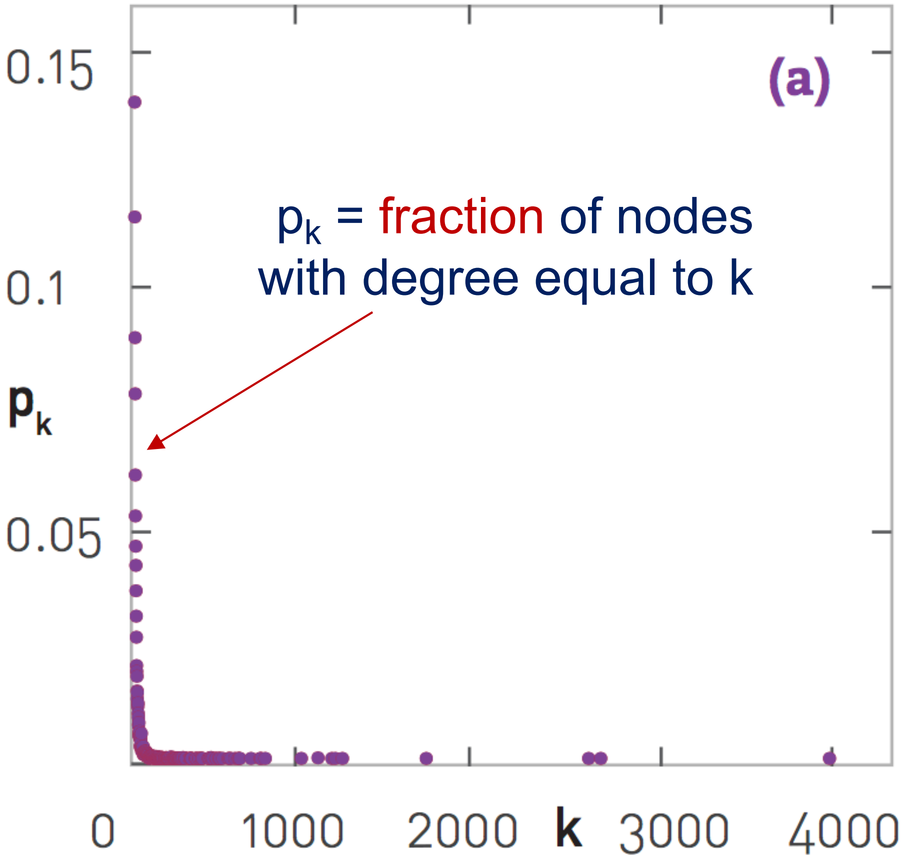
# Degree centrality

Counting the in/out degrees of nodes

# The degree distribution

## for an undirected network

LINEAR SCALE

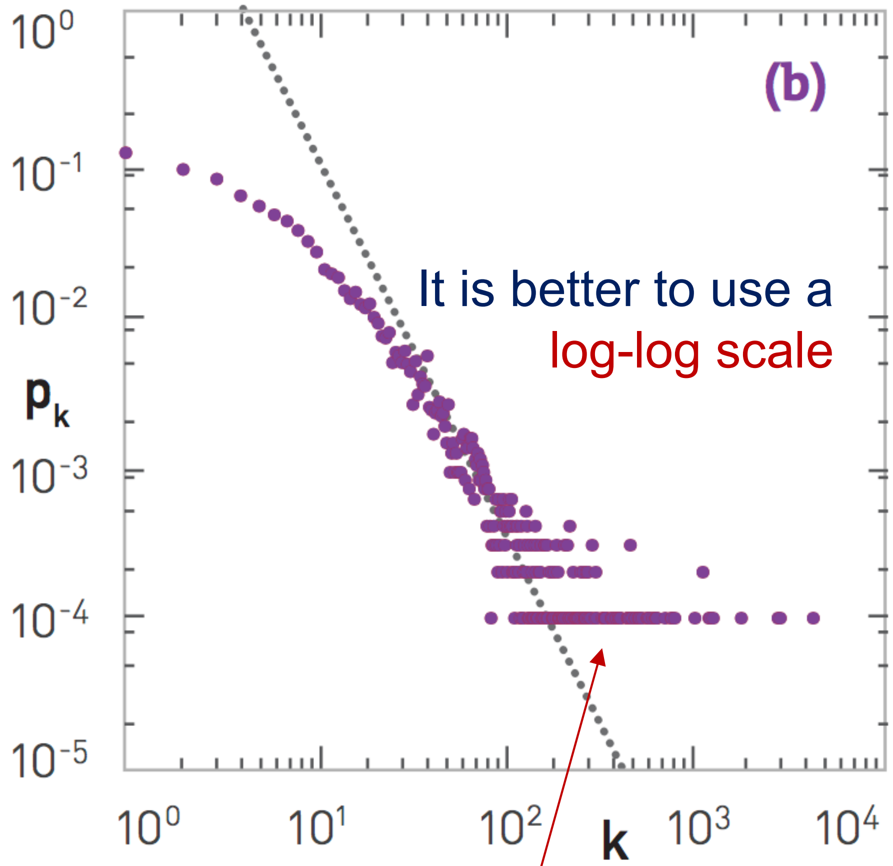



$p_k$ = fraction of nodes with degree equal to k

Wide range for the degree k !

Wide range for the probability $p_k$ !

LINEAR BINNING



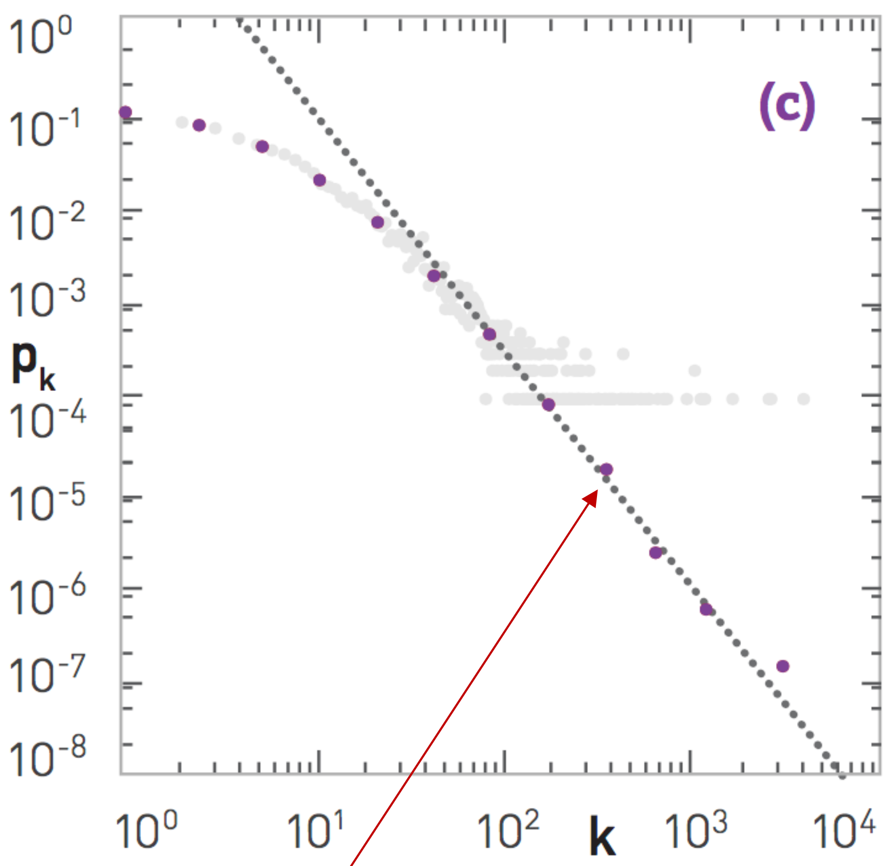

It is better to use a log-log scale

plateau

# Alternative log representations

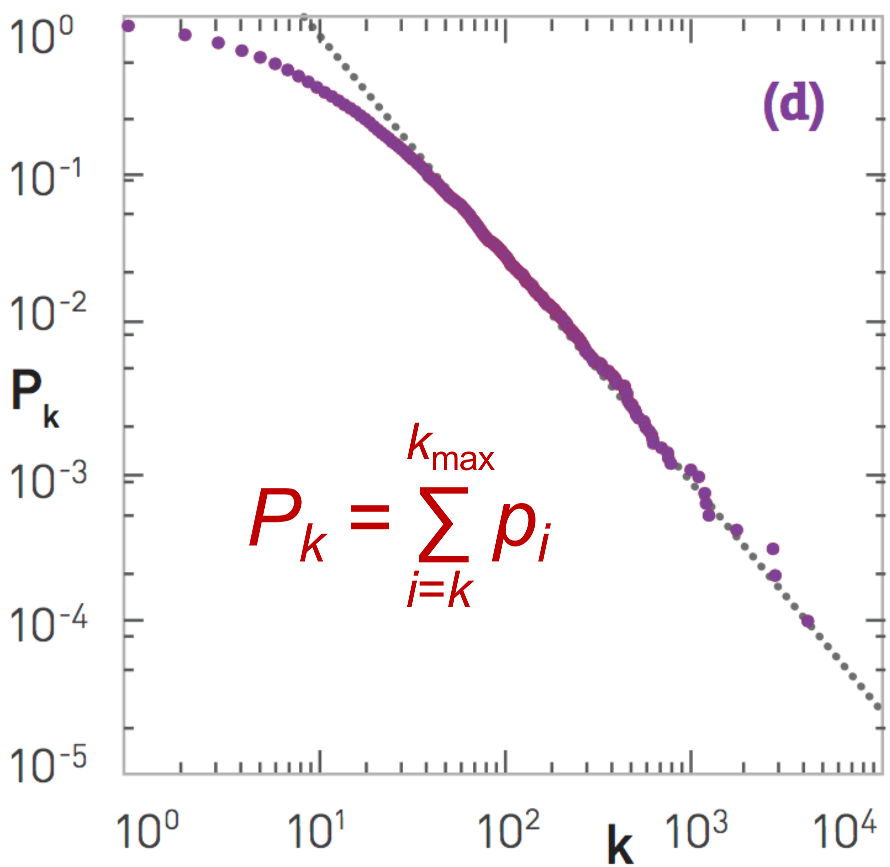
## for an undirected network

LOG-BINNING




$p_{k_i}$ = fraction of nodes with degree in the range $[k_i,k_{i+1})$ where $k_i$ are uniformly distributed in the log-domain, $k_{i+1}=k_i \cdot \Delta$

CUMULATIVE

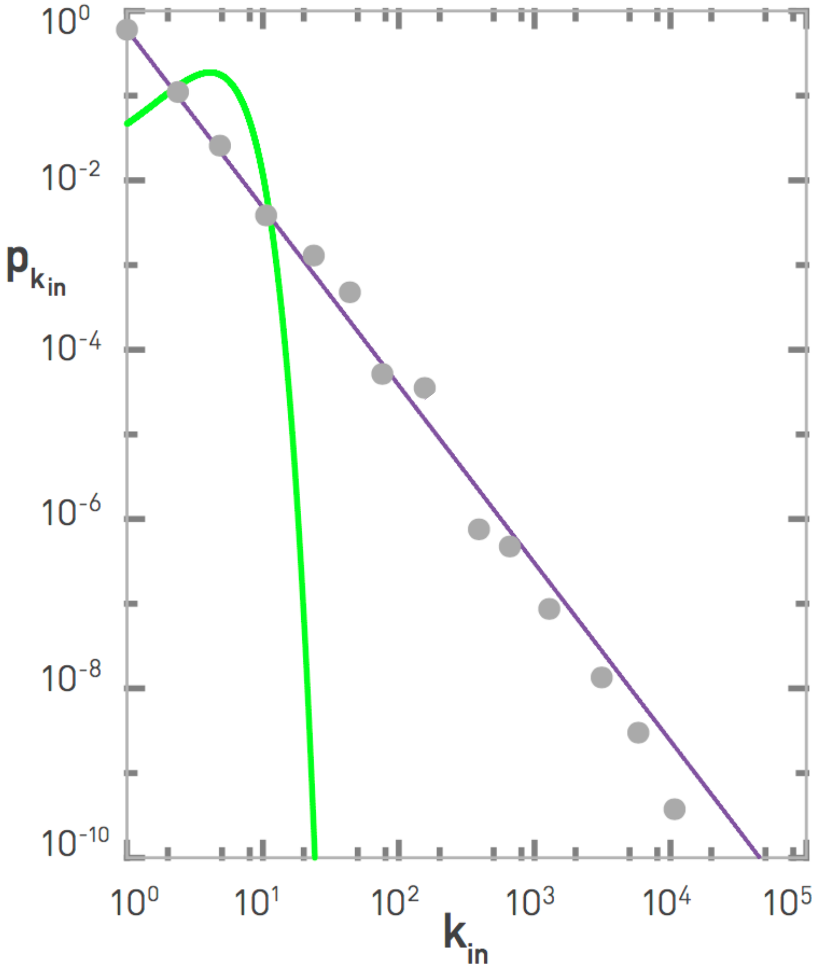


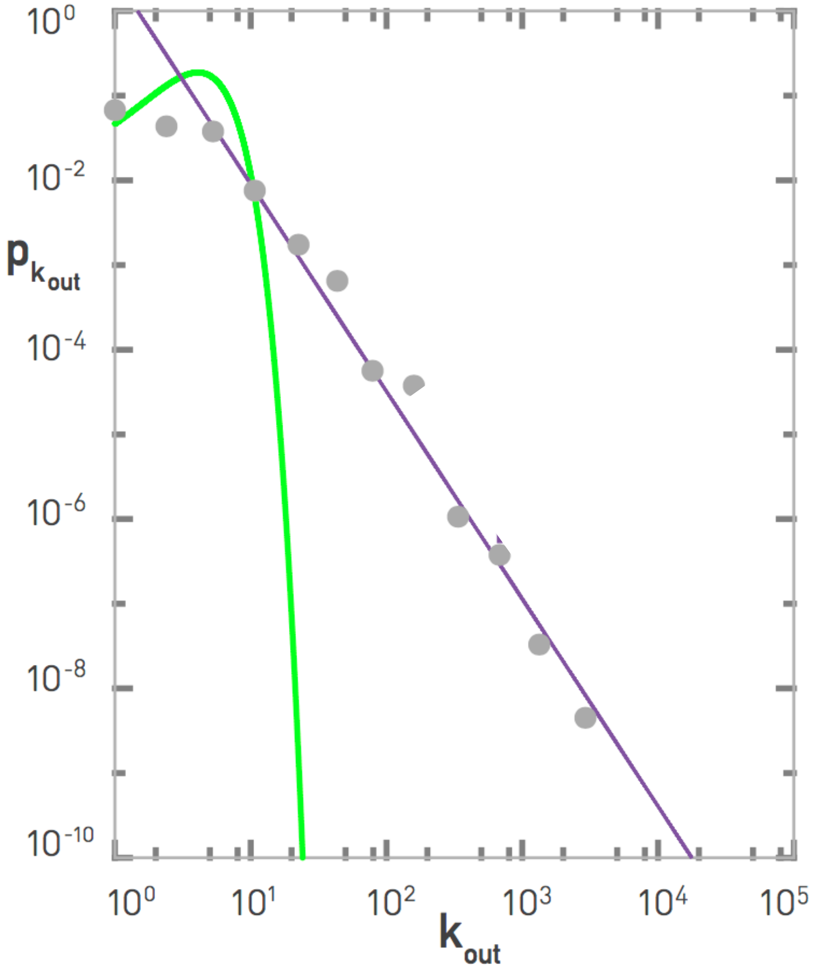

$$P_k = \sum_{i=k}^{k_{max}} p_i$$

complementary cumulative distr. function (CCDF)

# Two degree distributions

## for directed networks



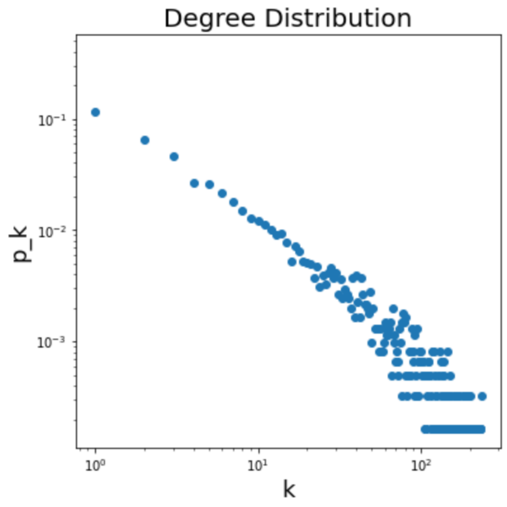

$p_{kin}$ = fraction of nodes with input degree equal to $k_{in}$

$p_{kout}$ = fraction of nodes with output degree equal to $k_{out}$

# Pseudocode example

https://snap.stanford.edu/data/wiki-Vote.html

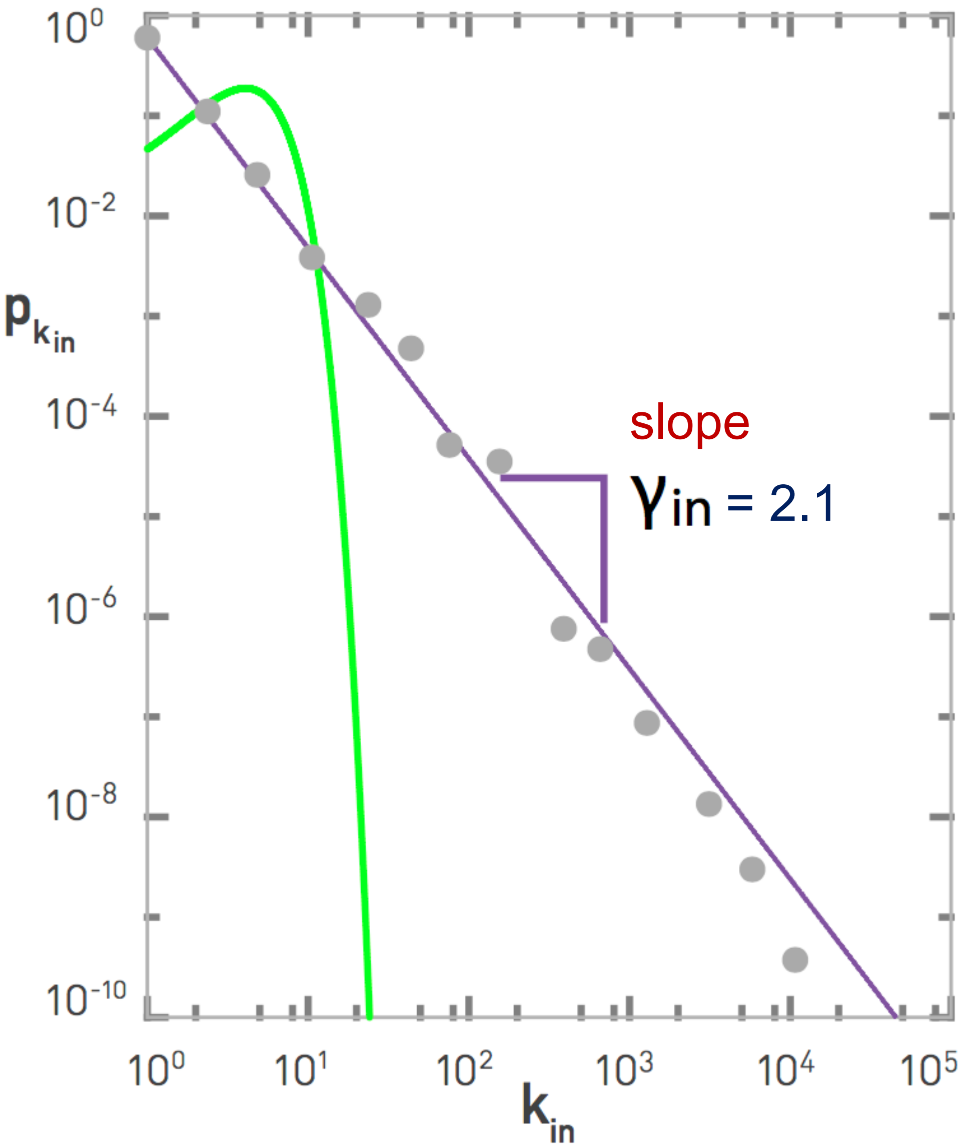
```
G = np.loadtxt('Wiki-Vote.txt').astype(int)

# adjacency matrix
N = np.max(G)
A = csr_matrix((np.ones(len(G)), (G[:, 1], G[:, 0])))


#distribution
which_deg = 0 # 0=out degree, 1=in degree
d = np.sum(A, which_deg) # out degree for each node
d = np.squeeze(np.asarray(d)) # from matrix to array
d = d[d>0] # avoid zero degree
k = np.unique(d) # degree samples
pk = np.histogram(d, k)[0] # occurrence of each degree
pk = pk/np.sum(pk) # normalize to 1
Pk = 1 - np.cumsum(pk) # complementary cumulative
```




```
fig = plt.figure()
plt.loglog(pk, 'o')
plt.title("Degree Distribution", size = 20)
plt.xlabel("k", size = 18)
plt.ylabel("p_k", size = 18)
plt.show()
```

# The power-law

## typical behaviour of social networks




many networks follow a power-law

$$\ln(p_k) = c - \gamma \cdot \ln(k)$$

$$\boxed{p_k = C \cdot k^{-\gamma}}$$

how to correctly estimate the slope $\gamma$ ?

Degree distribution $p_k = C\, k^{-\gamma}$

Constant $C$ is determined by the (approx.) normalization condition

$$\int_{k_{min}}^{\infty} p_k \, dk = C \cdot k_{min}^{-(\gamma-1)} / (\gamma-1) = 1$$

Target PDF $\boxed{p(k|\gamma) = (\gamma-1)/k_{min} \cdot (k/k_{min})^{-\gamma}}$

# ML estimate for the exponent $\gamma$

## the most reliable approach

**ML criterion**: find the $\gamma$ that best fits the data

$$\max_\gamma \sum_i \ln p(k_i|\gamma)$$

*where* $k_i$ is the measured degree of node $i$

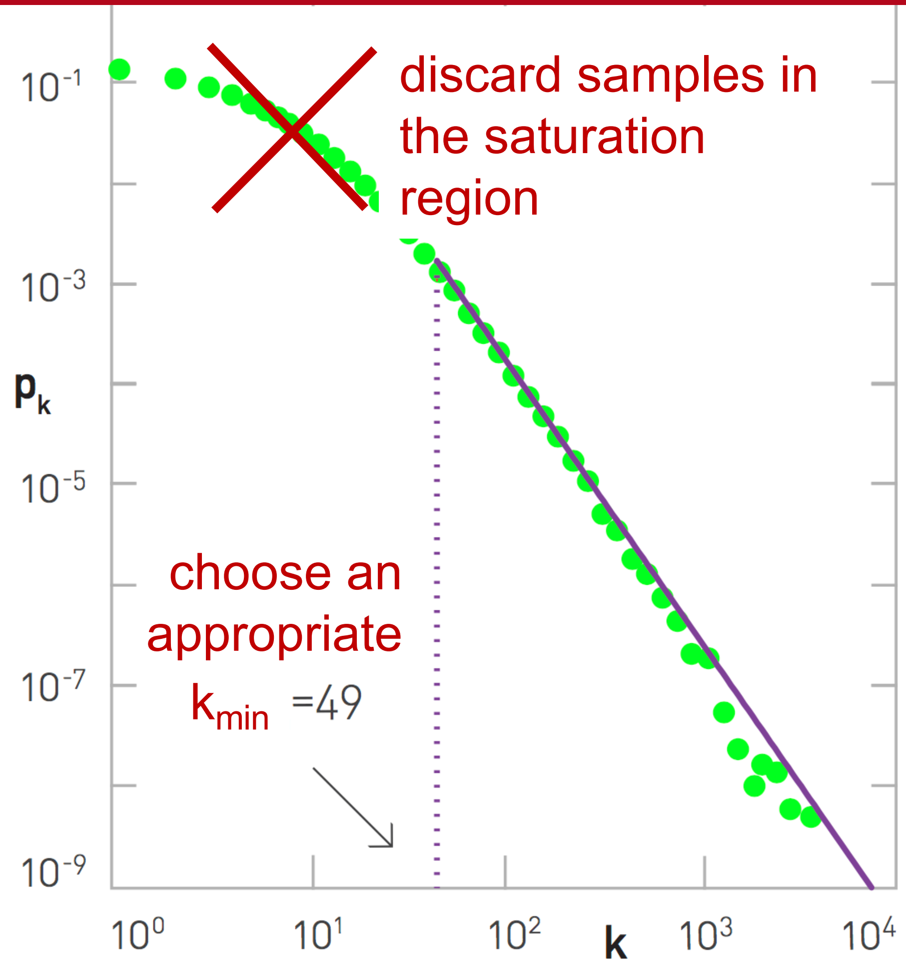
$$f(\gamma) = \sum \ln((\gamma-1)/k_{\min}) - \gamma \ln(k_i/k_{\min})$$

$$f'(\gamma) = \sum 1/(\gamma-1) - \ln(k_i/k_{\min}) = 0$$

$$\boxed{\gamma = 1 + \sum_i 1 / \sum_i \ln(k_i/k_{\min})}$$

# Pseudocode example

## to estimate the exponent




MATLAB

```
which_deg = 1; % 1 = out degree
d = full(sum(A,which_deg));
d2 = d(d>=kmin); % restrict range
ga = 1+1/mean(log(d2/kmin)); % estimate the exponent
```

UNIVERSITÀ DEGLI STUDI DI PADOVA

# The value of the exponent γ in real networks γ ∈ [2,5]

| NETWORK | $N$ | $L$ | $\langle k \rangle$ | $\gamma_{in}$ | $\gamma_{out}$ | $\gamma$ |
|---|---|---|---|---|---|---|
| Internet | 192,244 | 609,066 | 6.34 | - | - | 3.42* |
| WWW | 325,729 | 1,497,134 | 4.60 | 2.00 | 2.31 | - |
| Power Grid | 4,941 | 6,594 | 2.67 | - | - | Exp. |
| Mobile Phone Calls | 36,595 | 91,826 | 2.51 | 4.69* | 5.01* | - |
| Email | 57,194 | 103,731 | 1.81 | 3.43* | 2.03* | - |
| Science Collaboration | 23,133 | 93,439 | 8.08 | - | - | 3.35* |
| Actor Network | 702,388 | 29,397,908 | 83.71 | - | - | 2.12* |
| Citation Network | 449,673 | 4,689,479 | 10.43 | 3.03** | 4.00* | - |
| E. Coli Metabolism | 1,039 | 5,802 | 5.58 | 2.43* | 2.9 0* | - |
| Protein Interactions | 2,018 | 2,930 | 2.9 0 | - | - | 2.89* |

* = good statistical fit with a power-law

** = good fit for a power-law with an exponential cutoff

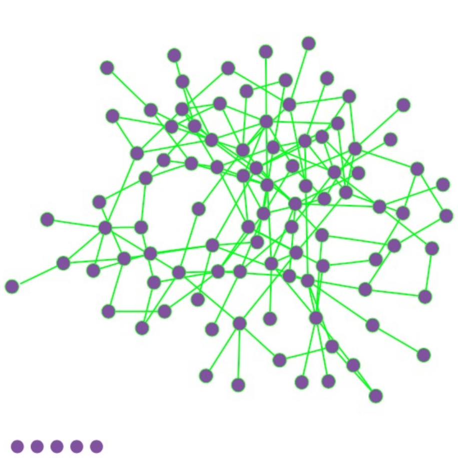
Exp = good fit with an exponential distribution $e^{-ak}$

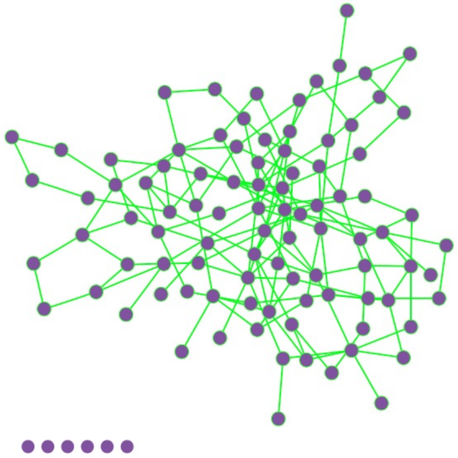
# Explaining the power-law

Preferential attachment

# Random networks

## Erdös-Rényi model 1959/60





- The random network is the simplest model:

  pick a probability *p*, with *0<p<1*

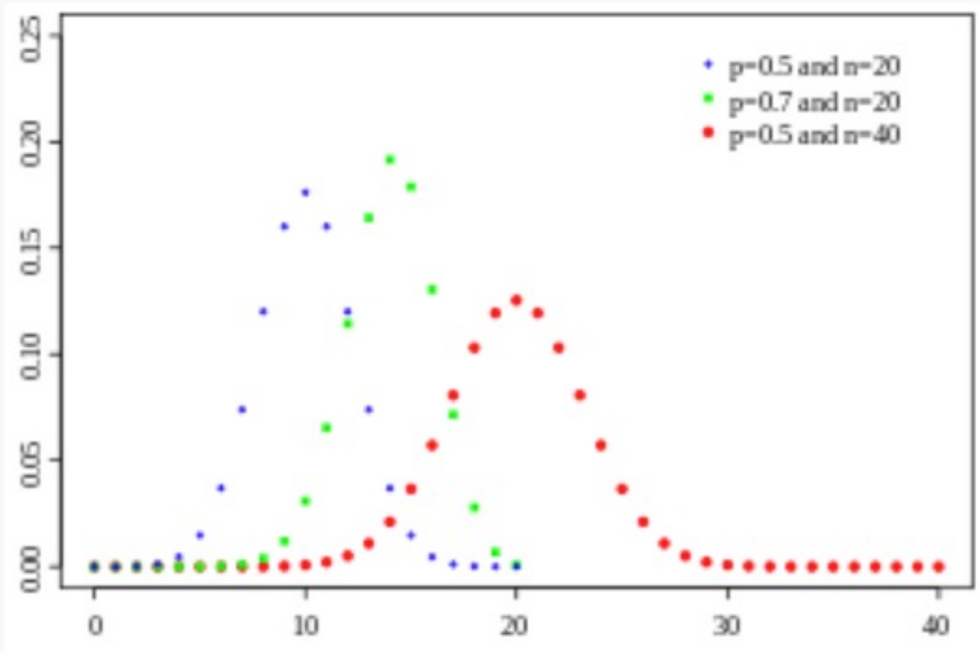
  activate each link (*i,j*) with probability p
- The number of links is variable
- There might be isolates
- Easy to calculate fundamental parameters

# Binomial distribution

## explains the degree distribution for random networks

| | |
|---|---|
| **Notation** | $B(n,p)$ |
| **Parameters** | $n \in \{0,1,2,\ldots\}$ – number of trials<br>$p \in [0,1]$ – success probability for each trial<br>$q = 1 - p$ |
| **Support** | $k \in \{0,1,\ldots,n\}$ – number of successes |
| **PMF** | $\binom{n}{k} p^k q^{n-k}$ |
| **CDF** | $I_q(n-k, 1+k)$ |
| **Mean** | $np$ |
| **Median** | $\lfloor np \rfloor$ or $\lceil np \rceil$ |
| **Mode** | $\lfloor (n+1)p \rfloor$ or $\lceil (n+1)p \rceil - 1$ |
| **Variance** | $npq$ |
| **Skewness** | $\frac{q-p}{\sqrt{npq}}$ |
| **Ex. kurtosis** | $\frac{1-6pq}{npq}$ |

Probability mass function




P(k;n,p) = probability that *k* out of *n* trials are positive, where each is positive with probability *p*

# Degree distribution
## in random networks

- The number of neighbours is binomially distributed

  $P(k;n,p)$ = probability that a node has exactly k neighbours, with number of possible neighbours $n = N\text{-}1$

- Average # of neighbours

  this defines $p$

  $$\langle k \rangle = (N\text{-}1)p \;\rightarrow\; \boxed{p = \langle k \rangle/(N\text{-}1)}$$

- Variance

  $p$ is usually very small (since $\langle k \rangle \ll N$)

  $$\sigma_x^2 = (N\text{-}1)p(1\text{-}p) \simeq \langle k \rangle$$

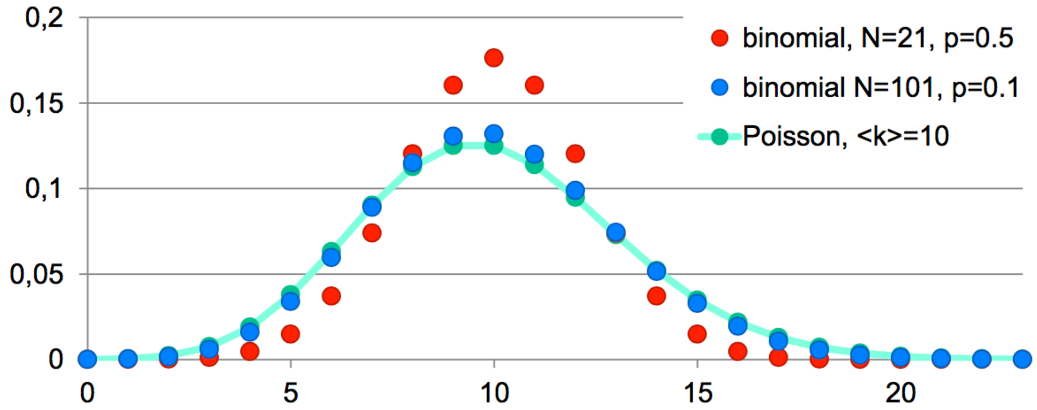
  tight around the mean

# Poisson approximation

## why random networks are called Poisson networks

- Poisson distribution (easier to use)

$$P\left[x=k\right]=\frac{m_x^k}{k!}\cdot e^{-m_x}$$




- Very good approximation of binomial for small *p* (and at small *k*)

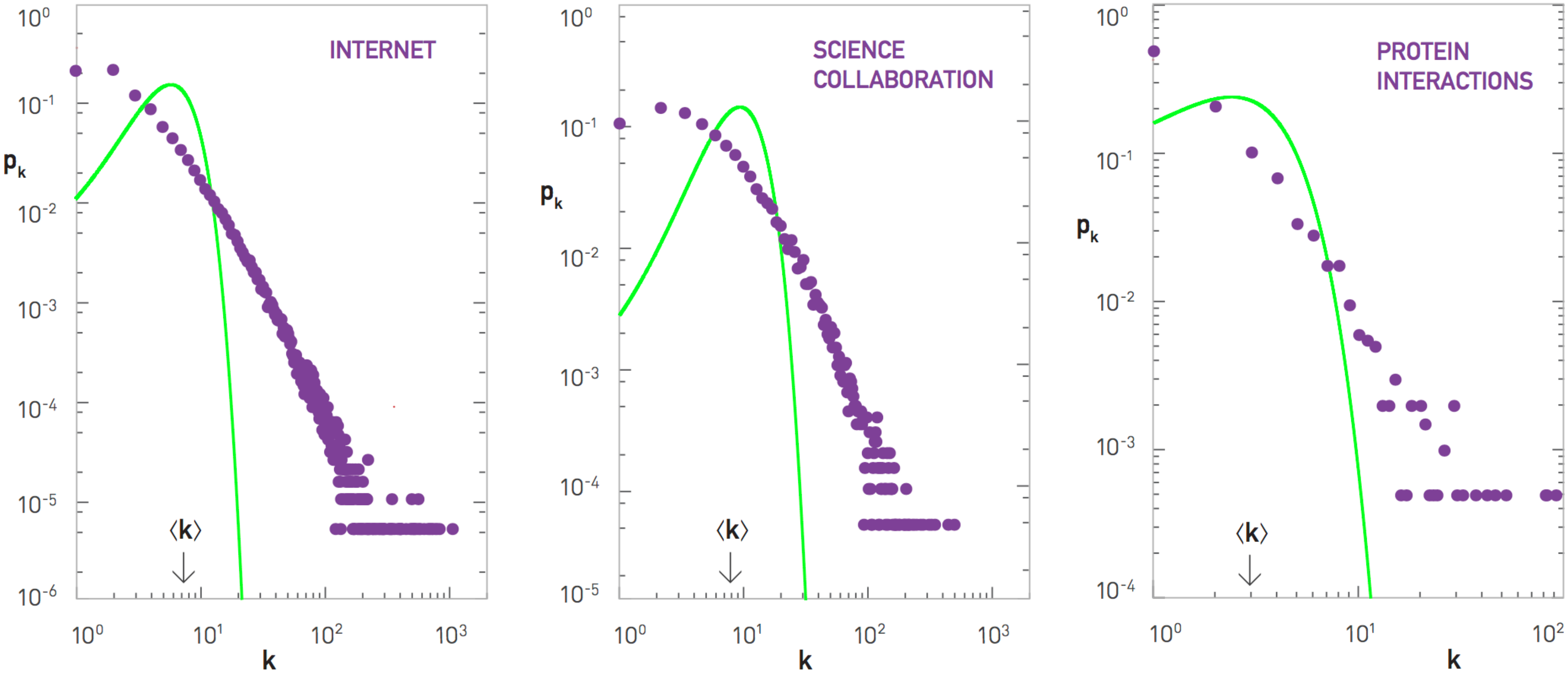
active part

*p*

$$P\left[x=k\right]=\underbrace{\frac{(n-k+1)\dots(n-1)n}{n^k}}_{\simeq 1}\cdot\frac{m_x^k}{k!}\cdot\underbrace{\left(1-\frac{m_x}{n}\right)^{n-k}}_{\simeq \text{const}}$$

# Are real networks Poisson?

no, they aren't

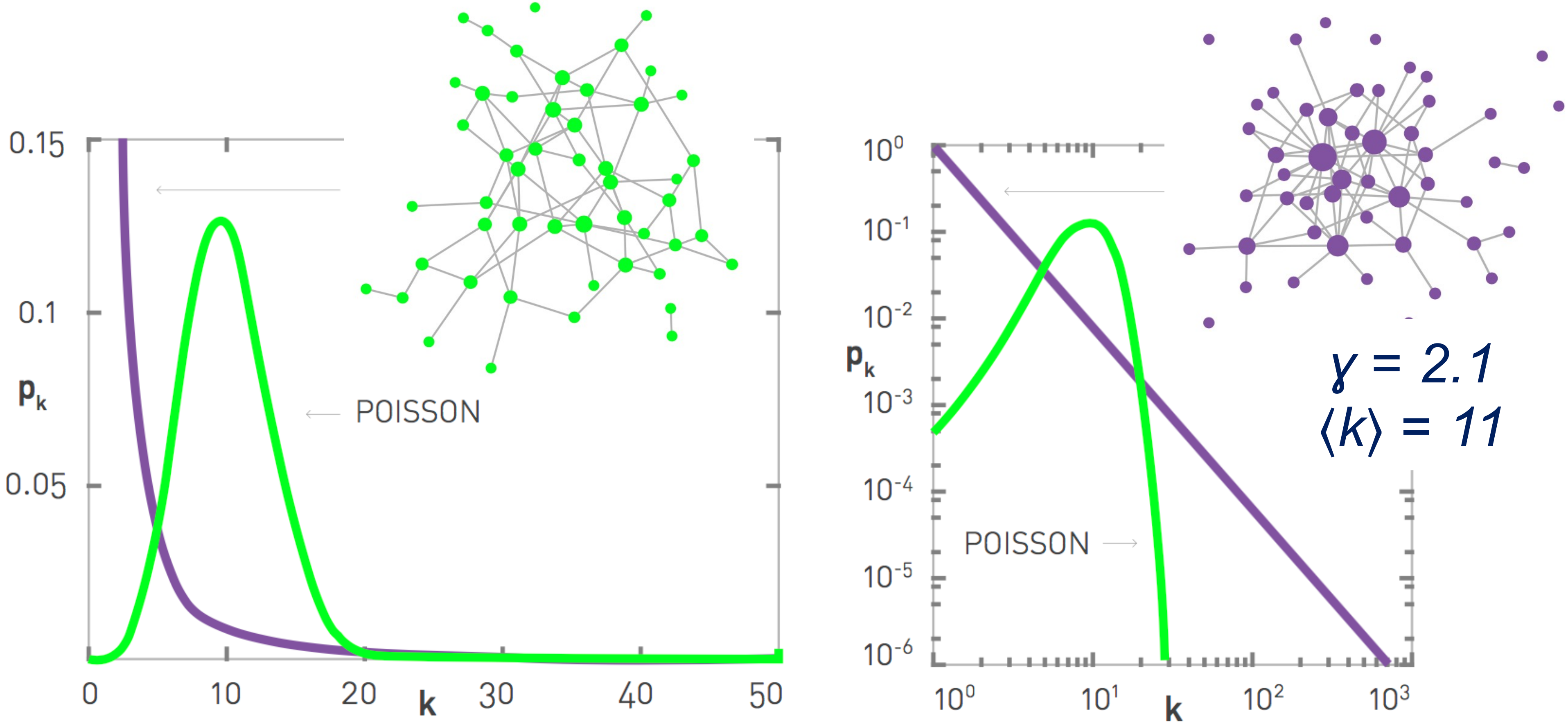



No! Poisson networks are deprived of hubs

… but, nevertheless, Poisson networks capture some aspects

# Poisson versus power law

## a comparison




Power-law is heavy tailed (presence of hubs) - like Weibull, lognormal, Lévy

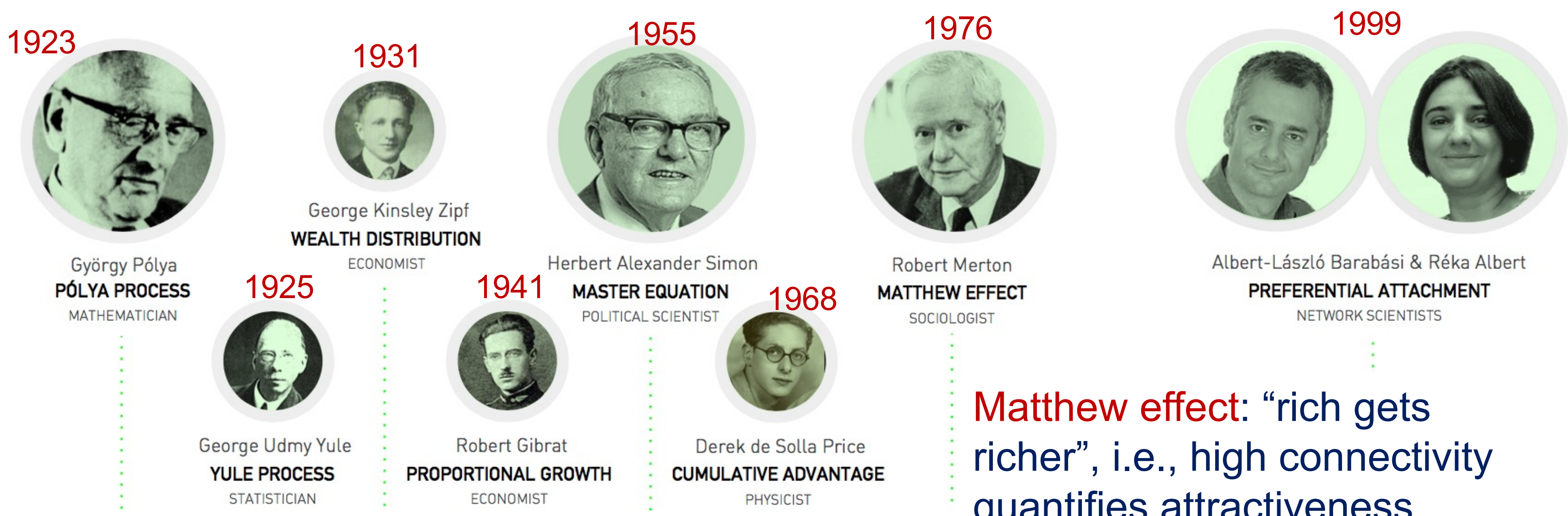
# Preferential attachment

a simple concept that (partially) explains the power-law

## Nodes link to the more connected nodes

e.g., think of www

## This idea has a long history




Matthew effect: “rich gets richer”, i.e., high connectivity quantifies attractiveness

# The Barabasi-Albert model

Barabási, Albert. "Emergence of scaling in random networks" (1999)

Start with $m_0$ nodes arbitrarily connected, with $\langle k \rangle$=m

- Growth

  add a node (the *N*th) with *m* links that connect the node to nodes in the network

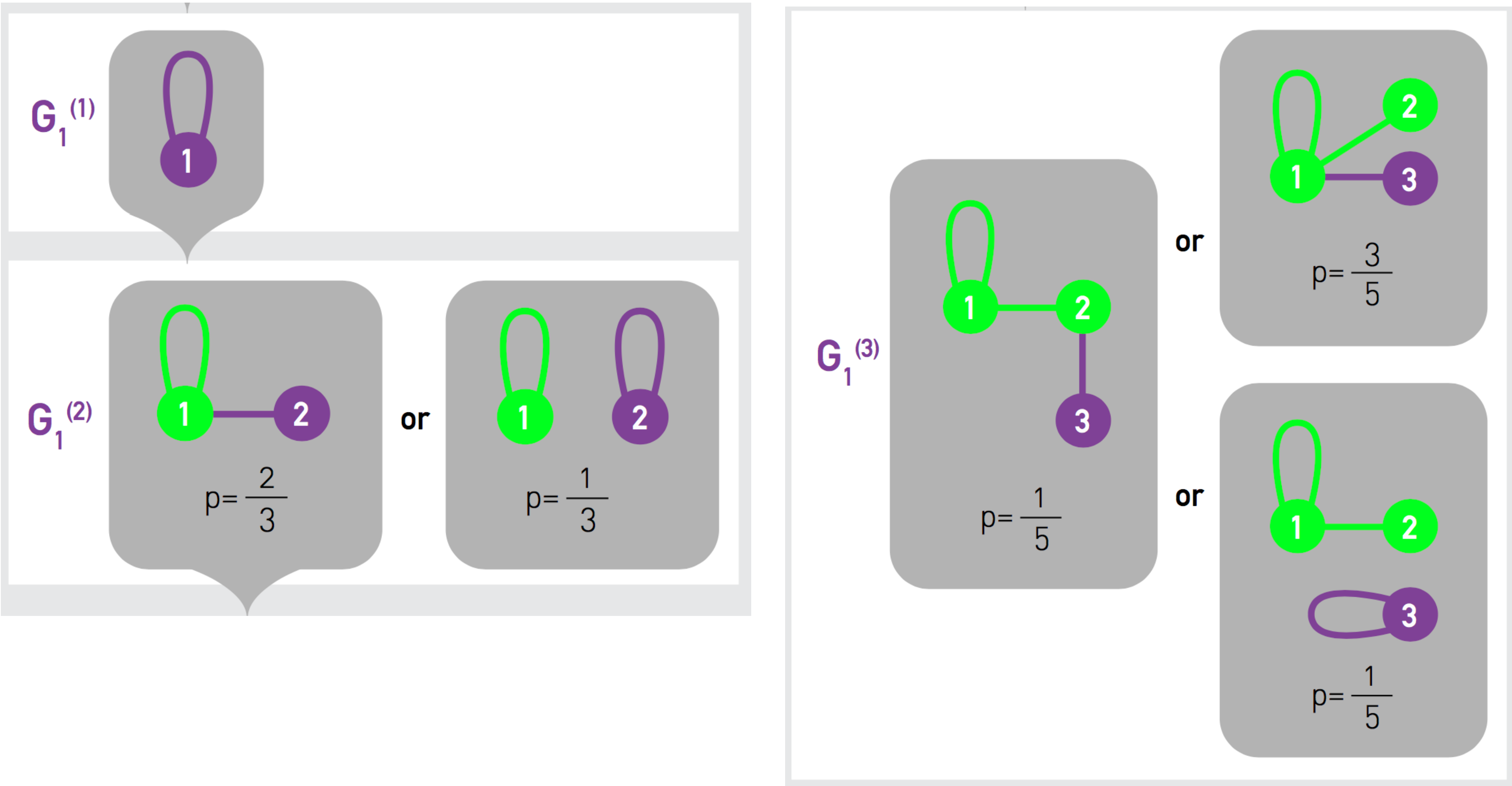
- Preferential attachment

  $p_i = k_i / C$ probability of connecting to node *i*

  $p_i = 1/C$ for self-loops

$$C = 1 + \sum k_i = 1 + 2(N-1)m$$

# An example
## with m=1



# Approximate analysis

## evolution of nodes degree

- Increase in the degree (at each step)

$$\Delta k_i \simeq \underbrace{m}_{\text{trials}} \cdot \underbrace{k_i / (1+2m(N\text{-}1))}_{\text{probability per trial}} \simeq k_i / 2N$$

- Approximation in the continuous domain

$$\Delta k_i \simeq dk_i/dN \rightarrow \boxed{dk_i/k_i \simeq \tfrac{1}{2}\, dN/N}$$

- Integration

$$\ln(k_i) = \tfrac{1}{2} \ln(N) + \text{cost.} \rightarrow k_i = c\, N^{1/2}$$

- Recalling that node $i$ joins the network at time $N = i$

$$k_i\,(N{=}i) = m \rightarrow \boxed{k_i\,(N) = m\,(\mathrm{N/i})^{1/2}}$$

½ is the dynamic exponent

# Approximate analysis
## degree distribution

- Recall $k_i = m\,(N/i)^{½}$
- The number of nodes with degree smaller than $k$ is

$$k_i < k \;\rightarrow\; m\,(N/i)^{½} < k$$

$$\rightarrow\; i > N\,(m/k)^2 \;\rightarrow\; \boxed{N - N\,(m/k)^2}$$

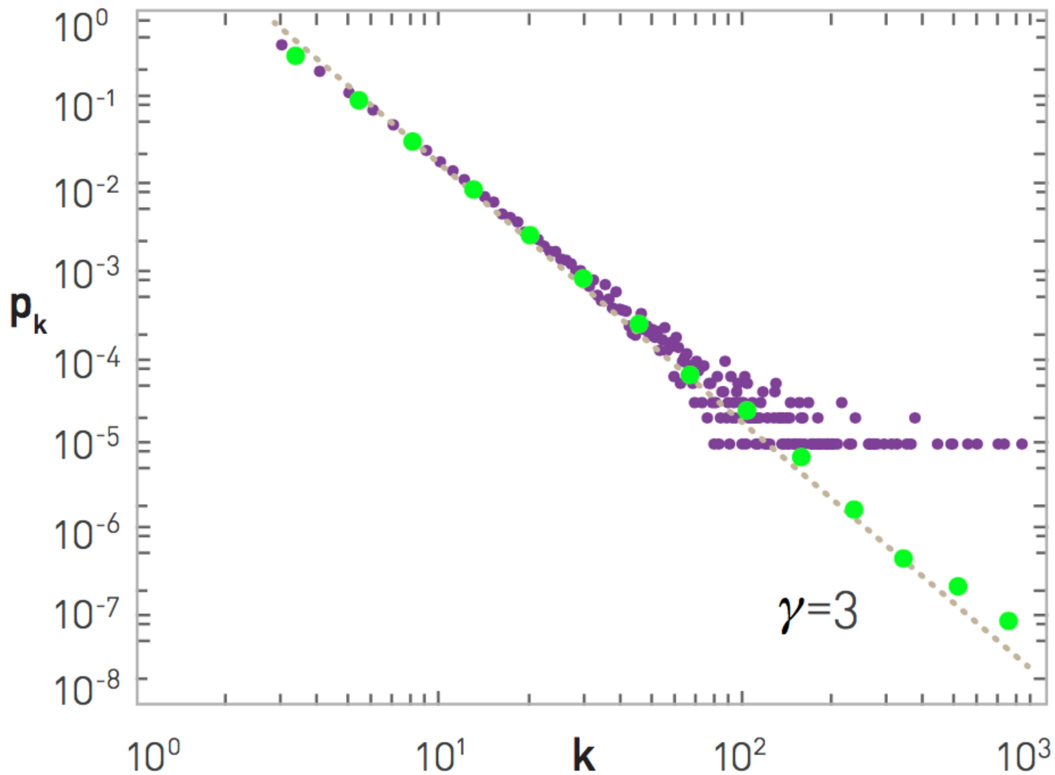
- CDF is $P_k = \mathrm{P}[k_i \le k] = 1 - (m/k)^2$
- The degree distribution is

$$\mathrm{d}P_k\,/\mathrm{d}k = \boxed{p_k = 2\,m^2\,/\,k^3}$$

# The Barabasi-Albert model

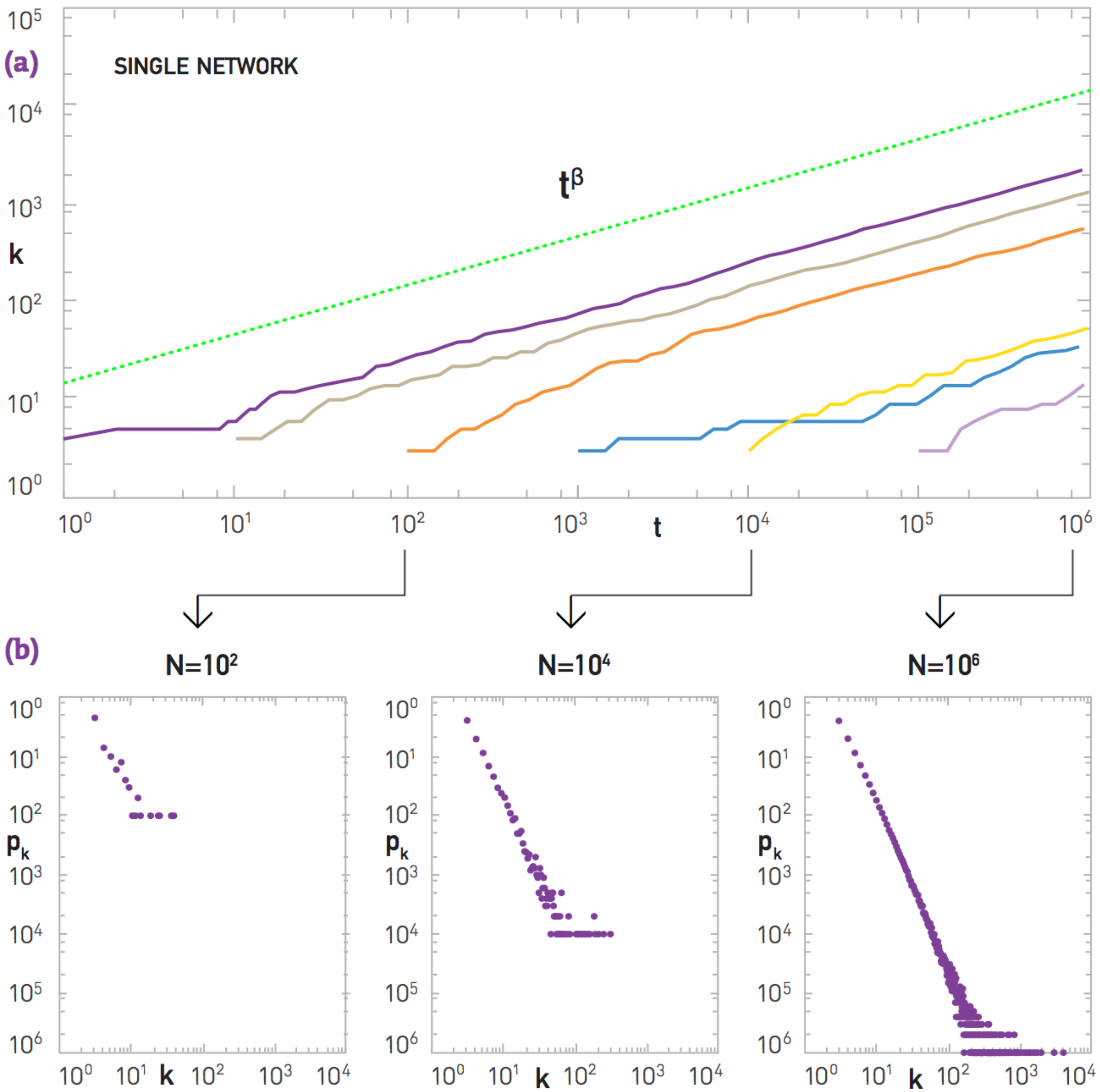
## wrap up

- Depending on the implementation there might be self/multiple links
- Most nodes have a small degree (exactly *m* for the youngest ones)
- Hubs appear
- The average degree is $\langle k \rangle = 2m$, and in fact $L = Nm = \frac{1}{2}\langle k \rangle N$
- The resulting degree distribution is always a power-law with exponent $\gamma = 3$



# The Barabasi-Albert model

## consequence of $k_i = m\,(N/i)^{1/2}$

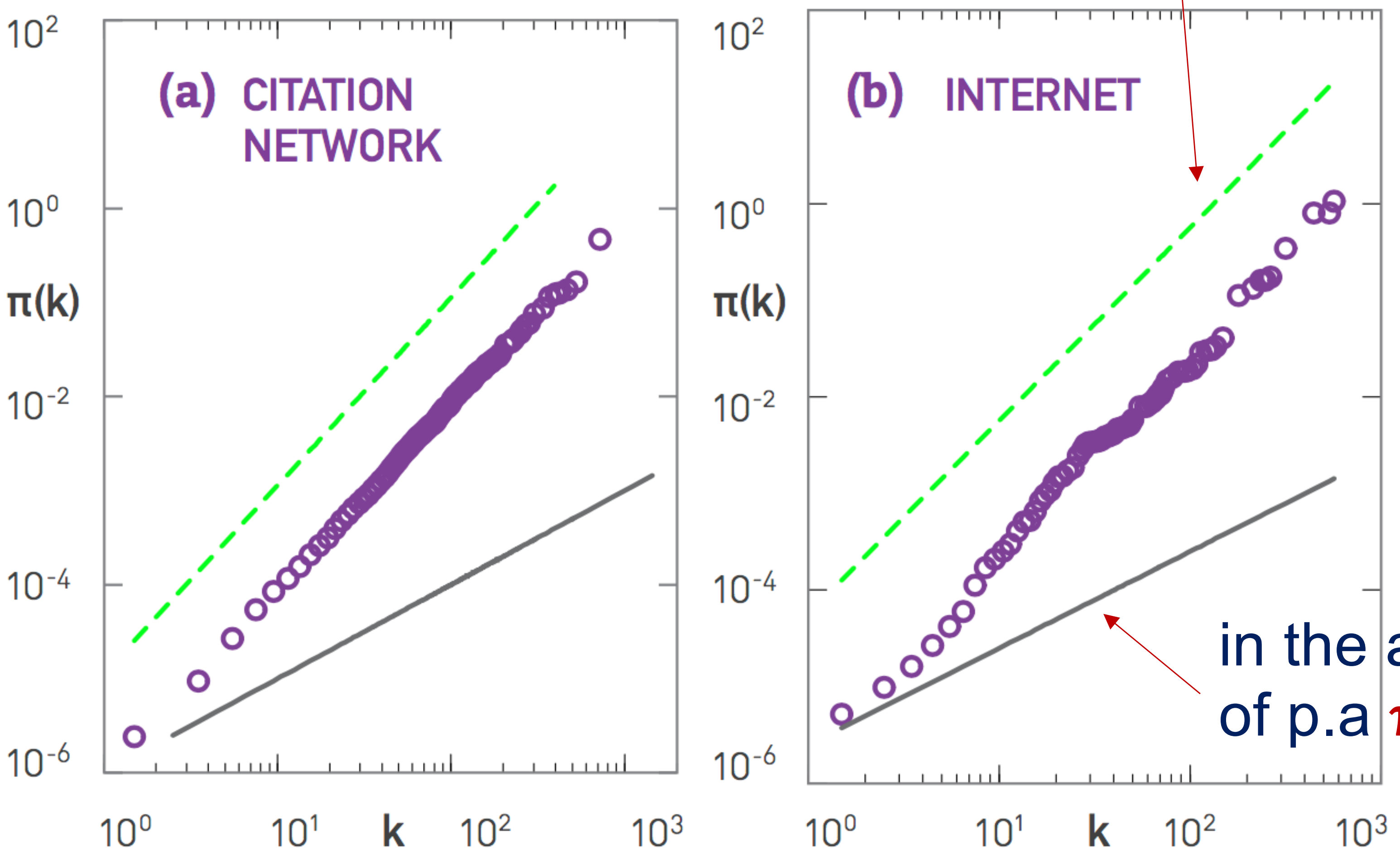



- all nodes follow the same dynamics
- the growth is sub-linear: nodes are competing with the others
- the earlier the node is added, the higher the degree – “first-mover advantage”
- older nodes acquire more links
- this explains the hub formation

# Measuring preferential attachment
## in real networks

$$\pi(k) = \sum_{k_i}^{k} \Delta k_i / \Delta N$$

under preferential attachment $\pi(k) \sim k^2$




in the absence of p.a $\pi(k) \sim k$

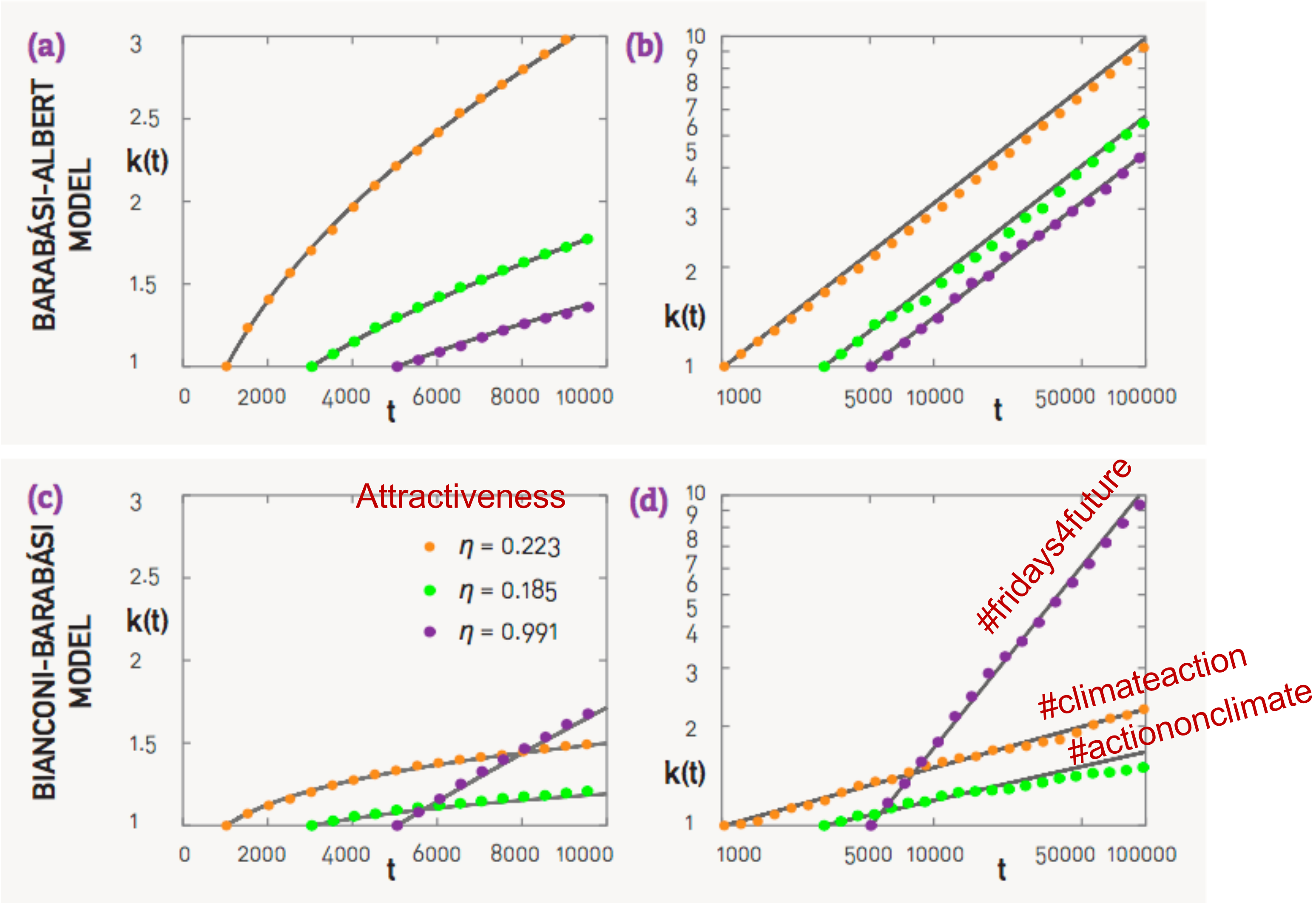
# The Bianconi-Barabasi model

Bianconi, Barabási. "Competition and multiscaling in evolving networks" (2001)

The model:

- Growth – at time step $N$ a new node $i$=$N$ is added with $m$ links and fitness $\eta_i$
- Attractiveness (or fitness) is a random number drawn from a given distribution $\rho(\eta)$ - a quality of the individual to attract links
- Preferential attachment - probability of linking to node $i$ is proportional to both the degree and the attractiveness, i.e., $p_i = k_i\eta_i / \sum k_j\eta_j$

## we guess $k_i \simeq m\ (N/i)^{\beta(\eta_i)}$ for some $\beta(\eta)$



# Approximate analysis
## starting point

- We guess $\boxed{k_i \simeq m\,(N/i)^{\beta(\eta_i)}}$

- Increase in the degree $\Delta k_i \simeq \underset{\text{trials}}{m} \cdot \underset{\text{probability per trial}}{k_i\,\eta_i / \sum k_j\,\eta_j}$
- We show that $\sum k_j\,\eta_j \simeq m\,N \cdot C$ (see proof)

# Approximate analysis
## the denominator

- Analysis of denominator $\sum k_i \eta_i$

  → average value wrt $\eta$

  → hypothesis $k_i \simeq m\,(N/i)^{\beta(\eta_i)}$

- $A = E[\sum_i k_i \eta_i] = \sum E[k_i \eta_i] \simeq \int_1^N E[k_i \eta_i]\, di$
- $E[k_i \eta_i] = \int m(N/i)^{\beta(\eta)}\, \eta \cdot \rho(\eta)\, d\eta$
- Swap integrals

$$A \simeq \int m\, N^{\beta(\eta)} \left[\int_1^N i^{-\beta(\eta)}\, di\right] \eta \cdot \rho(\eta)\, d\eta$$

- Integrate

constant $C$

$$A \simeq m\,N \cdot \boxed{\int \frac{(1 - \cancel{N^{\beta(\eta)-1}})}{1-\beta(\eta)}\, \eta\, \rho(\eta)\, d\eta}$$

negligible for large $N$ if $0 < \beta < 1$

# Approximate analysis
## evolution of nodes degrees

- We guess $\boxed{k_i \simeq m\,(N/i)^{\beta(\eta_i)}}$
- Increase in the degree $\Delta k_i \simeq m \cdot k_i\,\eta_i / \sum k_j \eta_j$
- It is $\sum k_j \eta_j \simeq m\,N \cdot C$

Hence:

1. By inspection of the above

$$\Delta k_i \simeq m\,(N/i)^{\beta(\eta_i)}\,\eta_i / N\,C$$

2. By continuum theory

$$\Delta k_i \simeq dk_i/dN \simeq m\,\beta(\eta_i)\,N^{\beta(\eta_i)-1}\,i^{-\beta(\eta_i)}$$

3. By combining the results $\boxed{\beta(\eta_i) \simeq \eta_i/C}$

We conclude $\boxed{k_i \simeq m\,(N/i)^{\eta_i/C}}$

# Approximate analysis

## constant C

$$\beta(\eta) \simeq \eta / C$$

$$C = \int \frac{\eta\, \rho(\eta)\, d\eta}{1-\beta(\eta)} \rightarrow \boxed{1 = \int_0^{\eta_{max}} (C - \eta)^{-1}\, \eta\, \rho(\eta)\, d\eta}$$

this identifies C for a given $\rho(\eta)$

it is $\boxed{C > \eta_{max}}$, i.e., $\beta < 1$, → the integral makes sense

growth with exponent <1

it also is $C \leq 2\eta_{max}$

# Approximate analysis
## degree distribution

Want to identify $P_k$ = P[$k_i \leq k$] = $1$ – P[$k_i > k$]

- $k_i > k$ and $k_i = m\ (N/i)^{\eta i/C}$ $\rightarrow$ $i < N\ (m/k)^{C/\eta i}$
- Hence P[$k_i > k|\eta_i$] = $(m/k)^{C/\eta i}$
- and P[$k_i \leq k|\eta_i$] = 1 - $(m/k)^{C/\eta i}$
- We have $P_k = 1 - \int (m/k)^{C/\eta}\ \rho(\eta)\ d\eta$

The degree distribution is

$$p_k = P_k' = \boxed{C \int_0^{\eta_{max}} k^{-(C/\eta+1)}\ m^{C/\eta}\ \eta^{-1}\ \rho(\eta)\ d\eta}$$

weighted combination of power laws with
exponent in [2,∞) since $\eta_{max} < C$

What if $\rho(\eta) = \delta(\eta-1)$ ?

- Coefficient $C = 2$ since

$$\int_0^{\eta_{max}} (C/\eta - 1)^{-1}\, \delta(\eta-1)\, d\eta = (C - 1)^{-1} = 1$$

- Exponential degree $k_i \simeq m\,(N/i)^{½}$

Degree distribution

$$p_k = C \int_0^{\eta_{max}} \eta^{-1}\, m^{C/\eta}\, k^{-(C/\eta+1)}\, \delta(\eta-1)\, d\eta = 2\, m^2\, k^{-3}$$

# Uniform fitness

## the model

What if $\rho(\eta) = 1$ and $\eta_{max} = 1$ ?

- Coefficient $C = 1.255$ since
  $$\int_0^1 (C/\eta - 1)^{-1} \, d\eta = 1 \;\rightarrow\; e^{-2/C} = 1-1/C$$
- Exponential degree $k_i \simeq m\,(N/i)^{\eta_i/C}$
- Each node has its own dynamic exponent !!!

Degree distribution

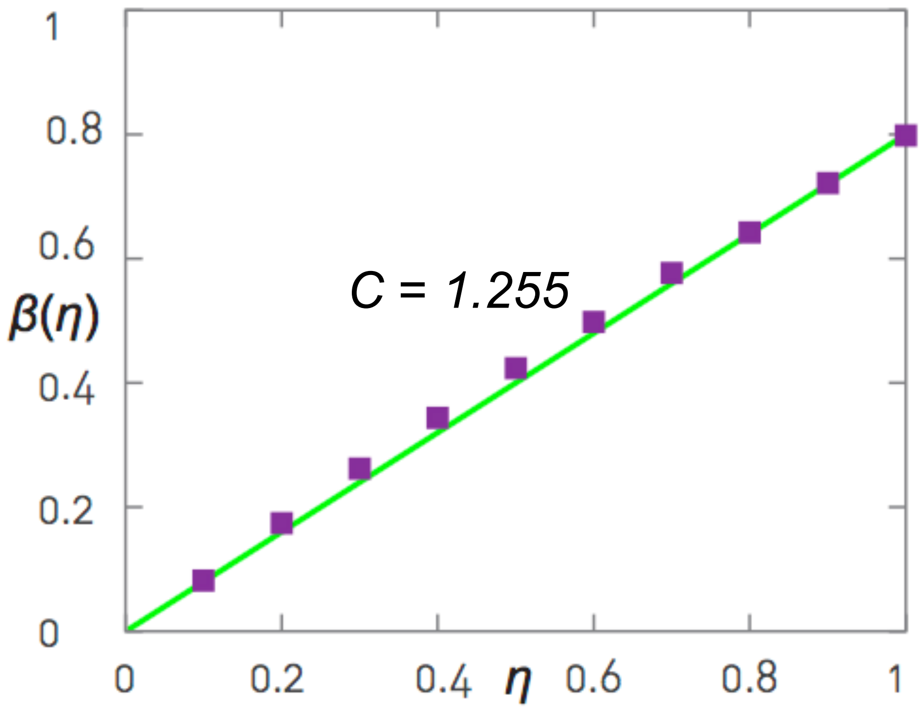
$$p_k = C/k \int_0^1 \eta^{-1}\, e^{-C\ln(k/m)/\eta}\, d\eta \sim \boxed{k^{-(1+C)} / \ln(k)}$$

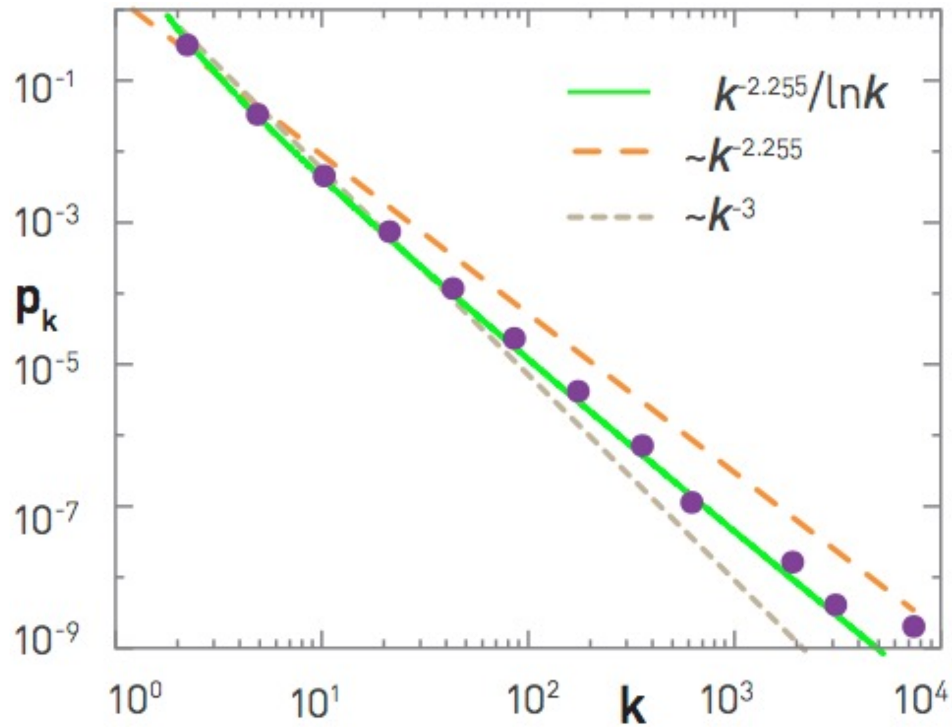
$e^{-b} - b\,E_1(b)$ , $b = C\ln(k/m)$

exponential integral $E_1$

# Uniform fitness

## the measured data



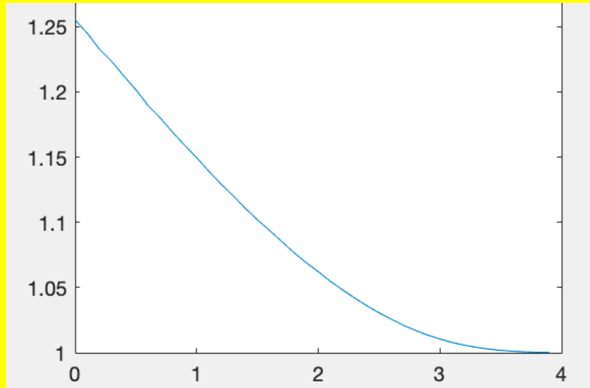

degree distribution $p_k \sim k^{-2.255} / \ln(k)$

corrective term

What if $\rho(\eta) = a\, e^{-a\eta} / (1-e^{-a})$ and $\eta_{max} = 1$ ?

- $C$ rapidly converges to C=1

$$\int_0^1 (C/\eta - 1)^{-1} \rho(\eta)\, d\eta = 1$$




- Exponential degree $k_i \simeq m\, (N/i)^{\eta_i/C}$
- Each node has its own dynamic exponent !!!

Degree distribution

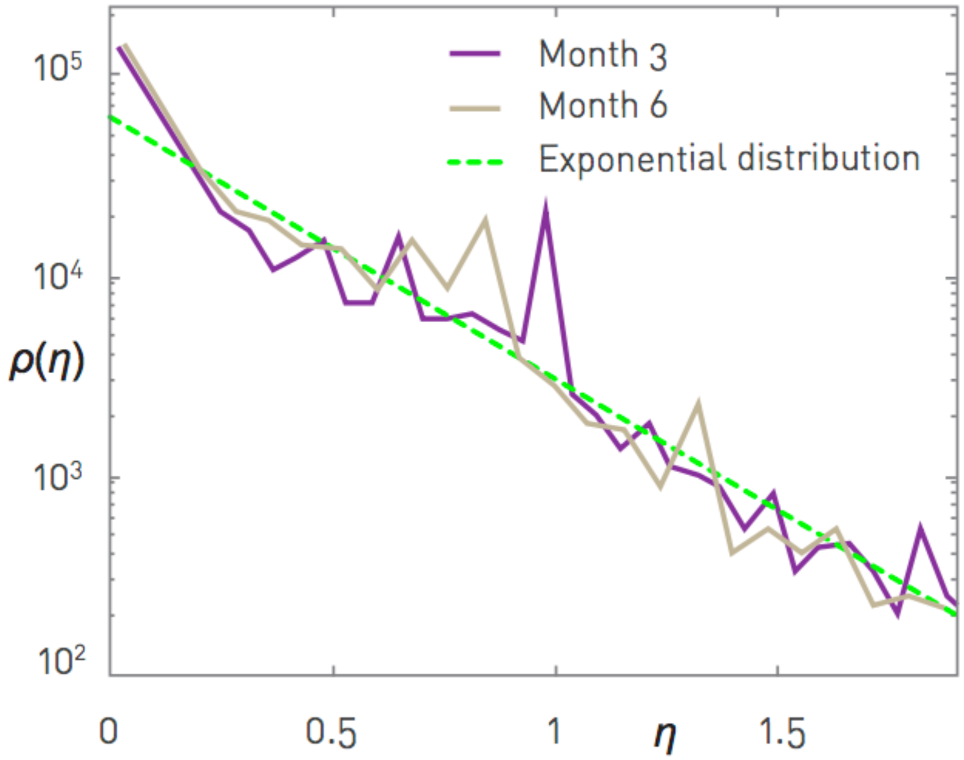
$$p_k = C/k \int_0^1 \eta^{-1}\, e^{-C \ln(k/m)/\eta}\, \rho(\eta)\, d\eta \sim \boxed{k^{-(1+C)} / \ln(k)}$$

exponential integral $E_1$

# Exponential fitness

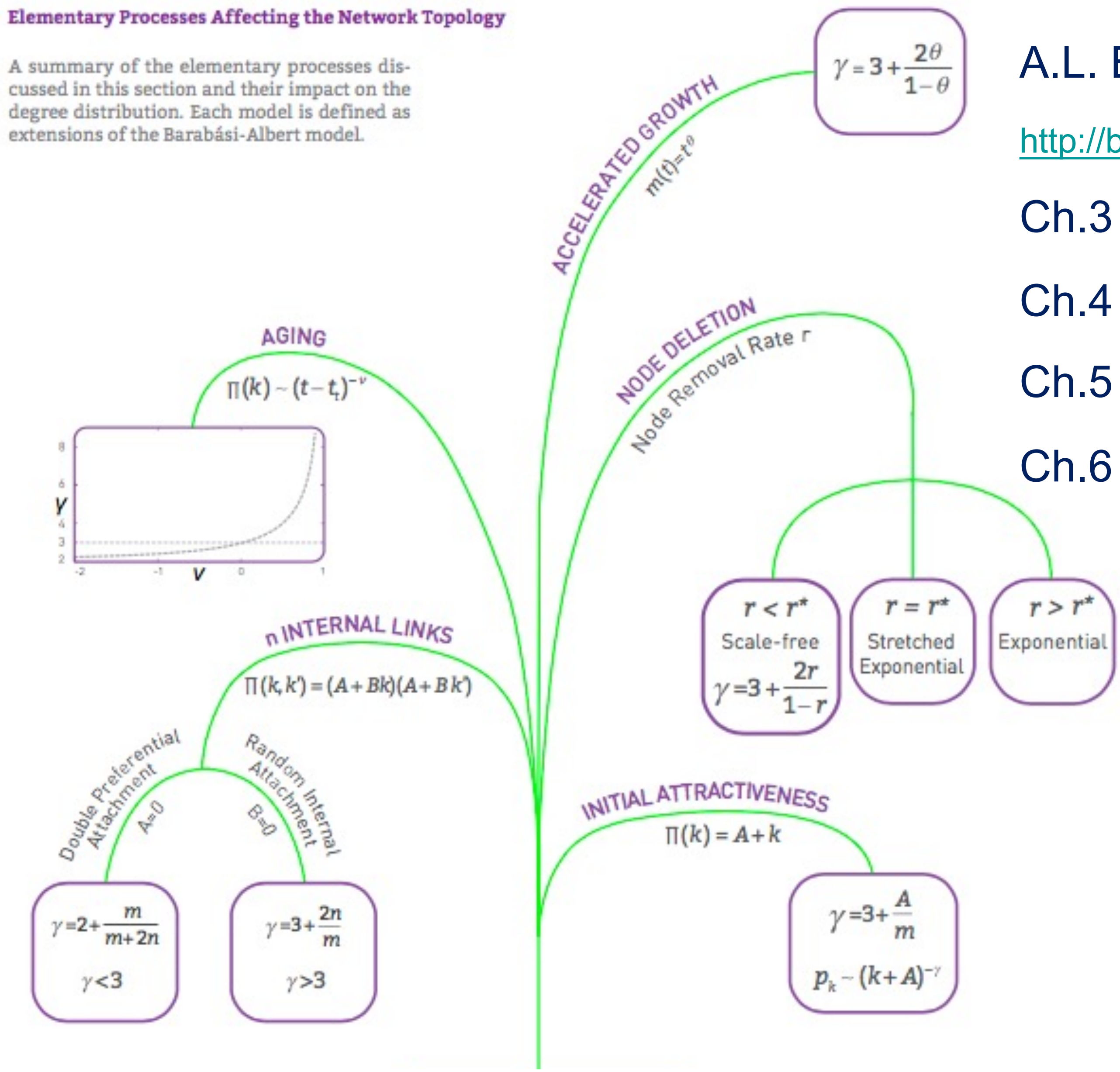
## the www

$a = 4.6$
$C = 1$




degree distribution $p_k \sim \boxed{k^{-2} / \ln(k)}$

# Other ideas for extension
## of the Albert-Barabasi model




A.L. Barabási, Network science

http://barabasi.com/networksciencebook

Ch.3 “Random networks”

Ch.4 “The scale-free property”

Ch.5 “The Barabási-Albert model”

Ch.6 “Evolving networks”

# Properties of the power-law

scale-free and random networks

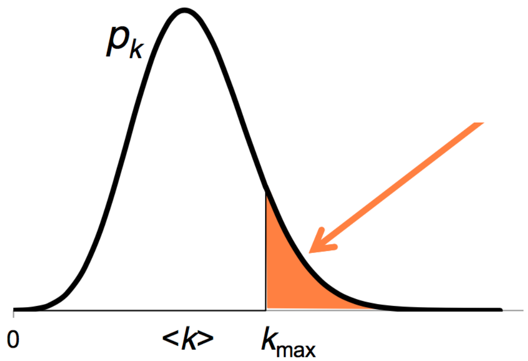
# The largest hub

## natural cutoff under the power-law

Degree distribution $p_k = C\, k^{-\gamma}$ with $C = (\gamma-1)\, k_{min}^{\gamma-1}$

The size of the largest hub is captured by

$$\int_{k_{max}}^{\infty} p_k\, dk = C \cdot k_{max}^{-(\gamma-1)} / (\gamma-1) = 1/N$$




$$\boxed{k_{max} = k_{min}\, N^{1/(\gamma-1)}}$$

is the natural cutoff
it explains large hubs

# Moments
## of the power-law

- $\langle k^n \rangle = \int_{k_{min}}^{k_{max}} k^n \, p_k \, dk$ with $p_k = C\,k^{-\gamma}$

  $= C \cdot (k_{max}^{\,n-\gamma+1} - k_{min}^{\,n-\gamma+1})/(n-\gamma+1)$

  $= C\,k_{min}^{\,n-\gamma+1} \cdot (N^{-1+n/(\gamma-1)} - 1) / (n-\gamma+1)$

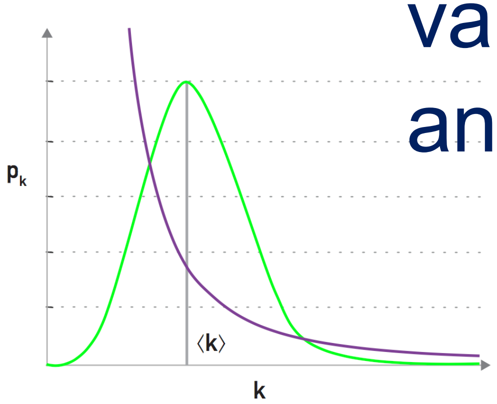
- They diverge with $N$ if $\gamma < n+1$

  mean ($n=1$) doesn't diverge for $\gamma \geq 2$

  variance ($n=2$) diverges for $\gamma < 3$

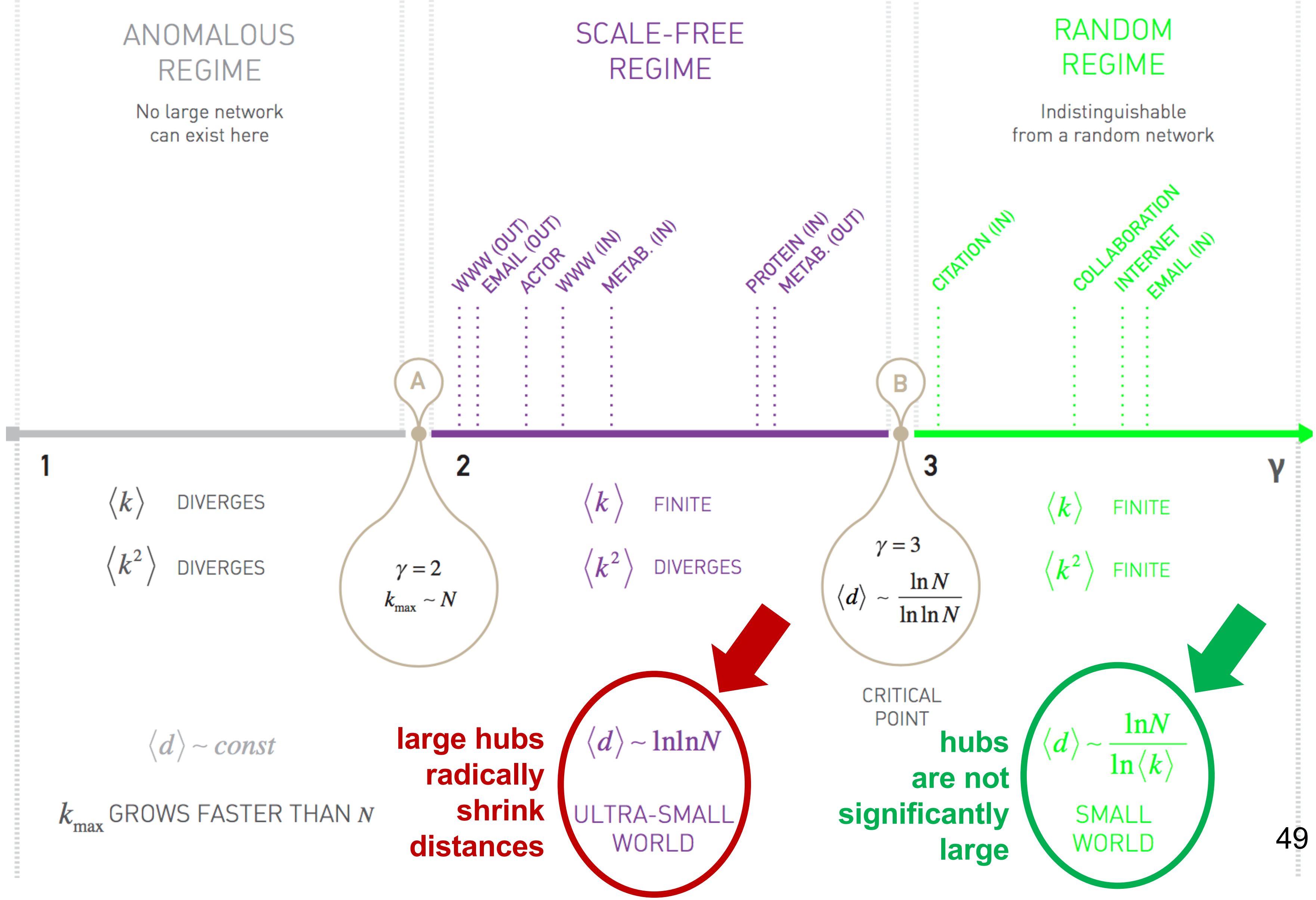
  and the network <u>does not have a scale</u>

  (scale-free regime)



# The scale-free regime

for 2<γ<3

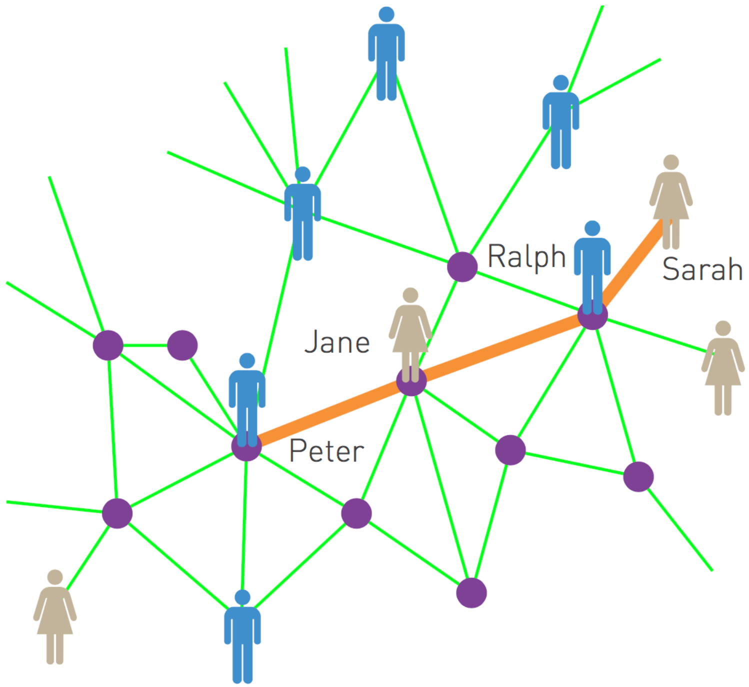


# Small world property

Watts, Strogatz, «*Collective dynamics of small-world networks*», (1998)

In real networks distance between two randomly chosen nodes is generally short

Milgram [1967]: *6 degrees of separation*




What does this mean?

We are more connected than we think

# Distances in random graphs

## theoretical result

- we reach $\langle k \rangle$ nodes in one hop, $\langle k \rangle^2$ in two, $\langle k \rangle^3$ in three, etc.
- an estimate of the average distance $\langle d \rangle$ is found by solving for $N = \langle k \rangle^{\langle d \rangle}$ to have

$$\langle d \rangle = \ln(N) / \ln(\langle k \rangle)$$

- $\langle d \rangle$ is often taken as an estimate of the network diameter $d_{max}$

e.g.: on earth we are $N=7 \cdot 10^9$ individuals,
with $\langle k \rangle$=1000 acquaintances each → $\langle d \rangle = 3.28$

UNIVERSITÀ DEGLI STUDI DI PADOVA

# Distances in random graphs

## fitting with real data

| NETWORK | $N$ | $L$ | $\langle k \rangle$ | $\langle d \rangle$ | $d_{max}$ | $\frac{\ln N}{\ln \langle k \rangle}$ |
|---|---|---|---|---|---|---|
| Internet | 192,244 | 609,066 | 6.34 | 6.98 | 26 | 6.58 ✓ |
| WWW | 325,729 | 1,497,134 | 4.60 | 11.27 | 93 | 8.31 ✓ |
| Power Grid | 4,941 | 6,594 | 2.67 | 18.99 | 46 | 8.66 |
| Mobile Phone Calls | 36,595 | 91,826 | 2.51 | 11.72 | 39 | 11.42 ✓ |
| Email | 57,194 | 103,731 | 1.81 | 5.88 | 18 | 18.4 |
| Science Collaboration | 23,133 | 93,439 | 8.08 | 5.35 | 15 | 4.81 ✓ |
| Actor Network | 702,388 | 29,397,908 | 83,71 | 3,91 | 14 | 3,04 ✓ |
| Citation Network | 449,673 | 4,707,958 | 10.43 | 11,21 | 42 | 5.55 |
| E. Coli Metabolism | 1,039 | 5,802 | 5.58 | 2.98 | 8 | 4.04 |
| Protein Interactions | 2,018 | 2,930 | 2.90 | 5.61 | 14 | 7.14 ✓ |

Very good fit ! Correct at least as order of magnitude

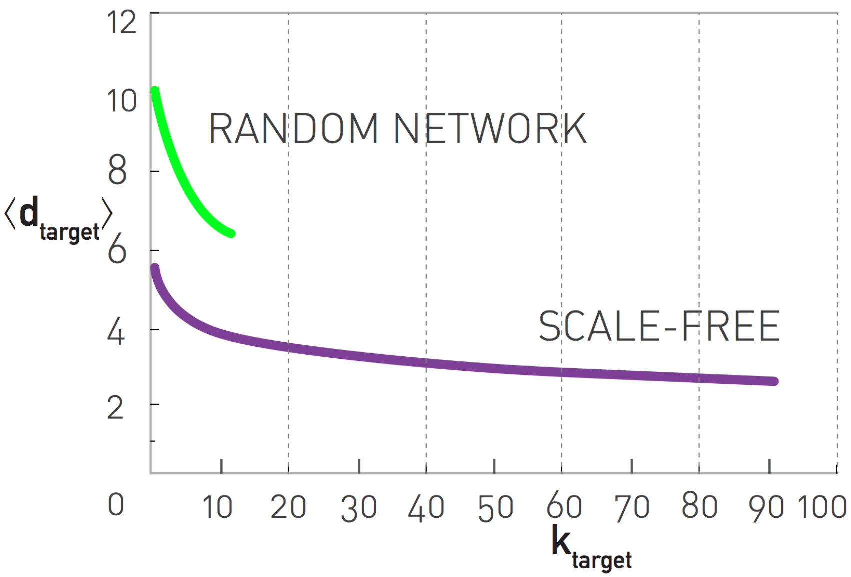
# Distances in scale-free networks

## the ultra-small-world

- The average distance increases as ln(ln($N$)), much slower than $N$ or ln($N$)

e.g. in www $N=7\cdot10^9$, ln($N$)=$22.7$, ln(ln($N$))=$3.12$ (very small)

distance of a node of degree $\langle k \rangle$ from a node of degree $k_{target}$




- The large hubs radically shrink the distance between nodes → ultra small world

# Curiosity

## wrong perception of the ultra-small-world

In many social experiments people avoided hubs for entirely perceptual reasons (e.g., they assumed they are busy, better use them only if really needed)

We live in a ultra-small-world, but we perceive that we are more distant from others than we really are!

# Friendship paradox

my friends are more popular than me (Feld 1991)

- Can be observed in the ultra-small-world under the presence of big hubs
- Rationale: a node is very likely to be connected to a big hub, having a very large number of connections
- # of friends (in the average) = $\langle k \rangle$
- # of friends of friends $\simeq N$

# Takeaways
## for degree centrality

- Do not use it for resizing nodes according to their importance (will use PageRank for this)
- Provide useful information in the form of a degree distribution
- Always plot degree distributions in the log scale
- Always evaluate their slope ɣ, but please use the ML approach: ɣ provides useful insights on the network
- Preferential attachment and attractiveness can be measured if you have temporal info on the network

# PageRank centrality

Google's approach to centrality

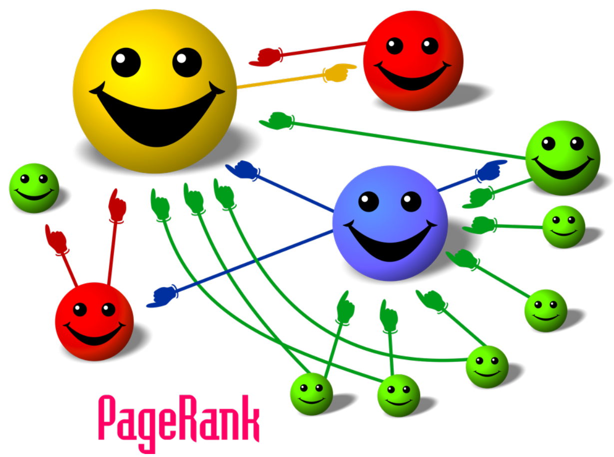
# How to organise the web?

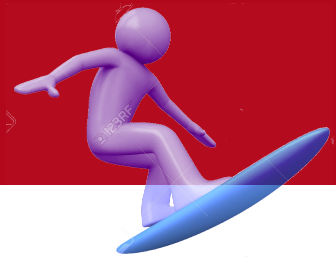
## links as votes

- the higher (and stronger) the number of incoming links, the more important a node
- the more important a node, the more valuable the output links

# The Google's view
quoting Google

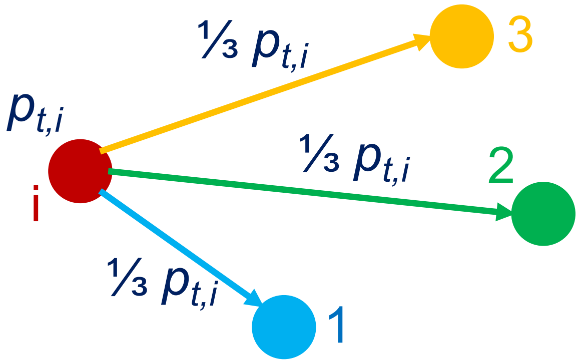
- PageRank works by counting the number and quality of links to a page to determine a rough estimate of how important the website is
- The underlying assumption is that more important websites are likely to receive more links from other websites



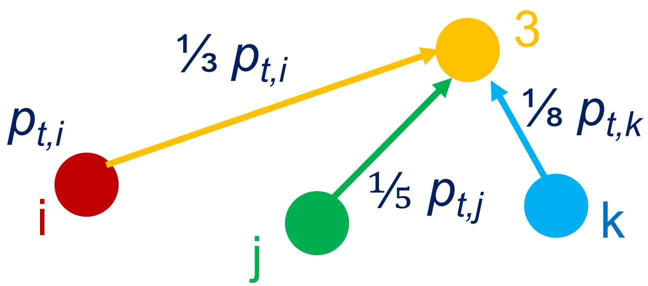


# A random walk on www
## the rationale behind PageRank

- at time t, a web surfer is at page *i* with probability $p_{t,i}$
- let the surfer choose with equal probability one of the sites linked by site *i*




$$p_{t+1,3} = \tfrac{1}{3}\, p_{t,i} + \tfrac{1}{5}\, p_{t,j} + \tfrac{1}{8}\, p_{t,k}$$

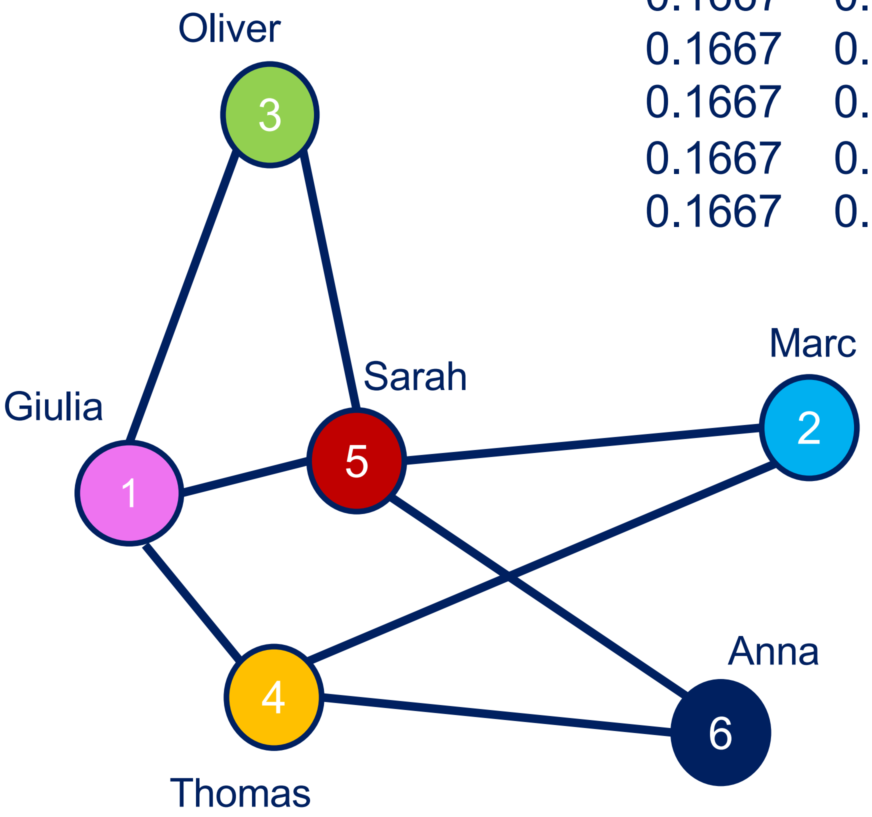



- this identifies a Markov chain
- after a while probabilities settle to a steady state = the PageRank vector

# Example

## of the random walk effect on a friends' network

| *t=1* | *2* | *3* | *4* | *5* |
|---|---|---|---|---|
| 0.1667 | 0.1806 | 0.1991 | 0.1723 | 0.2025 |
| 0.1667 | 0.0972 | 0.1505 | 0.1040 | 0.1436 |
| 0.1667 | 0.0972 | 0.1366 | 0.1179 | 0.1287 |
| 0.1667 | 0.2222 | 0.1574 | 0.2168 | 0.1614 |
| 0.1667 | 0.3056 | 0.2060 | 0.2851 | 0.2203 |
| 0.1667 | 0.0972 | 0.1505 | 0.1040 | 0.1436 |



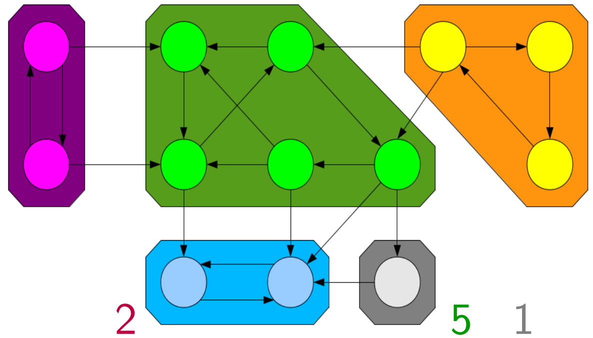

Equal to (normalized) degree centrality in undirected networks !!!

| *10* | *20* | *50* | *75* | *100* | |
|---|---|---|---|---|---|
| 0.1783 | 0.1848 | 0.1874 | 0.1875 | 0.1875 | Giulia |
| 0.1153 | 0.1222 | 0.1249 | 0.1250 | 0.1250 | Marc |
| 0.1242 | 0.1248 | 0.1250 | 0.1250 | 0.1250 | Oliver |
| 0.2020 | 0.1917 | 0.1876 | 0.1875 | 0.1875 | Thomas |
| 0.2649 | 0.2543 | 0.2501 | 0.2500 | 0.2500 | Sarah |
| 0.1153 | 0.1222 | 0.1249 | 0.1250 | 0.1250 | Anna |

Università
degli Studi
di Padova

# Matrix formalization
## of the random walk

- $\boldsymbol{p}_{t+1} = \boldsymbol{M}\,\boldsymbol{p}_t$
- $\boldsymbol{p}_t$ stochastic vector (positive entries which sum up to *1*)
- $\boldsymbol{M}$ normalized adjacency matrix (column stochastic)
- $\boldsymbol{M} = \boldsymbol{A}\,\mathrm{diag}^{-1}(\boldsymbol{d})$
- $\boldsymbol{d} = \boldsymbol{A}^{\mathrm{T}}\,\boldsymbol{1}$ output degree vector
- $\boldsymbol{p}_\infty = \boldsymbol{M}\,\boldsymbol{p}_\infty$ converges to an eigenvector of $\boldsymbol{M}$ (with eigenvalue *1*)
- $\mathbf{p}_\infty = \mathbf{d}$ for undirected networks where $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$



$$\begin{array}{ccc|cc|ccccc|c|cc} & & 3 & & 2 & & & & & 5 & 1 & & 2 \end{array}$$

$$\boldsymbol{M} = \left[\begin{array}{ccc|cc|ccccc|c|cc}
 & \frac{1}{3} & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
1 & & & & & & & & & & & & \\
\hline
 & & & & \frac{1}{2} & & & & & & & & \\
 & & & \frac{1}{2} & & & & & & & & & \\
\hline
 & \frac{1}{3} & & & & & & & & \frac{1}{2} & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \frac{1}{3} & & & & \\
 & \frac{1}{3} & & & & \frac{1}{2} & & & & & & & \\
 & & & & & & & \frac{1}{3} & & & & & \\
 & & & & \boxed{\frac{1}{2}} & & \boxed{1} & & \boxed{\frac{1}{3}} & & & & \\
\hline
 & & & & & & & \frac{1}{3} & & & 0 & & \\
\hline
 & & & & & & & \frac{1}{3} & \frac{1}{3} & & 1 & & 1 \\
 & & & & & & & & & \frac{1}{2} & & 1 & 
\end{array}\right]$$

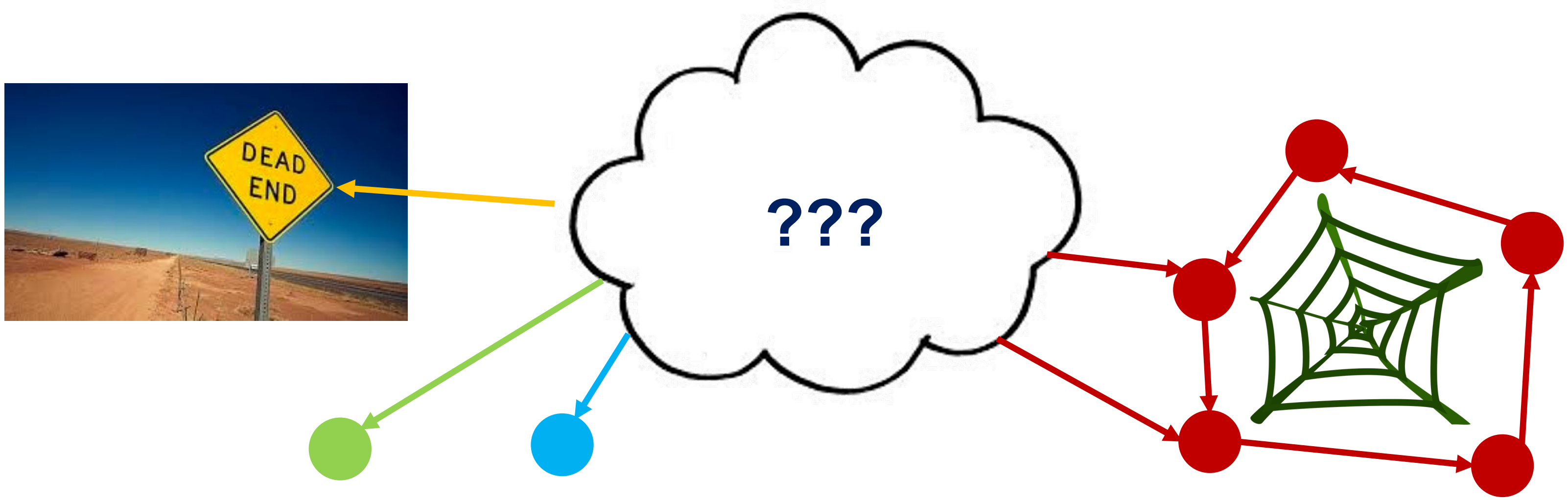
**columns sum to 1**

# Problems in the random walk

## dead ends and spider traps

With high probability the surfer ends in:

- Dead ends: some nodes do not have a way out = zero valued columns of ***M***
- Spider traps: some set of nodes do not have a way out, and further induce a periodic behaviour



# Teleportation

## as a method to overcome problems

Idea:

- the surfer does not necessarily move to one of the links of the page she/he is viewing
- with a certain probability, might jump to a random page
- $\boldsymbol{p}_{t+1} = c\, \boldsymbol{M}\, \boldsymbol{p}_t + (1\text{-}c)\, \boldsymbol{q}$

damping factor, typically $c = 0.85$, meaning that 85% of the times the surfer moves to one of the links of the page

the remaining $1 - c$ = 15% of the times the surfer moves at random according to a probability vector $\boldsymbol{q}$ independent of the node she/he is in, e.g., $\boldsymbol{q} = 1/N$ for uniform probability

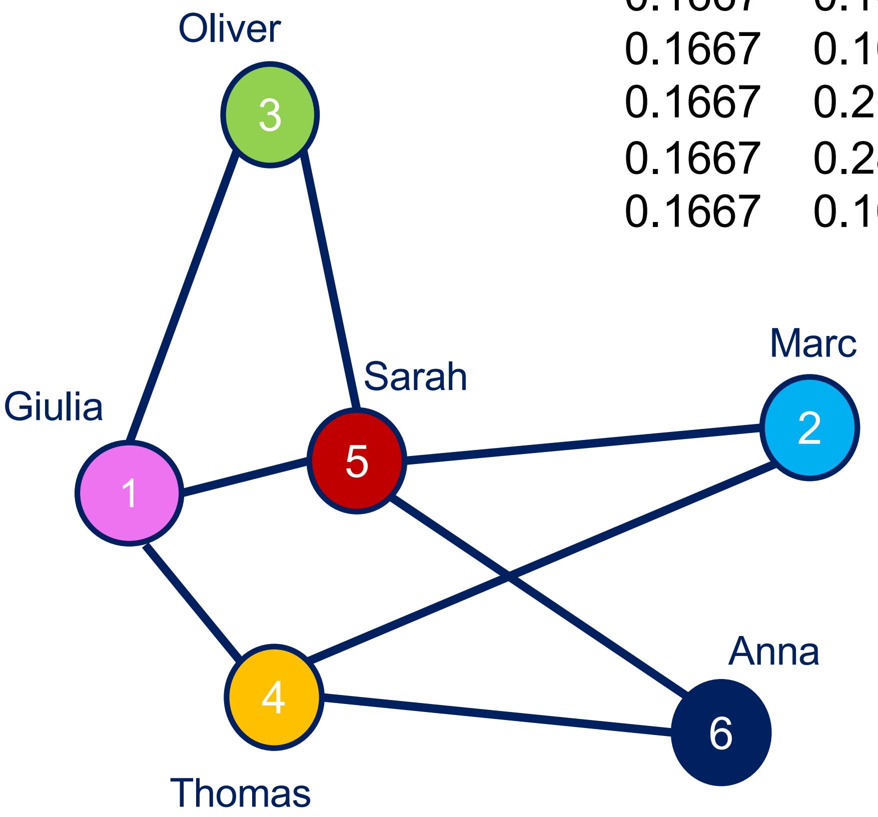
# PageRank with restart

## or simply PageRank

| | | |
|---|---|---|
| dead ends | $A_0$ | original adjacency matrix (can be fractional) |
| no dead ends | $A = A_0 + b\, e^T$ | teleportation; indicating vector of dead ends |
| normalization | $M = A \operatorname{diag}^{-1}(d)$, $\quad d = A^T 1$ | |
| no spider traps | $M_1 = c\, M + (1-c)\, q\, 1^T$ | equivalent formulation; matrix is no more sparse |
| Markov chain | $p_{t+1} = M_1\, p_t$ | |
| PageRank equation | $r = c\, M\, r + (1-c)\, q$, $\quad r = p_\infty$ | PageRank centrality vector |

# Example

## of PageRank with restart on a friends' network




| t=1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 0.1667 | 0.1785 | 0.1919 | 0.1754 | 0.1912 |
| 0.1667 | 0.1076 | 0.1461 | 0.1176 | 0.1382 |
| 0.1667 | 0.1076 | 0.1361 | 0.1246 | 0.1302 |
| 0.1667 | 0.2139 | 0.1671 | 0.2035 | 0.1746 |
| 0.1667 | 0.2847 | 0.2128 | 0.2614 | 0.2276 |
| 0.1667 | 0.1076 | 0.1461 | 0.1176 | 0.1382 |

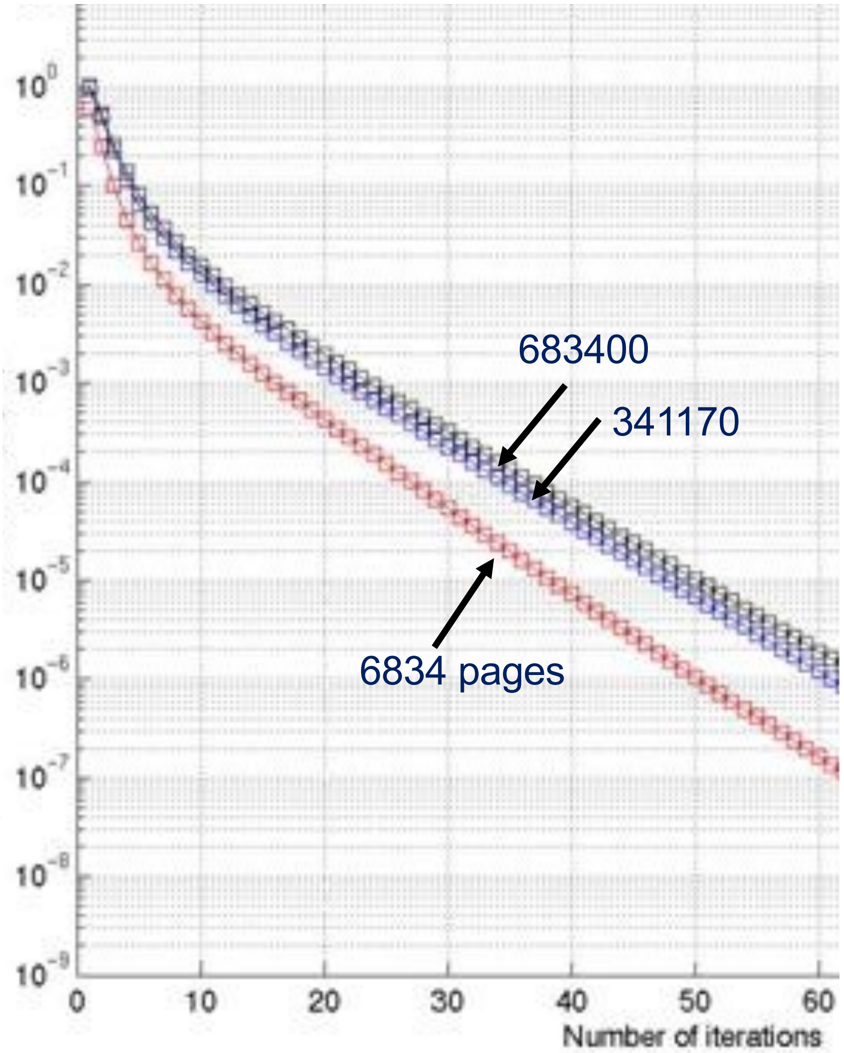
not anymore identical to degree centrality !!!

| 10 | 20 | 50 | 75 | 100 | |
|---|---|---|---|---|---|
| 0.1820 | 0.1839 | 0.1840 | 0.1840 | 0.1840 | Giulia |
| 0.1273 | 0.1293 | 0.1294 | 0.1294 | 0.1294 | Marc |
| 0.1283 | 0.1285 | 0.1285 | 0.1285 | 0.1285 | Oliver |
| 0.1902 | 0.1873 | 0.1871 | 0.1871 | 0.1871 | Thomas |
| 0.2449 | 0.2419 | 0.2417 | 0.2417 | 0.2417 | Sarah |
| 0.1273 | 0.1293 | 0.1294 | 0.1294 | 0.1294 | Anna |

# Convergence properies of PageRank

## an overview

- The PageRank vector is the probability $\boldsymbol{p}_t$ for large $t$
- It corresponds to the stationary behaviour of the Markov chain
- $\boldsymbol{p}_\infty$ is unique
- $\boldsymbol{p}_\infty$ is a stochastic vector (with positive entries summing to *1*)
- $\boldsymbol{p}_\infty$ depends on the choice of the teleportation vector $\boldsymbol{q}$ (and of $c$)
- $\boldsymbol{p}_\infty$ converges in few iterations, typically $\boldsymbol{p}_{40} \simeq \boldsymbol{p}_\infty$

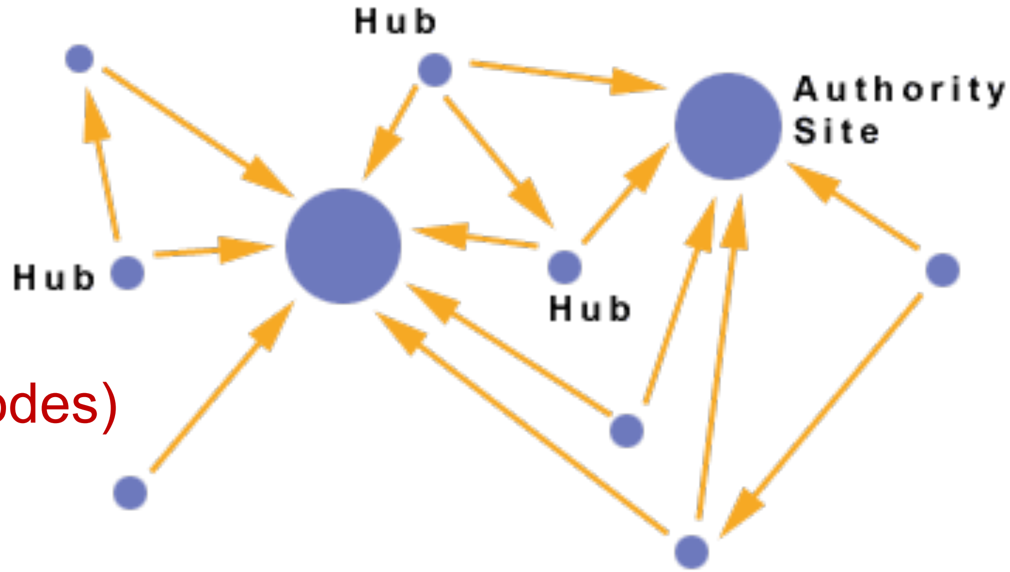


# Hubs and Authorities

what we can get from PageRank

## ❑ Authority (quality as a content provider)

nodes that contain useful information, or having a high number of edges pointing to them (e.g., course homepages)
= PageRank vector
(related to the in-degree of nodes)




## ❑ Hub (quality as an expert)

trustworthy nodes, or nodes that link to many authorities (e.g., course bulletin)
= PageRank vector starting from $\mathbf{A}_0^T$
(related to the out-degree of nodes)

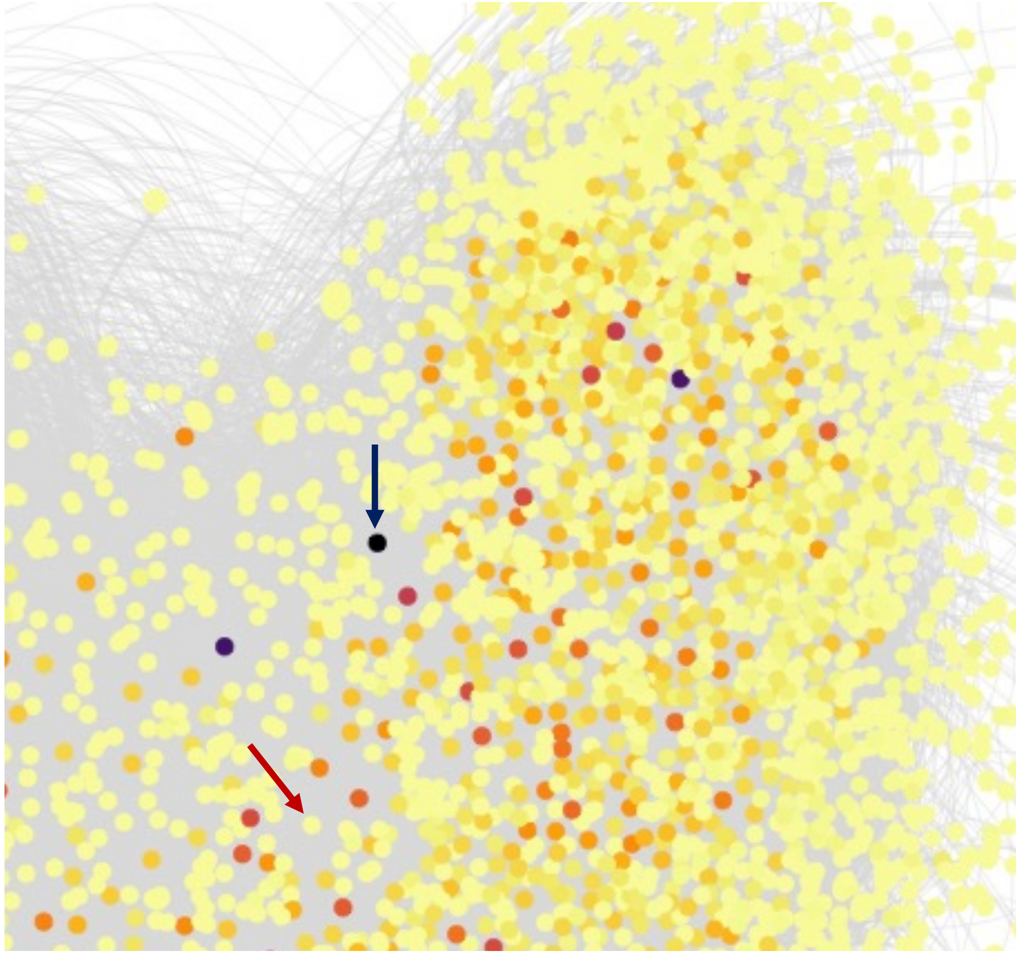
authority or hub?

# Example of PageRank centrality

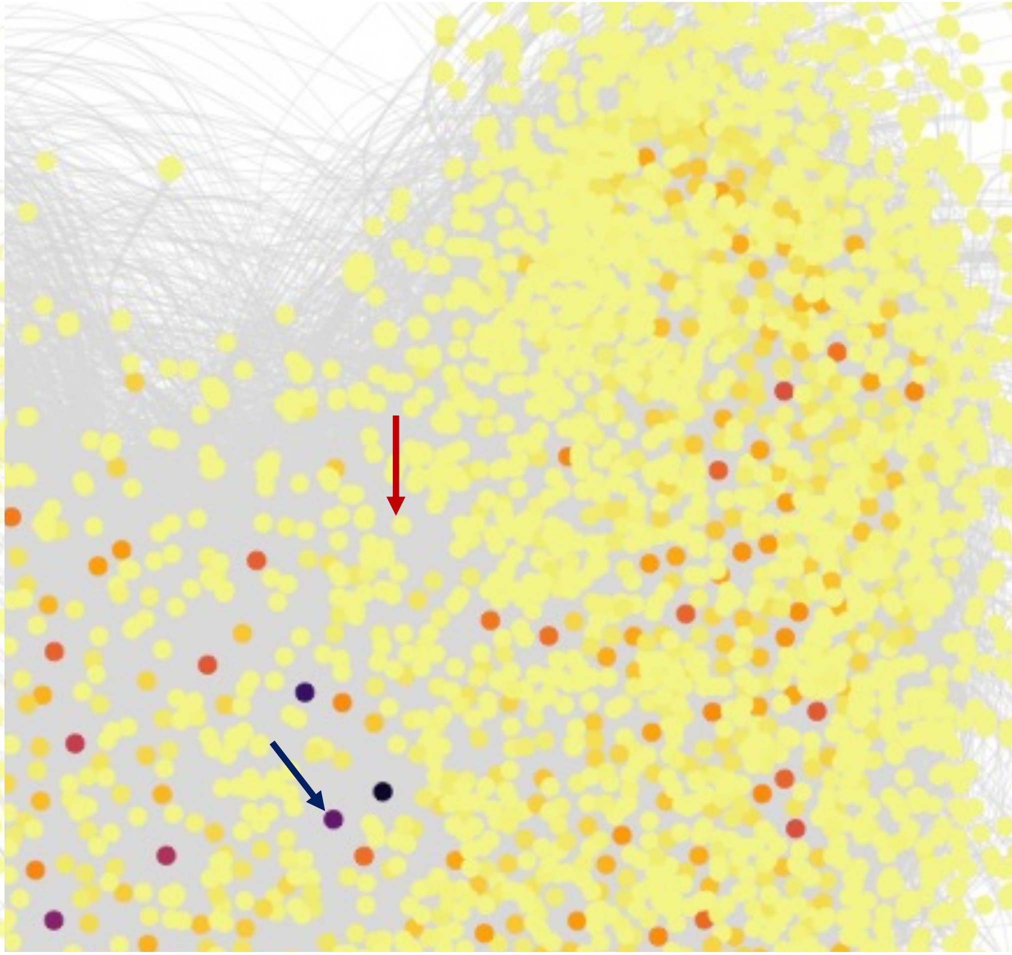
wikipedia administrator elections and vote history data

https://snap.stanford.edu/data/wiki-Vote.html

Authorities



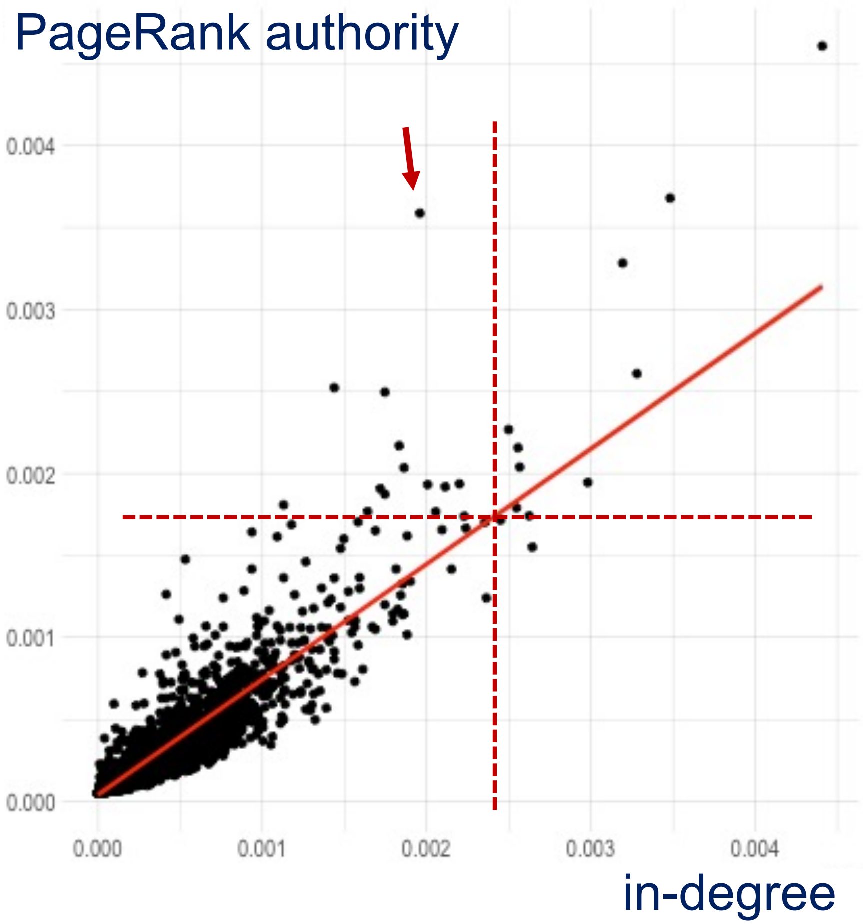
Hubs



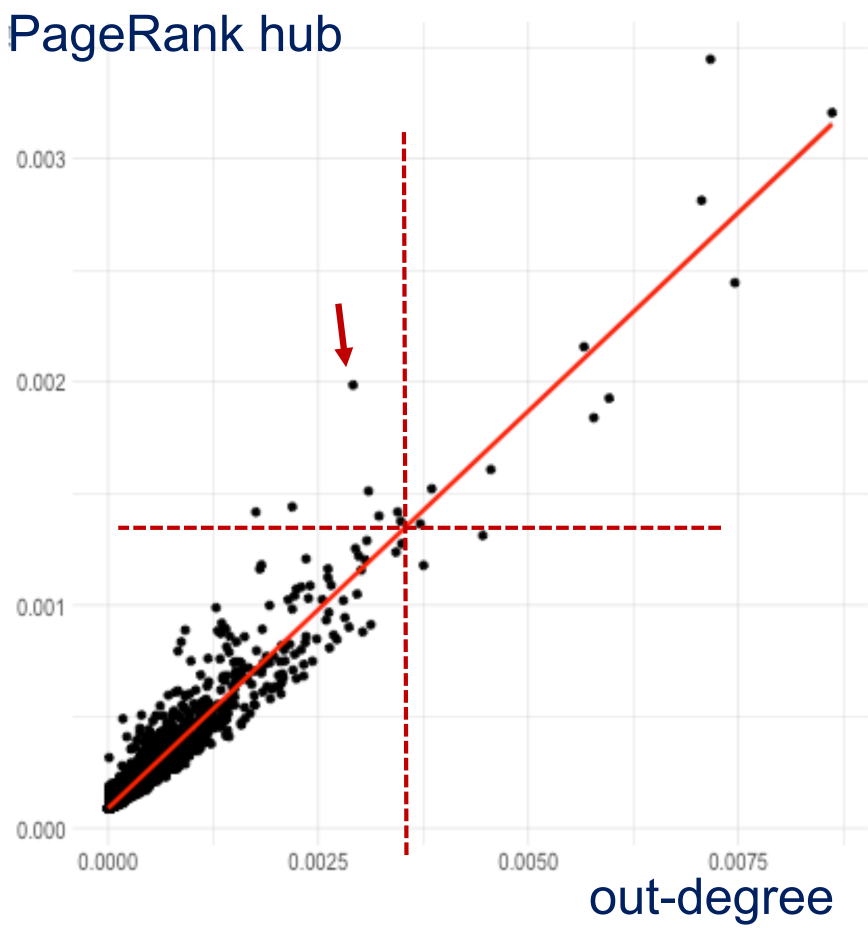
# PageRank versus degree centrality

## wikipedia administrator elections and vote history data

### Authorities



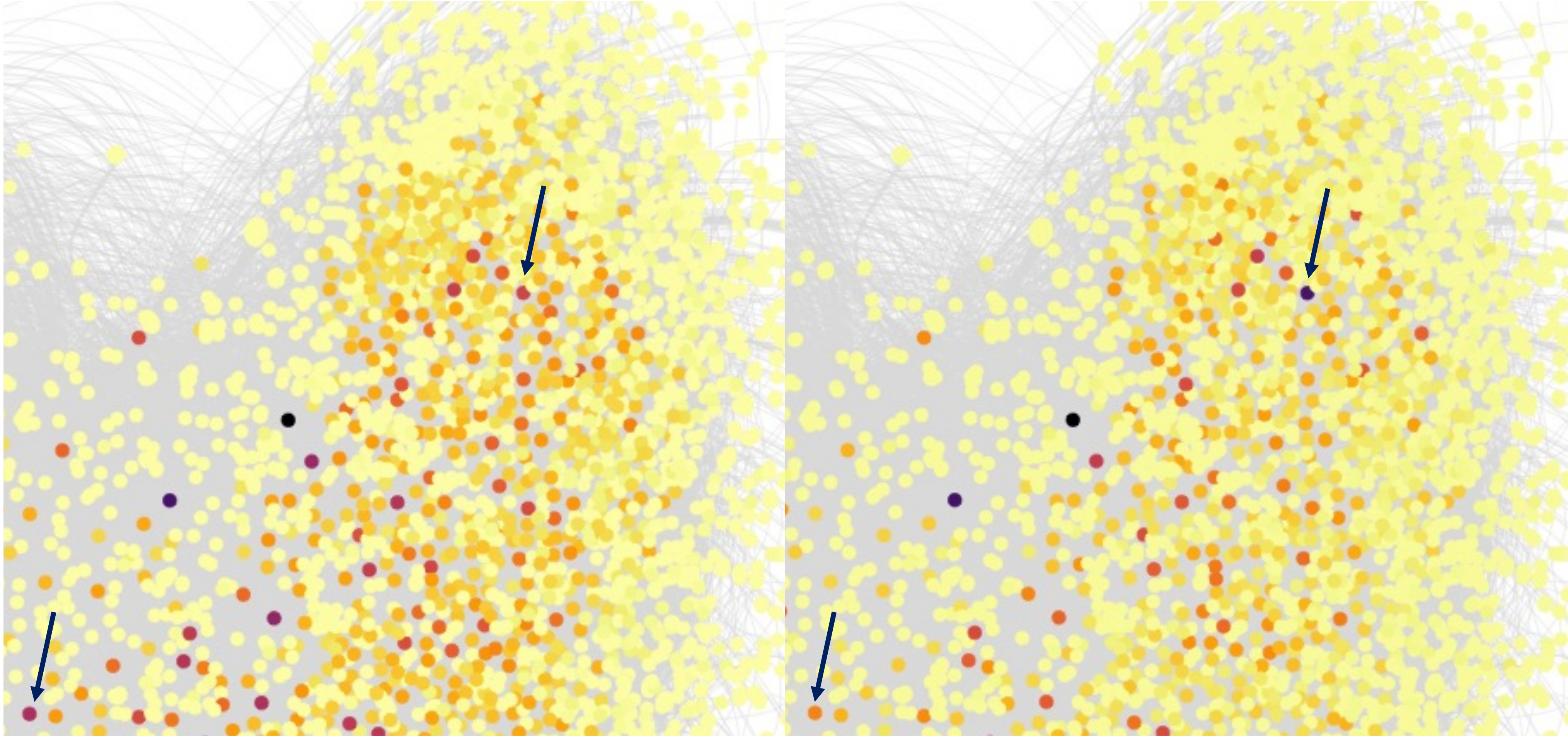

### Hubs



# PageRank versus degree authorities

wikipedia administrator elections and vote history data

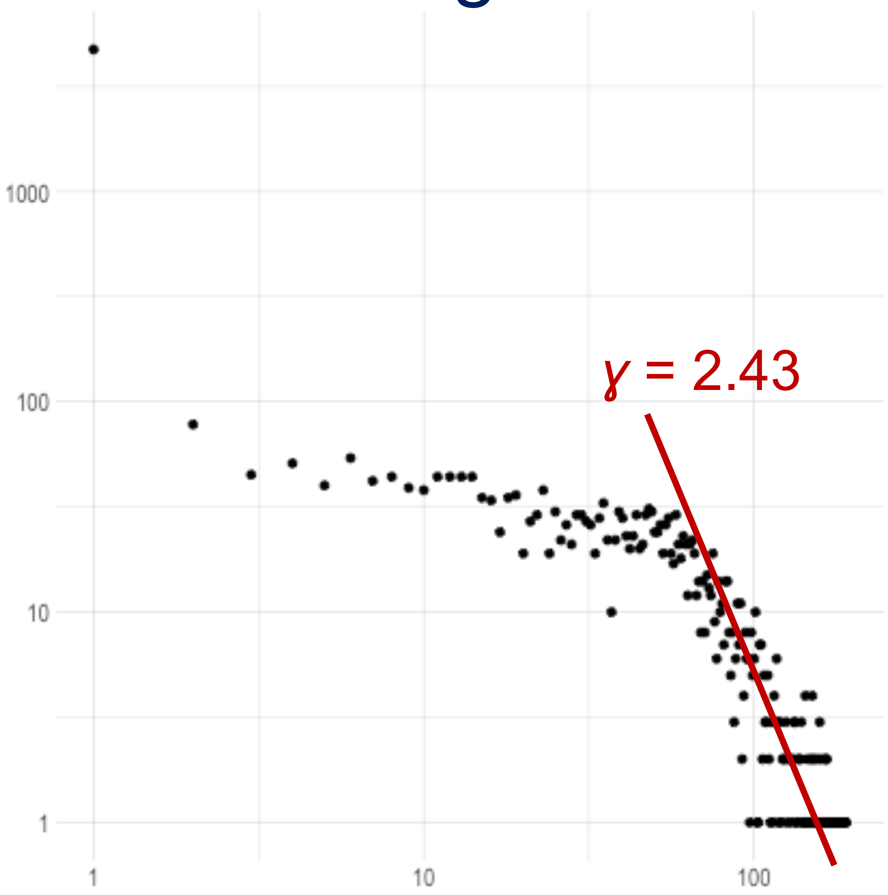
Degree PageRank



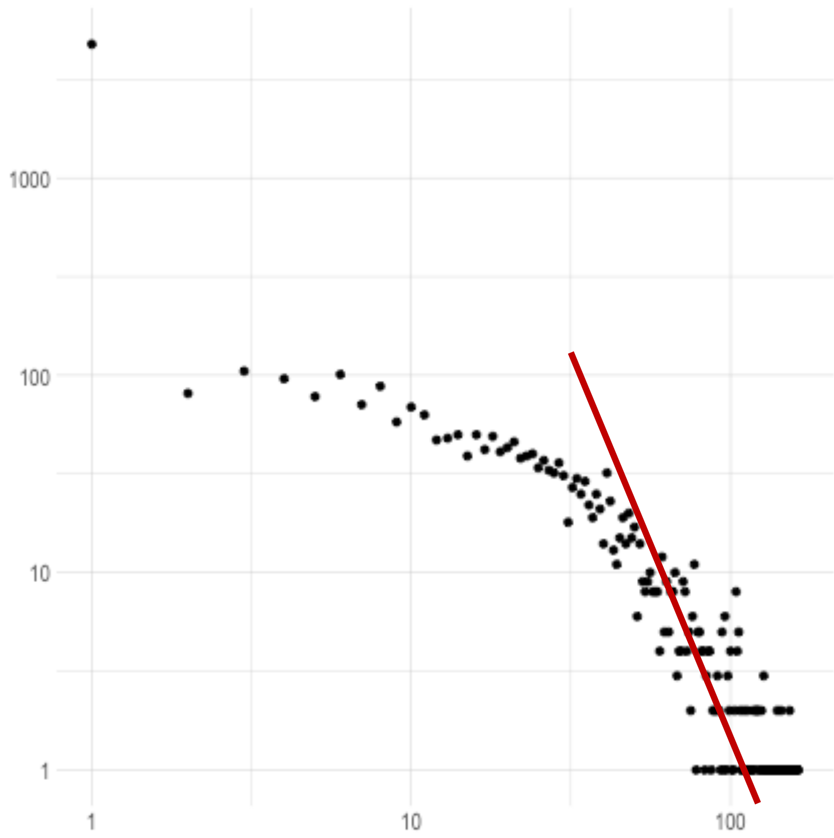
# PageRank versus degree authorities

## wikipedia administrator elections and vote history data

### Degree

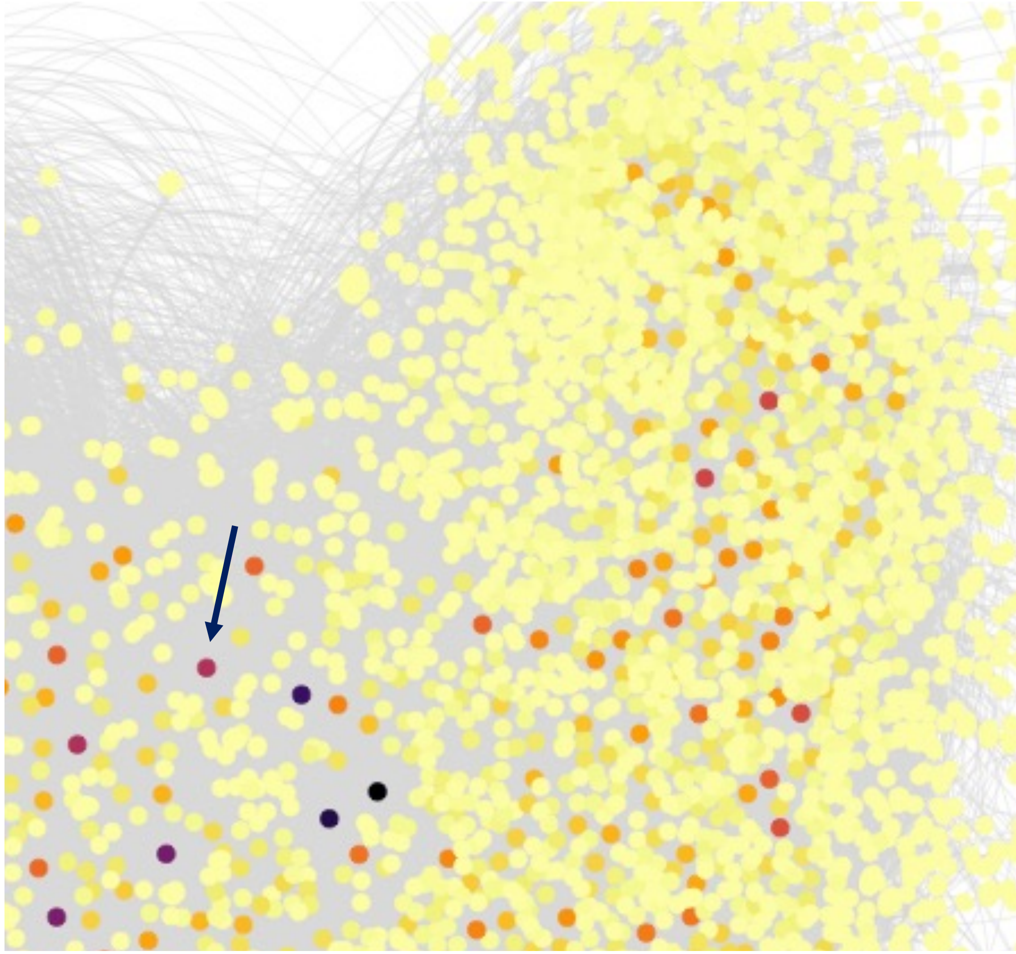


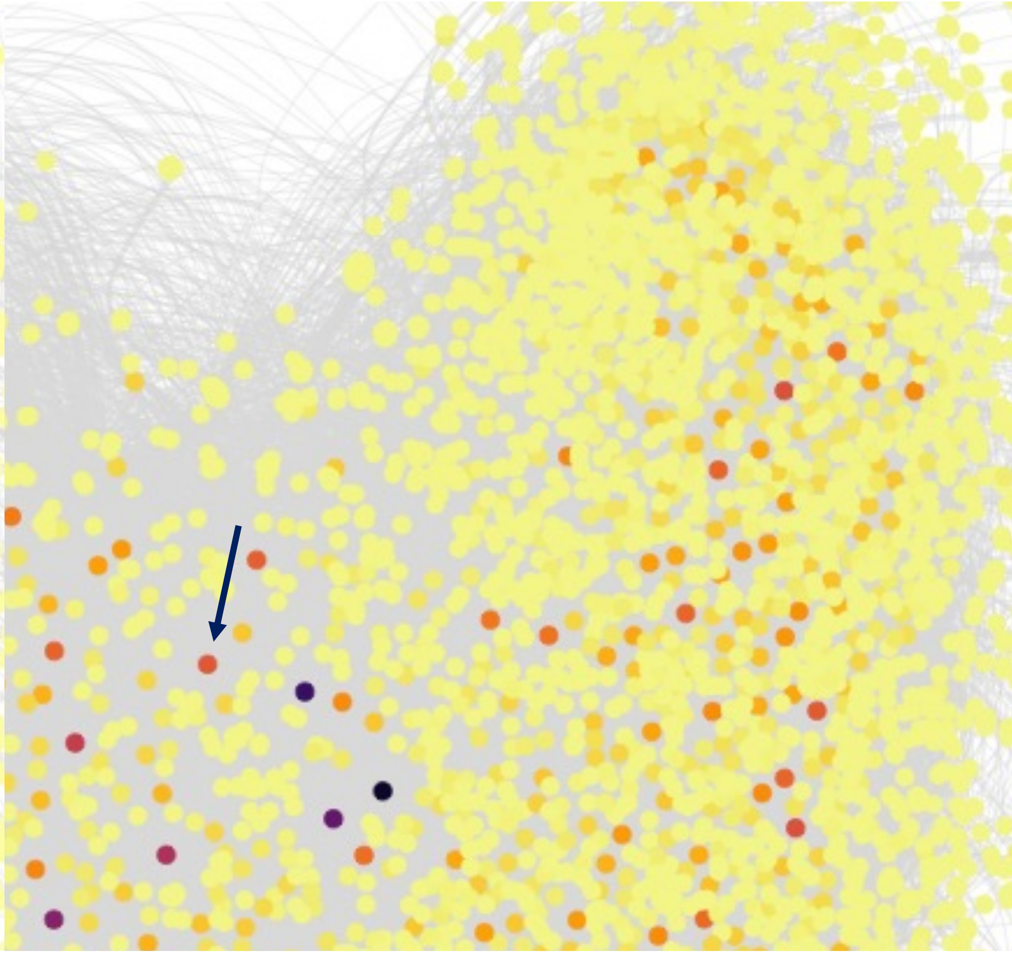

### PageRank



# PageRank versus degree hubs

## wikipedia administrator elections and vote history data

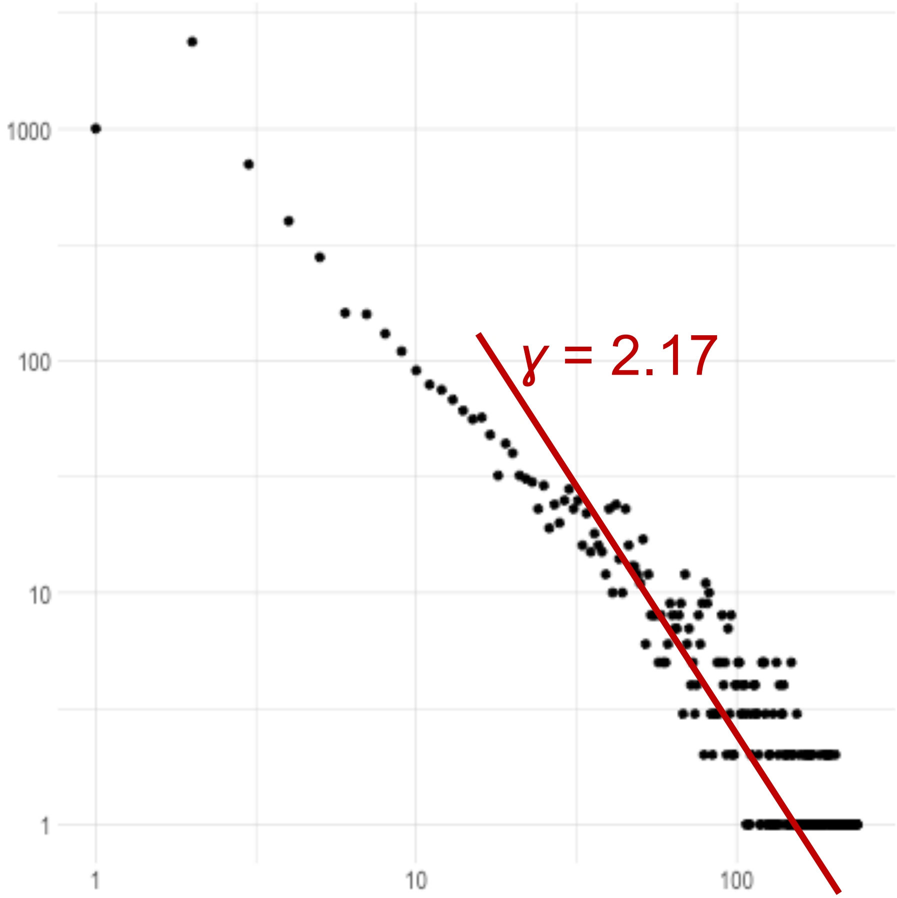


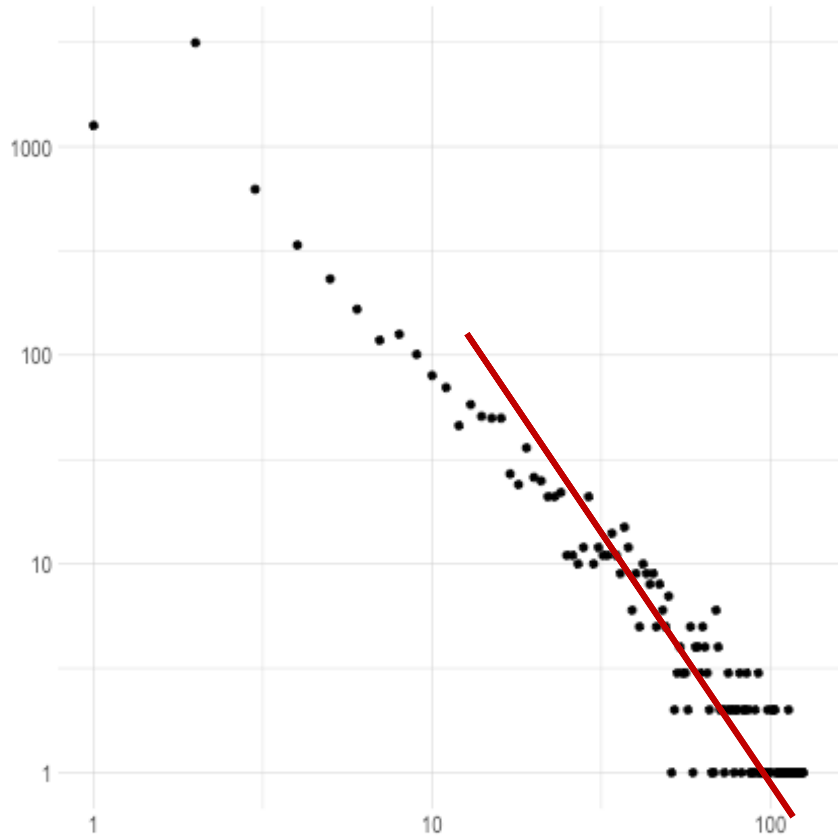
# PageRank versus degree hubs

wikipedia administrator elections and vote history data

## Degree



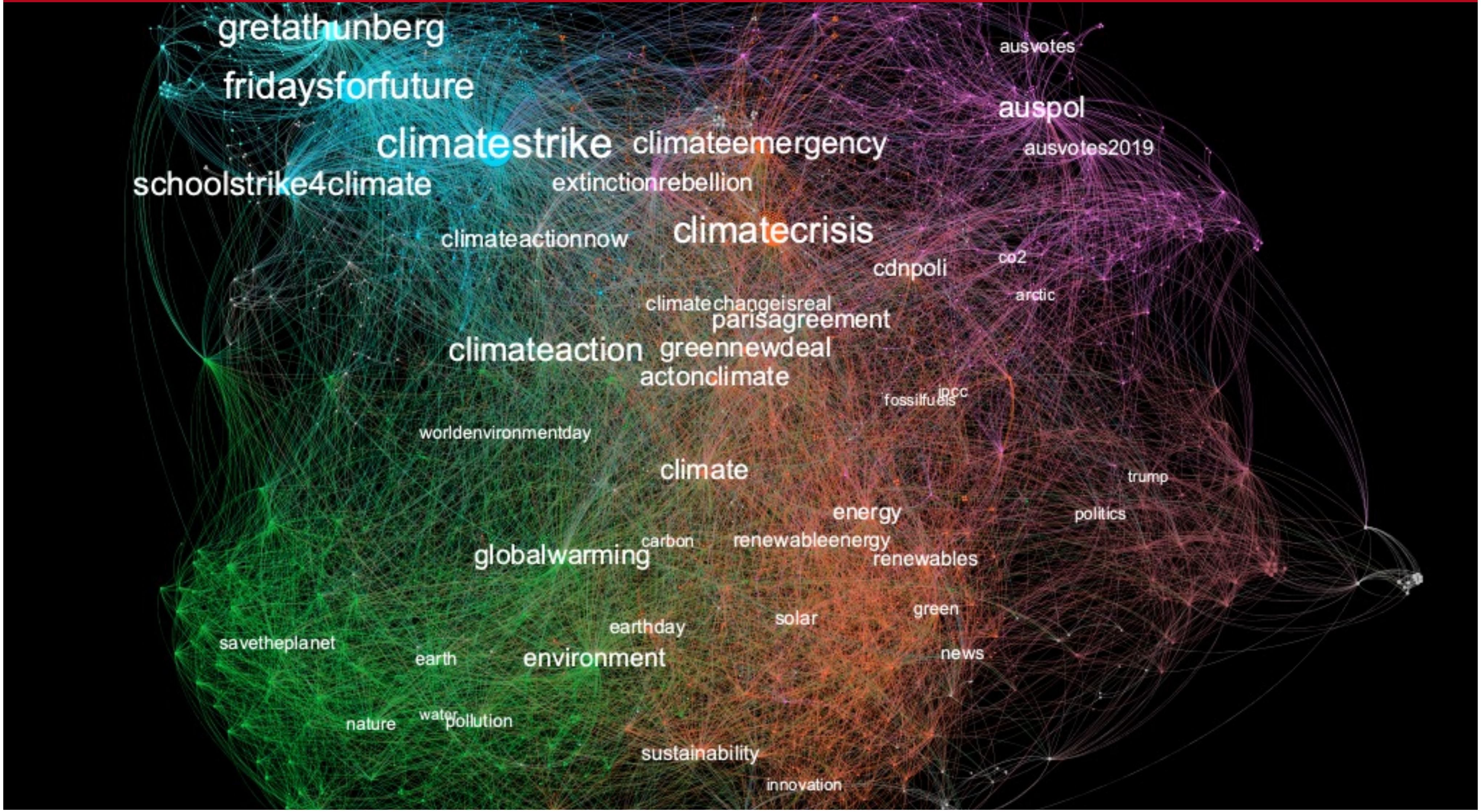

## PageRank



UNIVERSITÀ DEGLI STUDI DI PADOVA

# PageRank on a semantic network

2019 hashtag network related to #climatechange (from Twitter, after #gretathunberg)



# Readings
## on PageRank

- Brin and Page, “The anatomy of a large-scale hypertextual web search engine,” 1998
- Page, Brin, Motwani, Winograd, “The PageRank Citation Ranking: Bringing Order to the Web,” 1999

http://ilpubs.stanford.edu/422/1/1999-66.pdf

Google Scholar

**https://scholar.google.com/**

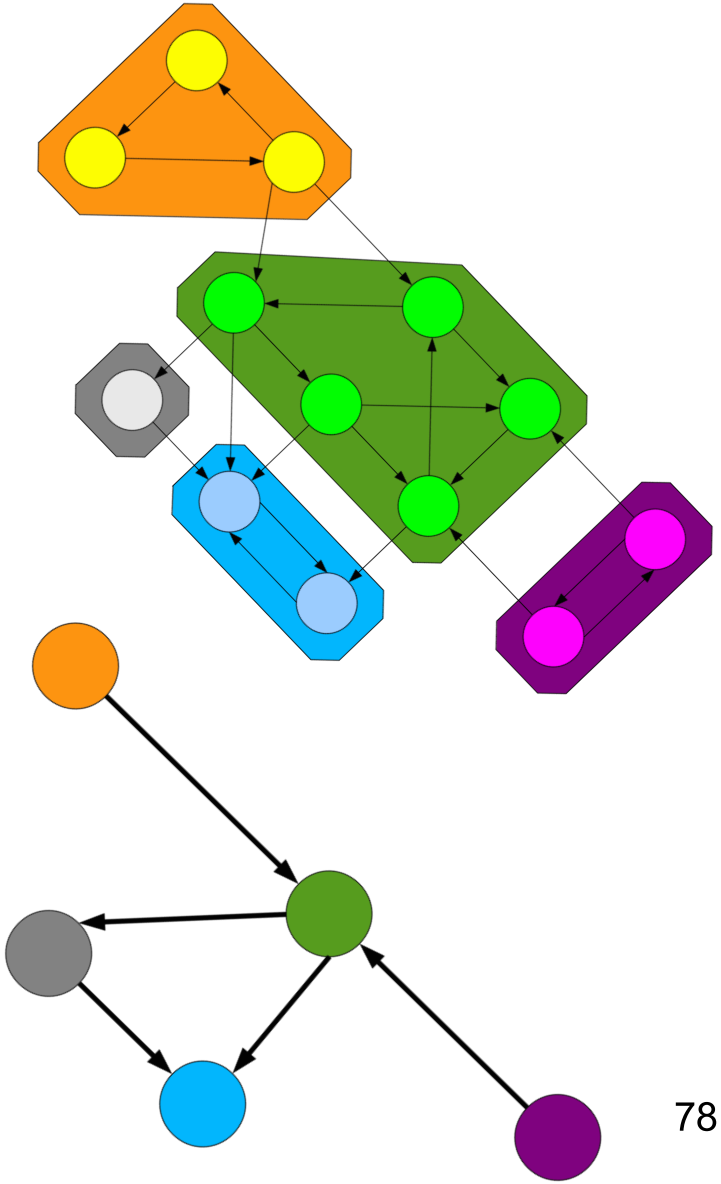
# Convergence properties

of PageRank power iterations

# The condensation graph

## ordering an adjacency matrix

- Strong connectivity induces a partition in disjoint strongly connected sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_K$
- By reinterpreting the sets as nodes we obtain a condensation graph $\mathcal{G}^*$ where $i \rightarrow j$ is an edge if a connection exists between sets $\mathcal{V}_i \rightarrow \mathcal{V}_j$



# Properties of the condensation graph

## ordering an adjacency matrix

- $\mathcal{G}^*$ does not contain cycles

  otherwise the sets in the cycle would be strongly connected

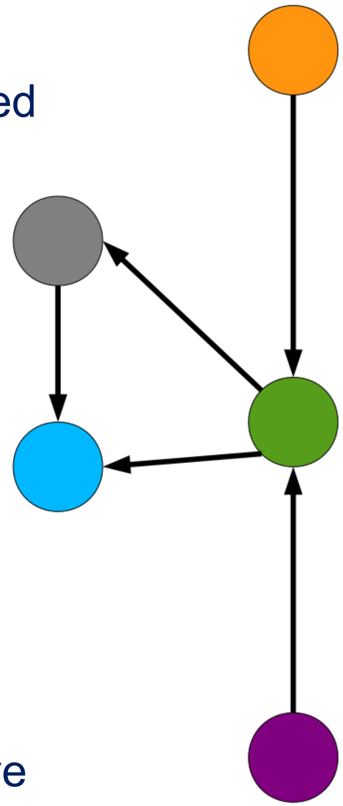
- $\mathcal{G}^*$ has at least one root and one leaf

  and every node in the graph can be reached from one of the roots

- $\mathcal{G}^*$ allows a particular reordering

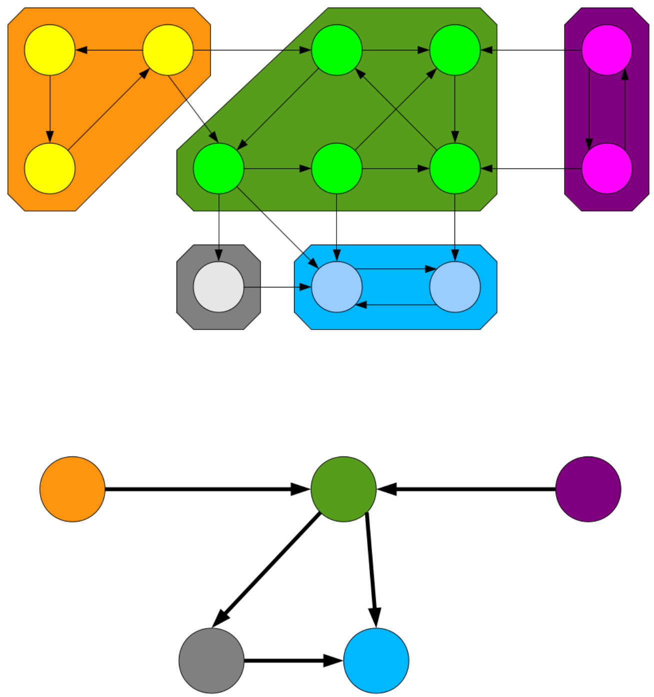
  where node $n_i$ does not reach any of the nodes $n_j$ with $j<i$

  procedure: identify a root $n_1$ and remove it from the network, then identify a new root; cycle until all nodes have been selected



# Matrix representation
## of the condensation graph

The condensation graph ordering induces a block-lower-triangular matrix structure on the adjacency matrix

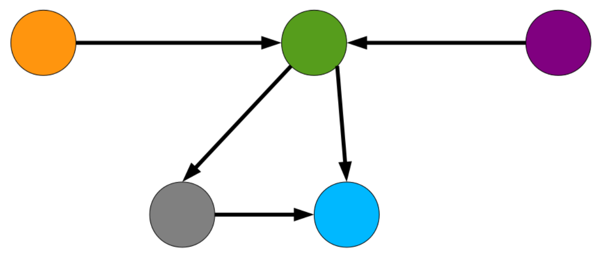


$$
\begin{array}{c}
\begin{array}{ccccc} 3 & 2 & 5 & 1 & 2 \end{array}\\
M=\left[\begin{array}{ccc|cc|ccccc|c|cc}
 & \frac{1}{3} & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
1 & & & & & & & & & & & & \\
\hline
 & & & & \frac{1}{2} & & & & & & & & \\
 & & & \frac{1}{2} & & & & & & & & & \\
\hline
 & \frac{1}{3} & & & & & & & & \frac{1}{2} & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \frac{1}{3} & & & & \\
 & \frac{1}{3} & & & & \frac{1}{2} & & & & & & & \\
 & & & & & & & \frac{1}{3} & & & & & \\
 & & & & \frac{1}{2} & & 1 & & \frac{1}{3} & & & & \\
\hline
 & & & & & & & \frac{1}{3} & & & 0 & & \\
\hline
 & & & & & & & \frac{1}{3} & \frac{1}{3} & & 1 & & 1 \\
 & & & & & & & & & \frac{1}{2} & & 1 &
\end{array}\right]
\end{array}
$$

blocks in the diagonal are irreducible = no block-diagonal form !

# Perron-Frobenius theorem
## of the condensation graph

the eigenvalues of the diagonal blocks, except for the leaves, lie inside the unit circle, i.e., $|\lambda|<1$

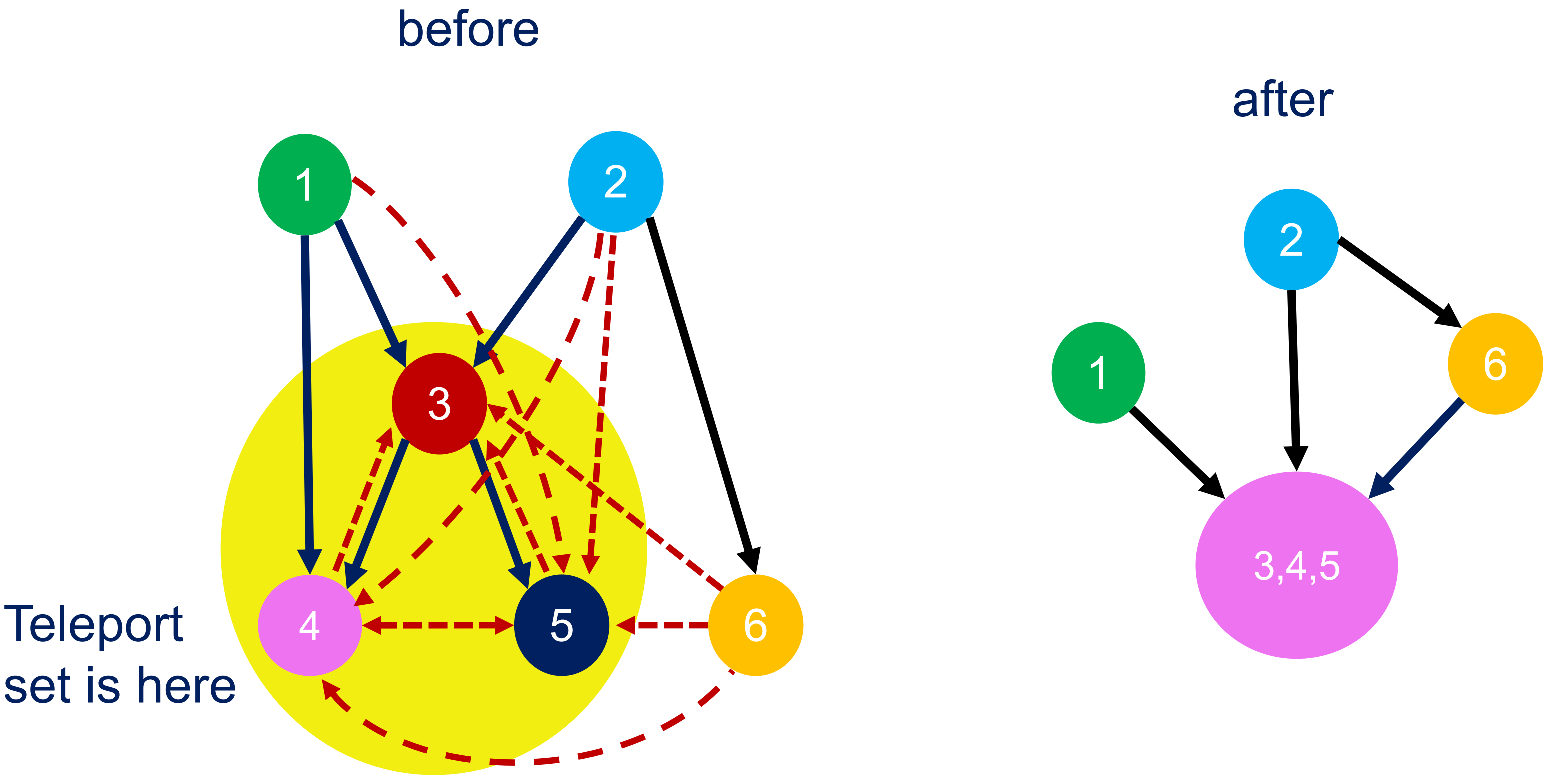


$$M=\begin{array}{c}
\begin{array}{ccc|cc|ccccc|c|cc} & & 3 & & 2 & & & & & 5 & 1 & & 2 \end{array}\\
\left[\begin{array}{ccc|cc|ccccc|c|cc}
 & \frac{1}{3} & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & \\
1 & & & & & & & & & & & & \\
\hline
 & & & & \frac{1}{2} & & & & & & & & \\
 & & & \frac{1}{2} & & & & & & & & & \\
\hline
 & \frac{1}{3} & & & & & & & & \frac{1}{2} & & & \\
 & & & \frac{1}{2} & & \frac{1}{2} & & & \frac{1}{3} & & & & \\
 & \frac{1}{3} & & & & \frac{1}{2} & & & & & & & \\
 & & & & & & & \frac{1}{3} & & & & & \\
 & & & & \frac{1}{2} & & 1 & & \frac{1}{3} & & & & \\
\hline
 & & & & & & & \frac{1}{3} & & & 0 & & \\
\hline
 & & & & & & & \frac{1}{3} & \frac{1}{3} & & 1 & & 1 \\
 & & & & & & & & & \frac{1}{2} & & 1 &
\end{array}\right]
\end{array}$$

each leaf-block has at least one eigenvalue in the unit circle; $\lambda=1$ is always available, the others are distinct

# The teleportation effect

## it implies only one leaf




Hence $M_1$ carries only one eigenvector associated with the eigenvalue $\lambda=1$

# Lemma

## on generalized eigenvectors

- PageRank matrix $\boldsymbol{M_1} = c\,\boldsymbol{M} + (1\text{-}c)\,\boldsymbol{q}\,\boldsymbol{1}^{\mathrm{T}}$
- Normalization property $\boldsymbol{1}^{\mathrm{T}}\,\boldsymbol{M_1} = \boldsymbol{1}^{\mathrm{T}}$
- Jordan form $\boldsymbol{M_1} = \boldsymbol{V}\,\boldsymbol{J}\,\boldsymbol{V}^{-1}$

carries the right (generalized) eigenvectors $\mathbf{e}_i$ of $\mathbf{M}_1$

carries the eigenvalues of $\mathbf{M}_1$

$$\boldsymbol{J} = \begin{bmatrix} \boxed{\begin{matrix}\lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1\end{matrix}} & & & & \\ & \boxed{\begin{matrix}\lambda_2 & 1 \\ & \lambda_2\end{matrix}} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \ddots & \\ & & & & \boxed{\begin{matrix}\lambda_n & 1 \\ & \lambda_n\end{matrix}} \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{1}^{\mathrm{T}}\,\boldsymbol{M_1}\,\boldsymbol{V} &= \boldsymbol{1}^{\mathrm{T}}\,\boldsymbol{V} \\ &= \boldsymbol{1}^{\mathrm{T}}\,\boldsymbol{V}\,\boldsymbol{J} \end{aligned} \quad \Rightarrow \quad \underbrace{\boldsymbol{1}^{\mathrm{T}}\,\boldsymbol{V}}_{\rho}\;\underbrace{(\boldsymbol{J} - \boldsymbol{I})}_{\text{only one value is 0}} = \boldsymbol{0}$$

Hence $\boldsymbol{1}^{\mathrm{T}}\,\mathbf{e}_i = 0$ for $i>1$, i.e., except for the eigenvector associated with eigenvalue 1

# Main result
## for the eigenstructure of the PageRank matrix

$$\underbrace{\boldsymbol{M}_1\, \boldsymbol{e}_i = c\, \boldsymbol{M}\, \boldsymbol{e}_i}_{\text{same eigenvalues of } \mathbf{M}\text{, but multiplied by c !!!}} + (1-c)\, \boldsymbol{q}\, \cancel{\boldsymbol{1}^T}\, \boldsymbol{e}_i \qquad \text{for } i>1$$

- $\boldsymbol{M}_1$ has one eigenvalue equal to 1
- The remaining eigenvalues satisfy $|\lambda| \leq c$

Haveliwala and Kamvar, “The second eigenvalue of the Google matrix,” 2003

http://ilpubs.stanford.edu:8090/582/1/2003-20.pdf

# Convergence properties
## of the PageRank power iteration

$$\boldsymbol{p}_t = \boldsymbol{M}_1 \boldsymbol{p}_{t-1} = \boldsymbol{M}_1^t \boldsymbol{p}_0 = \boldsymbol{V} \boldsymbol{J}^t \boldsymbol{V}^{-1} \boldsymbol{p}_0$$

gets large for high multiplicity

max eigenvalue multiplicity

- $\|\boldsymbol{p}_t - \boldsymbol{p}_\infty\|_2 \lesssim K c^t t^{m-1} \sim K c^t$
- Triangular inequality: $\|\boldsymbol{p}_{t+1} - \boldsymbol{p}_t\|_2 \lesssim 2K c^t$
- Precision $\varepsilon$: $\|\boldsymbol{p}_{t+1} - \boldsymbol{p}_t\|_2 < \varepsilon$
- Iterations required: $\boxed{t = [\ln(2/\varepsilon) + \ln(K)] / \ln(1/c)}$

precision $10^{-3}$ → 7.6

c=0.85 → 1/ln(1/c) = 6

Is usually small
→ fast algorithm

# Local PageRank

measuring similarity/closeness among nodes

# Measuring closeness: LocalPageRank

## for the eigenstructure of the PageRank matrix

### Idea

- Measure similarity / closeness to node *i* by applying PageRank with teleport set $S=\{i\}$, i.e., with $\mathbf{q} = \boldsymbol{\delta}_i$

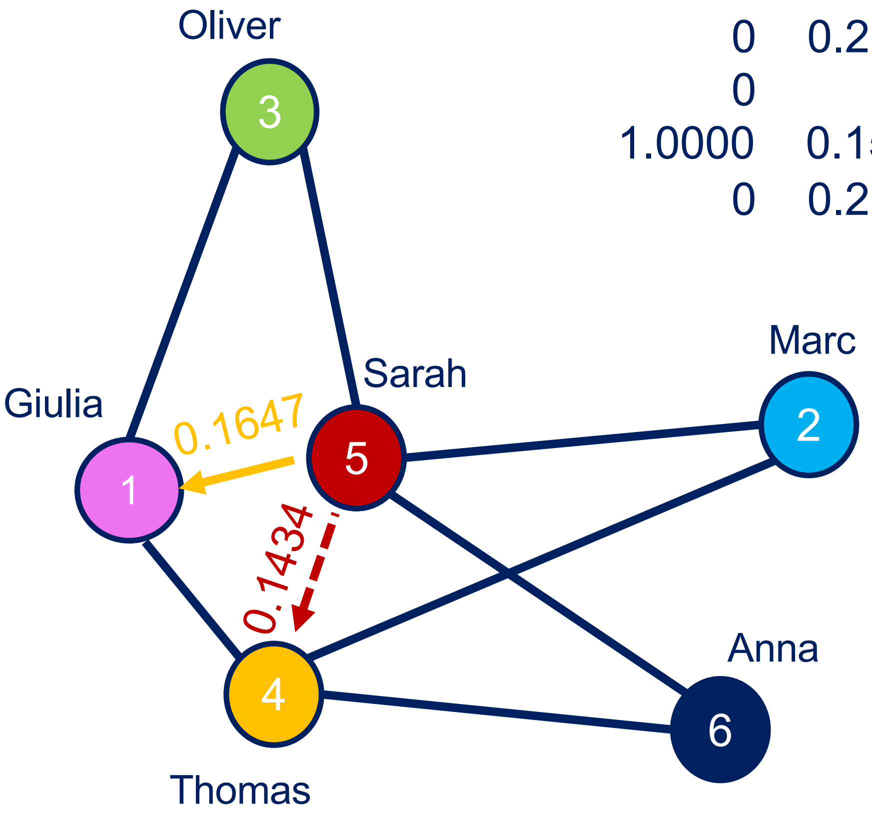
### Result

- Measures direct and indirect multiple connections, their quality, degree or weight

# Example

## who's Sara's best friend?

| t=1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 0 | 0.2125 | 0.1222 | 0.2096 | 0.1290 |
| 0 | 0.2125 | 0.0319 | 0.1705 | 0.0708 |
| 0 | 0.2125 | 0.0921 | 0.1369 | 0.1127 |
| 0 | 0 | 0.2408 | 0.0617 | 0.2043 |
| 1.0000 | 0.1500 | 0.4811 | 0.2508 | 0.4125 |
| 0 | 0.2125 | 0.0319 | 0.1705 | 0.0708 |

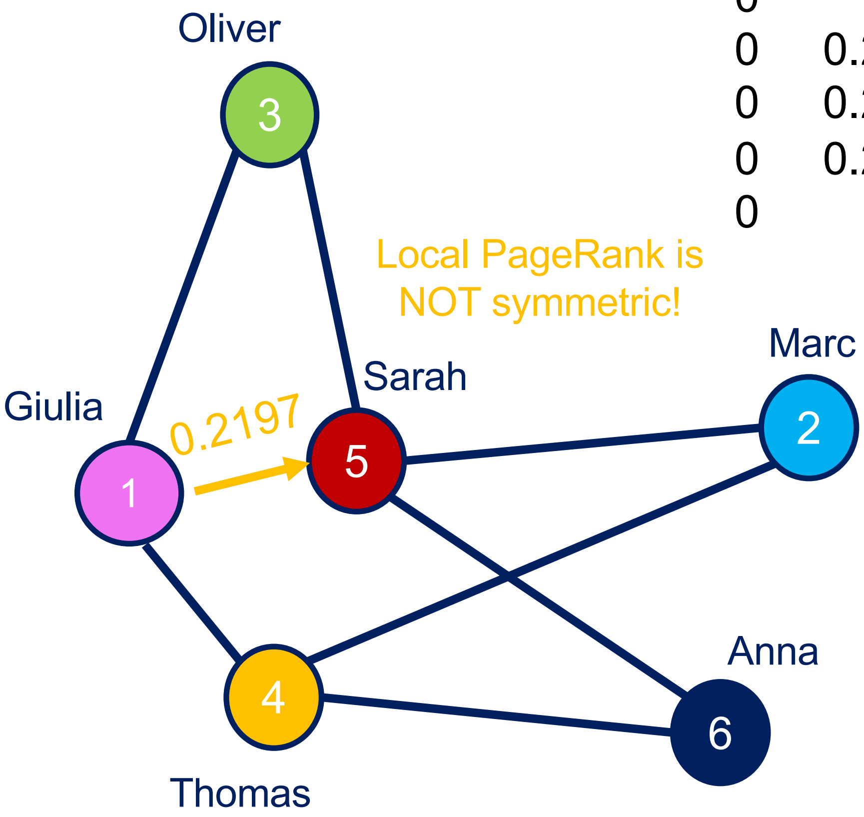



| 10 | 20 | 50 | 75 | 100 | |
|---|---|---|---|---|---|
| 0.1743 | 0.1653 | 0.1647 | 0.1647 | 0.1647 | Giulia |
| 0.1238 | 0.1144 | 0.1138 | 0.1138 | 0.1138 | Marc |
| 0.1206 | 0.1199 | 0.1199 | 0.1199 | 0.1199 | Oliver |
| 0.1285 | 0.1426 | 0.1434 | 0.1434 | 0.1434 | Thomas |
| 0.3290 | 0.3435 | 0.3444 | 0.3444 | 0.3444 | Sarah |
| 0.1238 | 0.1144 | 0.1138 | 0.1138 | 0.1138 | Anna |

# Example

## who's Giulia's best friend?

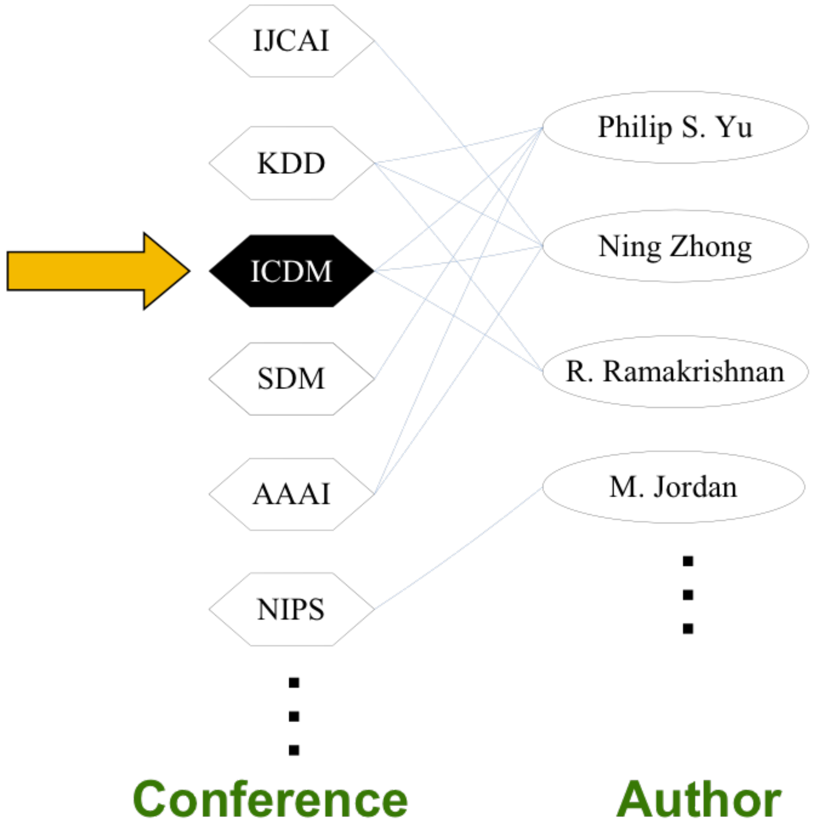
| t=1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 1.0000 | 0.1500 | 0.4109 | 0.2403 | 0.3404 |
| 0 | 0 | 0.1405 | 0.0467 | 0.1262 |
| 0 | 0.2833 | 0.1027 | 0.1510 | 0.1275 |
| 0 | 0.2833 | 0.0425 | 0.2358 | 0.1078 |
| 0 | 0.2833 | 0.1629 | 0.2795 | 0.1719 |
| 0 | 0 | 0.1405 | 0.0467 | 0.1262 |



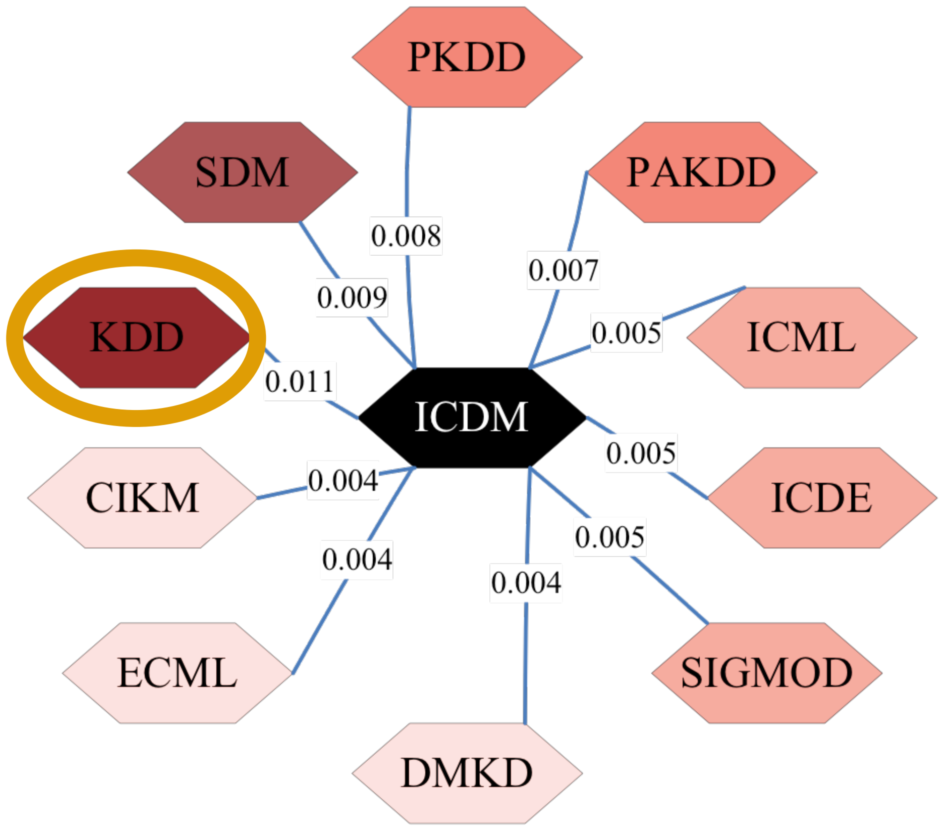

| 10 | 20 | 50 | 75 | 100 | |
|---|---|---|---|---|---|
| 0.2909 | 0.2985 | 0.2989 | 0.2989 | 0.2989 | Giulia |
| 0.0848 | 0.0926 | 0.0931 | 0.0931 | 0.0931 | Marc |
| 0.1309 | 0.1313 | 0.1314 | 0.1314 | 0.1314 | Oliver |
| 0.1763 | 0.1645 | 0.1638 | 0.1638 | 0.1638 | Thomas |
| 0.2324 | 0.2204 | 0.2197 | 0.2197 | 0.2197 | Sarah |
| 0.0848 | 0.0926 | 0.0931 | 0.0931 | 0.0931 | Anna |

# Example

## what is the most related conference to ICDM?




### Top 10 ranking results



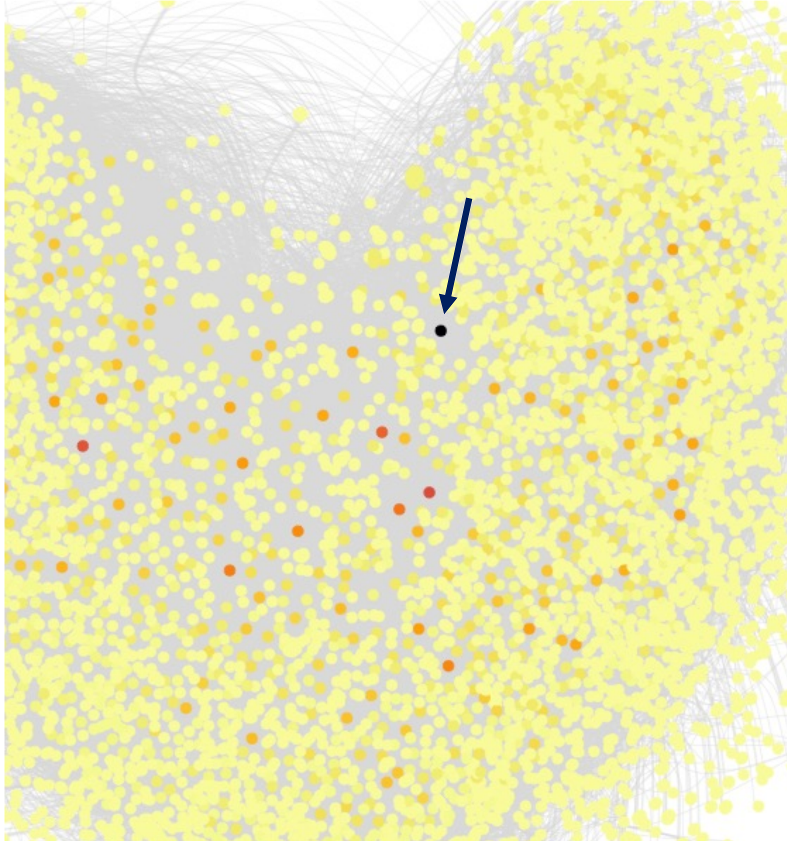

ICDM = international conf. on data mining
KDD = knowledge discovery and data mining

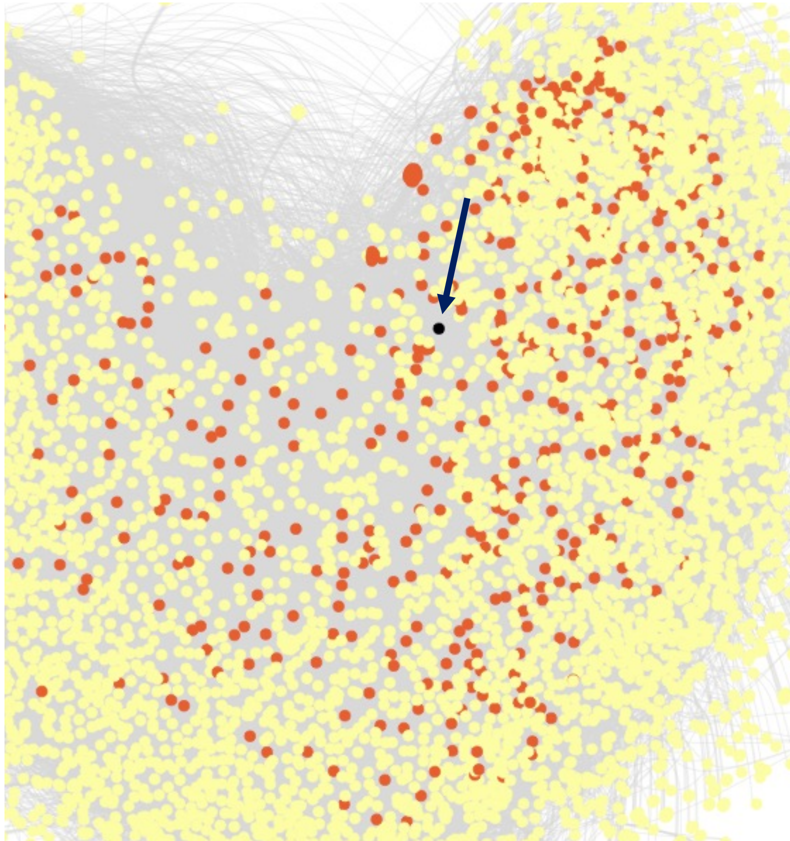
# Local PageRank versus degree

## authorities

Local PageRank

1-hop out-neighbours





neighbours authority score =
local node → neighbours

# On the complexity of Local PageRank

## approximate PageRank

Andersen, Chung, Lang, “Local graph partitioning using PageRank vectors,” 2006

https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=4031383

use institutional Sign In with your unipd credentials

# Approximate PageRank algorithm

## the push operation

- Start from $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{v} = \boldsymbol{q}$
- To all the nodes *i* satisfying $v_i > \varepsilon\, d_i / D$ apply the **push** operation

precision

degree of node i

sum of the degrees

u constantly increases

$$\boldsymbol{u}^+ = \boldsymbol{u} + (1\text{-}c)\,\boldsymbol{\delta}$$

$$\boldsymbol{v}^+ = \boldsymbol{v} - \boldsymbol{\delta} + c\,\boldsymbol{M}\,\boldsymbol{\delta}$$

v always positive

only one active element in position i with value $v_i$

Returns $\boldsymbol{u} \simeq \boldsymbol{r}$ with precision $|\boldsymbol{r} - \boldsymbol{u}|_1 < \varepsilon$

It is simple

# Linearity of PageRank

## to build a lemma for the proof

column stochastic matrix $\mathbf{1}^T \mathbf{M} = \mathbf{1}^T$

- PageRank equation $\mathbf{r}_q = c\,\mathbf{M}\,\mathbf{r}_q + (1-c)\,q$

stochastic ranking vector
$\mathbf{1}^T \mathbf{r}_q = 1,\ \mathbf{r}_q \geq 0$

stochastic Teleport vector
$\mathbf{1}^T \mathbf{q} = 1$

- Alternative equation $\mathbf{r}_q = (\mathbf{I} - c\,\mathbf{M})^{-1}\,(1-c)\,\mathbf{q}$

linear in q

$$\mathbf{r}_{au+bv} = a\,\mathbf{r}_u + b\,\mathbf{r}_v$$

# Modifying the PageRank equation

## the lemma for the proof

one-step random walk

- PageRank equation $\boldsymbol{r}_q = c\, \boldsymbol{r}_{Mq} + (1\text{-}c)\, \boldsymbol{q}$

$\Downarrow$

- $\boldsymbol{r}_q = (\boldsymbol{I} - c\, \boldsymbol{M})^{-1}\, (1\text{-}c)\, \boldsymbol{q}$
- $\boldsymbol{r}_q = (1\text{-}c)\, \Sigma\, (c\, \boldsymbol{M})^k\, q$
- $\boldsymbol{M}\, r_q = (1\text{-}c)\, \Sigma\, (c\, \boldsymbol{M})^k\, \boldsymbol{M}\, \boldsymbol{q}$
- $\boldsymbol{M}\, \boldsymbol{r}_q = \boldsymbol{r}_{Mq}$

# Main property of push: $\boldsymbol{r}_q = \boldsymbol{u} + \boldsymbol{r}_v$

almost there

- At starting point $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{v} = \boldsymbol{q}$ imply $\boldsymbol{r}_q = \boldsymbol{0} + \boldsymbol{r}_q$
- The following steps are proved by induction

$$\boldsymbol{u}^+ = \boldsymbol{u} + (1-c)\,\delta$$
$$\boldsymbol{v}^+ = \boldsymbol{v} - \delta + c\,\boldsymbol{M}\,\delta$$

$$\Downarrow$$

$$\boldsymbol{u}^+ + \boldsymbol{r}_{v+} = \boldsymbol{u} + (1-c)\,\delta + \overbrace{\boldsymbol{r}_v - \boldsymbol{r}_\delta + \underbrace{c\,\boldsymbol{r}_{M\delta}}_{\boldsymbol{r}_\delta - (1-c)\,\delta}}^{\text{by linearity}}$$

$$\Downarrow$$

$$\boldsymbol{u}^+ + \boldsymbol{r}_{v+} = \boldsymbol{u} + \boldsymbol{r}_v = \boldsymbol{r}_q$$

# Precision guarantee: $|\boldsymbol{r}_q - \boldsymbol{u}|_1 < \varepsilon$
and the result is proved

- The push property implies $\boldsymbol{r}_q = \boldsymbol{u} + \boldsymbol{r}_v$
- Hence $|\boldsymbol{r}_q - \boldsymbol{u}|_1 = |\boldsymbol{r}_v|_1 = \boldsymbol{1}^T \boldsymbol{r}_v$

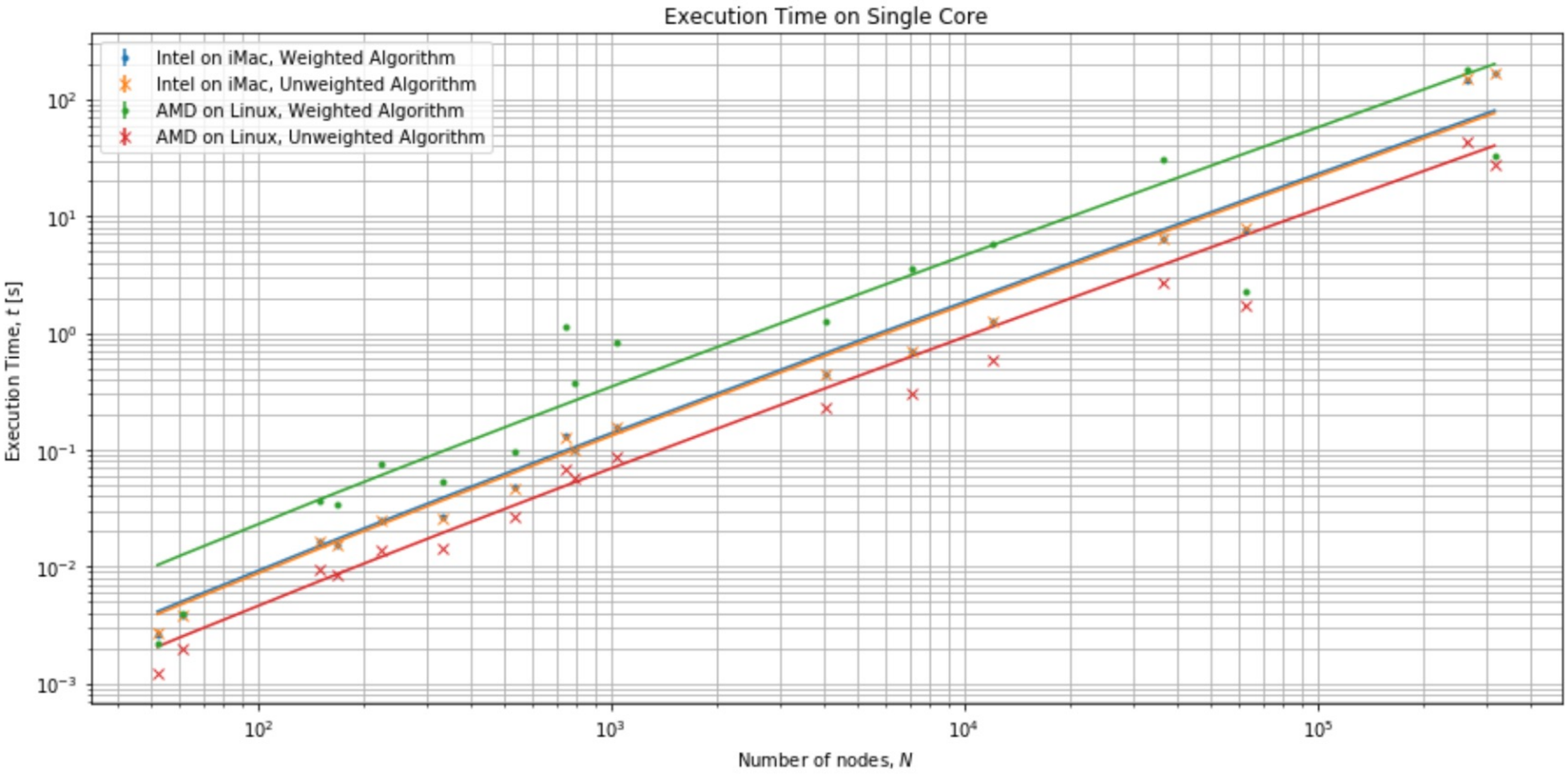
- The PageRank equation is $\boldsymbol{r}_v = c\, \boldsymbol{M}\, \boldsymbol{r}_v + (1-c)\, \boldsymbol{v}$
- Hence $\boldsymbol{1}^T \boldsymbol{r}_v = c\, \underbrace{\boldsymbol{1}^T \boldsymbol{M}}_{\boldsymbol{1}^T}\, \boldsymbol{r}_v + (1-c)\, \boldsymbol{1}^T \boldsymbol{v}$ so that $\boldsymbol{1}^T \boldsymbol{r}_v = \boldsymbol{1}^T \boldsymbol{v}$

As a result $|\boldsymbol{r}_q - \boldsymbol{u}|_1 = \boldsymbol{1}^T \boldsymbol{v} < \Sigma\, \varepsilon\, d_i / D = \varepsilon$

# Scalability properties

## of Local PageRank using Approximate PageRank

(Francesco Barbato & Tommaso Boccato, 2020)




Quasi-linear behaviour = scalability of Local PageRank

# Beware of the Lazy PageRank

which is suggested in the paper

- Lazy PageRank $\boldsymbol{r} = \underbrace{a\, \boldsymbol{M}_2}_{\boldsymbol{M}_2 = b\, \boldsymbol{I} + (1-b)\, \boldsymbol{M}}\, \boldsymbol{r} + (1-a)\, \boldsymbol{q}$

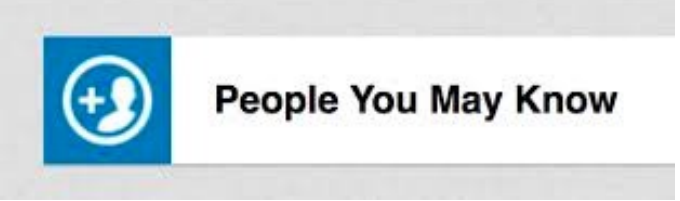
- Lazy because a fraction *b* of the times the surfer stays where she/he is

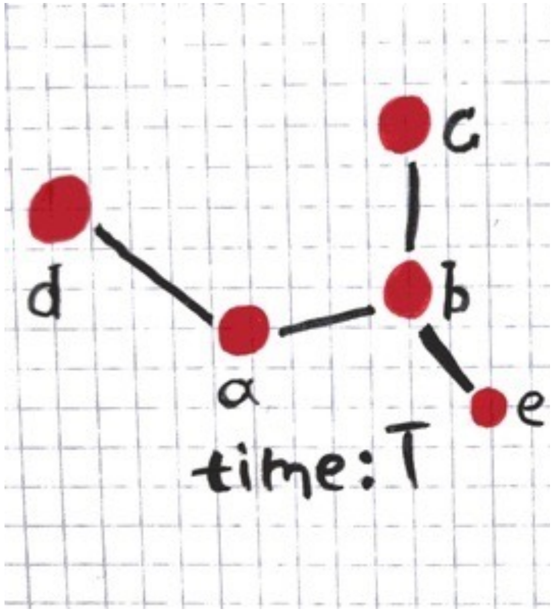
- Equivalent to $\boldsymbol{r} = \underbrace{c\, \boldsymbol{M}}_{c = a(1-b)/(1-ab) < a}\, \boldsymbol{r} + (1-c)\, \boldsymbol{q}$

$$c = a(1-b)/(1-ab) \underbrace{< a}_{\text{slower algorithm, as its convergence speed depends on a>c, better use c directly!}}$$

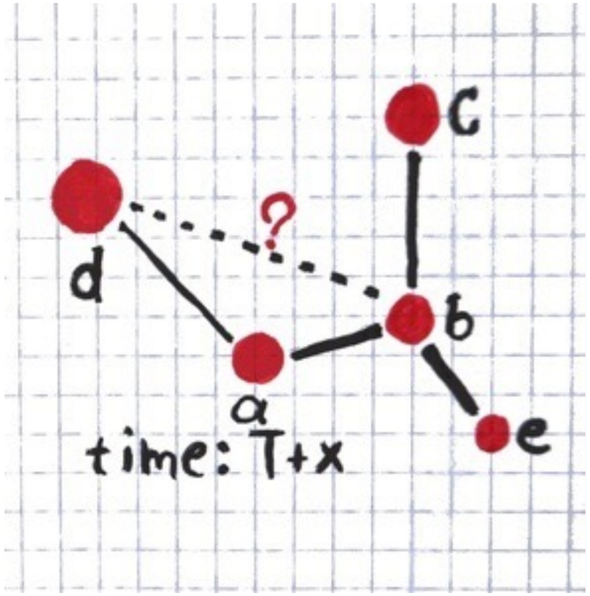
# Application #1
## the link prediction task

Recommendation in social networks

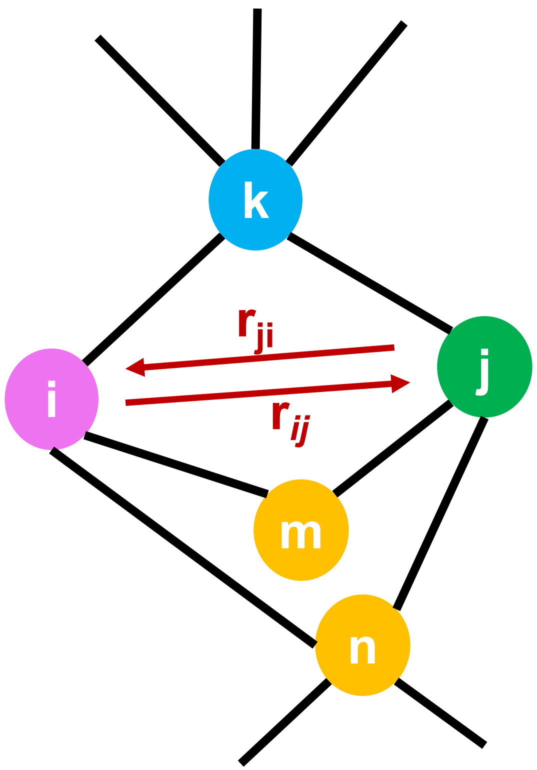



Given a graph at time T, can we output a ranked list of links that are predicted to appear in the graph at time T+x ?



# Application #1
## random walk with restart (RWR) method




Local PageRank vector

teleportation to node *i*

$$\mathbf{r}_i = c\,\mathbf{M}\,\mathbf{r}_i + (1-c)\,\boldsymbol{\delta}_i$$

Likelihood of activating the link (i,j)

$$\boxed{L_{RWR}(i,j) = r_{ij} + r_{ji}}$$

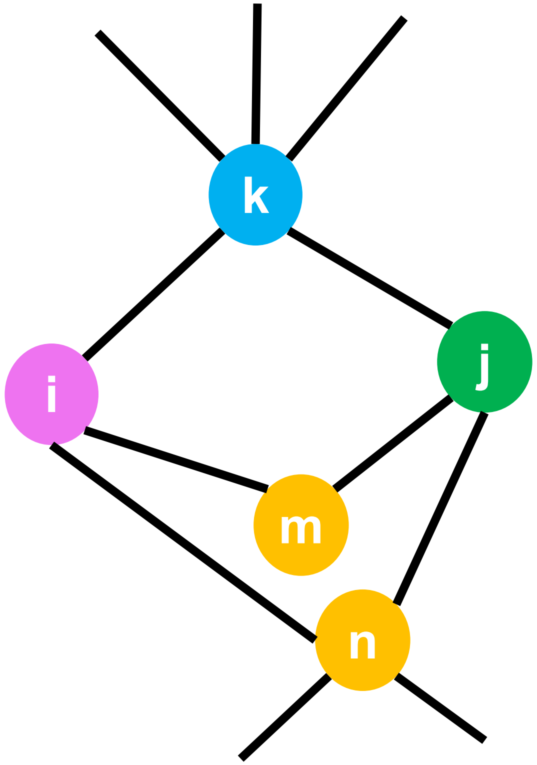
Select the highest values of $L_{RWR}$ for recommendation pourposes

# Application #1

## the resorse allocation (RA) counterpart

$$L_{RA}(i,j) = \sum_{k \in N_i \cap N_j} 1/d_k$$

common neighbours




related to a two-hop RWR

$$\mathbf{r}_i \simeq (1\text{-c}) \sum_{n=0}^{2} (\text{c}\,\mathbf{M})^n\, \boldsymbol{\delta}_i$$

to have

$$r_{ij} \simeq (1\text{-c})\, c^2 / d_i\, L_{RA}(i,j)$$

$$L_{RWR}(i,j) \simeq (1\text{-c})\, c^2\, (1/d_i + 1/d_j)\, L_{RA}(i,j)$$

# Application #1
## performance metrics

fraction of links correctly guessed
(out of 100 recomendations)

| **Precision** | CN | RA | LP | ACT |
|---|---|---|---|---|
| USAir | 0.59 | 0.64 | 0.61 | 0.49 |
| NetScience | 0.26 | 0.54 | 0.30 | 0.19 |
| Power | 0.11 | 0.08 | **0.13** | 0.08 |
| Yeast | 0.67 | 0.49 | 0.68 | 0.57 |
| C.elegans | 0.12 | 0.13 | **0.14** | 0.07 |

| | RWR | HSM | LRW | SRW |
|---|---|---|---|---|
| USAir | 0.65 | 0.28 | 0.64(3) | **0.67**(3) |
| NetScience | **0.55** | 0.25 | 0.54(2) | 0.54(2) |
| Power | 0.09 | 0.00 | 0.08(2) | 0.11(3) |
| Yeast | 0.52 | 0.84 | **0.86**(3) | 0.73(9) |
| C.elegans | 0.13 | 0.08 | **0.14**(3) | **0.14**(3) |

Among the best performance in social networks

But not strikingly good compared to simpler methods (e.g., RA = resource allocation)

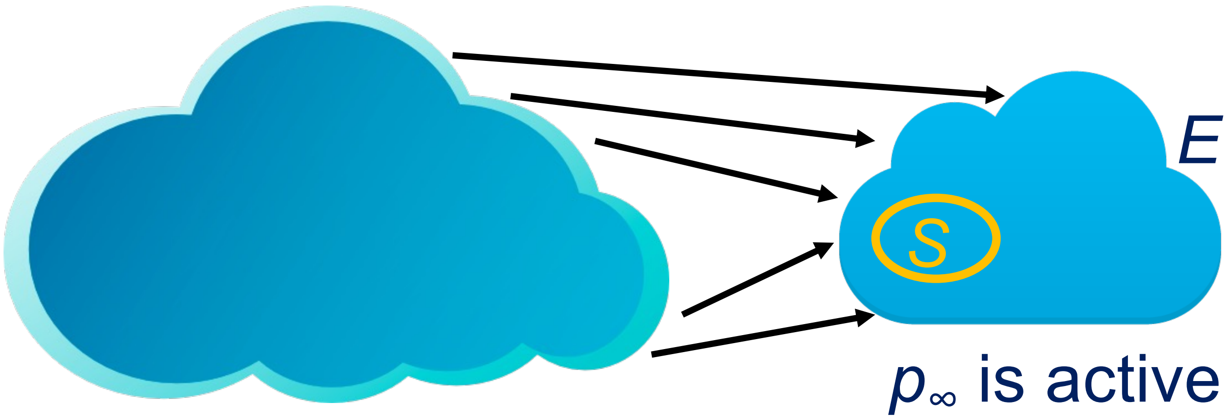
# Application #2

## TopicSpecific PageRank

- **Bias** the random walk towards a **topic specific teleport set** *S* of nodes, i.e., make sure that *q* is active in *S* only
- *S* should contain only pages that are relevant to the topic

Result

- The random walk **deterministically** ends in a small set *E*, containing *S*, and being in some sense close to it




$p_\infty$ is active in E only

# Application #2

## assigning documents to topics in semantic networks

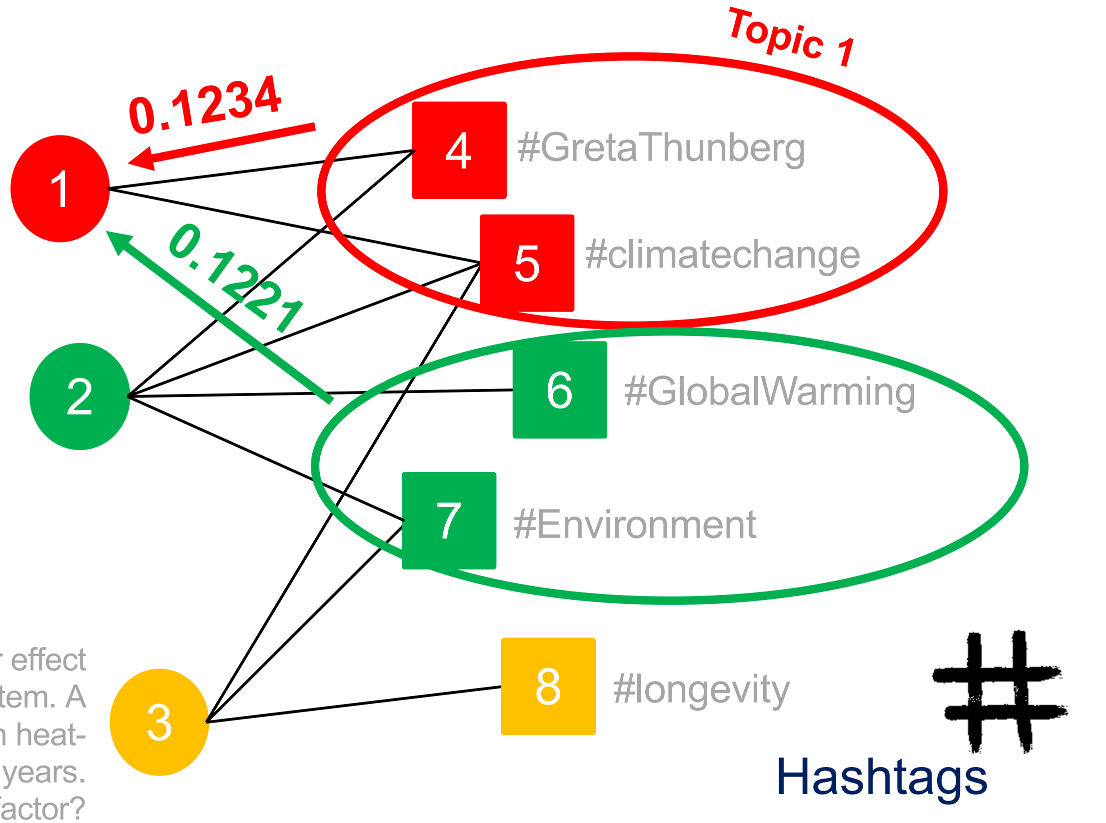
**Tweet 1** is assigned
to **Topic 1** !!!

Tweets

those who think they are crazy enough to change the world eventually do. #climatechange #ClimateCrisis #ClimateAction #GretaThunberg #Greta

Hopefully these kids will succeed where past generations have failed. #TheResistance #FBR #ClimateChange #Environment #GlobalWarming #GretaThunberg

The #environment can have a major effect on the human cardiovascular system. A new study has found an increase in heat-induced #heartattack risk in recent years. Could #ClimateChange be a risk factor? #longevity



# Signed PageRank

modifications for signed networks

# PageRank in signed networks

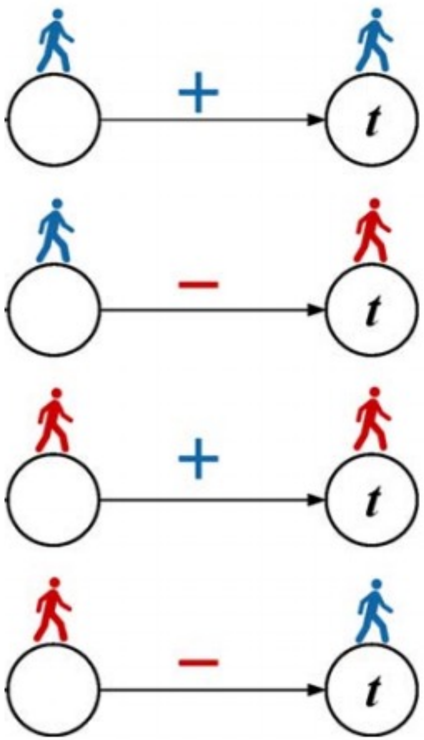
Jung, Jim, Sael, Kang, “Personalized ranking in signed networks using signed random walk with restart,” 2016

https://ieeexplore.ieee.org/iel7/7837023/7837813/07837935.pdf

- Identify + (favourable) and – (adversarial) paths, i.e., ranking values $\boldsymbol{r}_+$ and $\boldsymbol{r}_-$ for positive and negative surfers
- Extract positive $\boldsymbol{A}_+$ and negative $\boldsymbol{A}_-$ contributions to $\boldsymbol{A} = \boldsymbol{A}_+ - \boldsymbol{A}_-$
- Normalize the absolute value, to get $\boldsymbol{M}_+$ and $\boldsymbol{M}_-$ (with normalized $\boldsymbol{M}_+ + \boldsymbol{M}_-$)
- Run a signed random walk

$$\boldsymbol{r}_+ = c\,\boldsymbol{M}_+\,\boldsymbol{r}_+ + c\,\boldsymbol{M}_-\,\boldsymbol{r}_- + (1-c)\,\boldsymbol{q}$$

$$\boldsymbol{r}_- = c\,\boldsymbol{M}_-\,\boldsymbol{r}_+ + c\,\boldsymbol{M}_+\,\boldsymbol{r}_-$$



# Signed PageRank

## power iteration

damping factor

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_+ & \boldsymbol{M}_- \\ \boldsymbol{M}_- & \boldsymbol{M}_+ \end{bmatrix}$$ (column) normalized adjacency matrix

$$\boldsymbol{r} = c\,\boldsymbol{M}\,\boldsymbol{r} + (1-c)\,\boldsymbol{q}_0$$

teleportation vector

PageRank vector (centrality) $\boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_+ \\ \boldsymbol{r}_- \end{bmatrix}$

$$\boldsymbol{q}_0 = \begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{0} \end{bmatrix}$$

signed centrality outcome $\boldsymbol{r}_{+-} = \boldsymbol{r}_+ - \boldsymbol{r}_-$

$$\boldsymbol{r}_{+-} = c\,\boldsymbol{M}_{+-}\,\boldsymbol{r}_{+-} + (1-c)\,\boldsymbol{q}$$

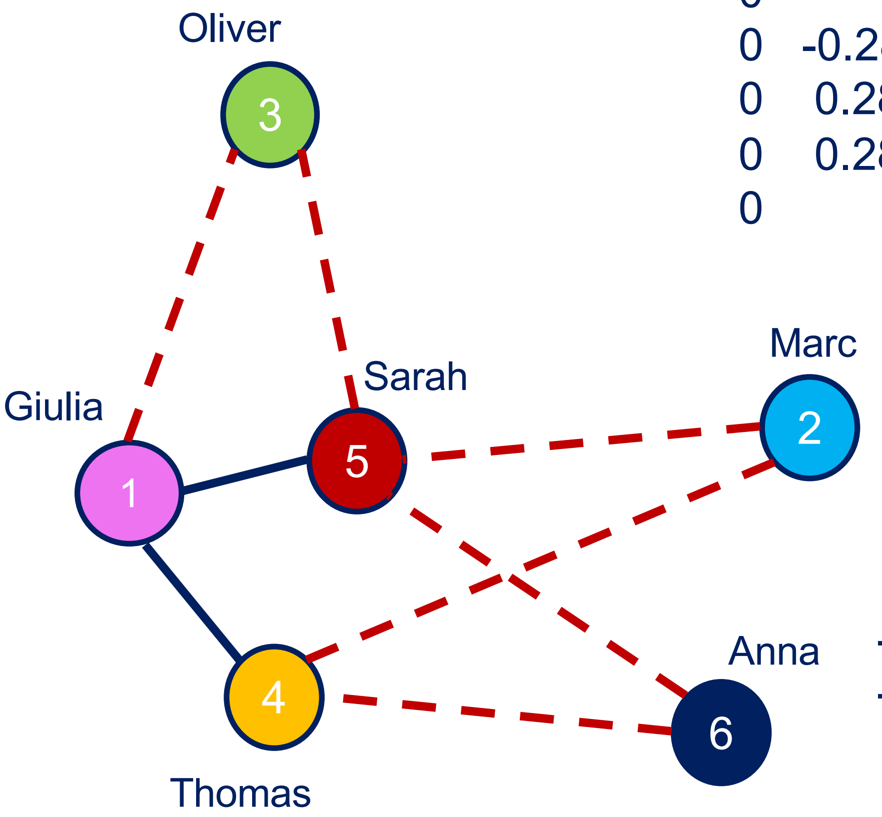
can be signed

$$\boldsymbol{M}_{+-} = \boldsymbol{A}\,\mathrm{diag}^{-1}(|\boldsymbol{A}|^T\boldsymbol{1})$$

Università degli Studi di Padova

# Example

## who’s Giulia’s best friend?




| *t=1* | *2* | *3* | *4* | *5* |
|---|---|---|---|---|
| 1.0000 | 0.1500 | 0.4109 | 0.2403 | 0.3404 |
| 0 | 0 | -0.1405 | -0.0467 | -0.1262 |
| 0 | -0.2833 | -0.1027 | -0.1510 | -0.1275 |
| 0 | 0.2833 | 0.0425 | 0.2358 | 0.1078 |
| 0 | 0.2833 | 0.1629 | 0.2795 | 0.1719 |
| 0 | 0 | -0.1405 | -0.0467 | -0.1262 |

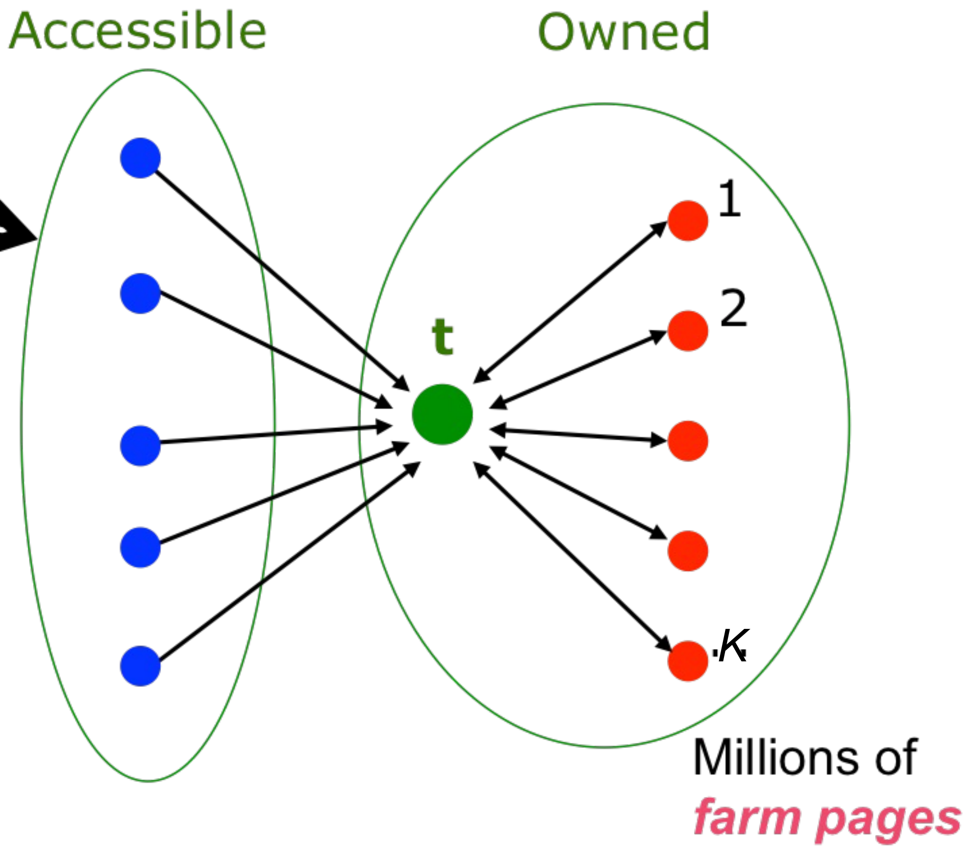
| *10* | *20* | *50* | *75* | *100* | |
|---|---|---|---|---|---|
| 0.2909 | 0.2985 | 0.2989 | 0.2989 | 0.2989 | Giulia |
| -0.0848 | -0.0926 | -0.0931 | -0.0931 | -0.0931 | Marc |
| -0.1309 | -0.1313 | -0.1314 | -0.1314 | -0.1314 | Oliver |
| 0.1763 | 0.1645 | 0.1638 | 0.1638 | 0.1638 | Thomas |
| 0.2324 | 0.2204 | 0.2197 | 0.2197 | 0.2197 | Sarah |
| -0.0848 | -0.0926 | -0.0931 | -0.0931 | -0.0931 | Anna |

# Preventing spamming

on the role of the teleport vector

# Spam farm

## how to boost PageRank for a web page


1. Get as many links as possible from accessible pages (e.g., blog comments pages)

2. Construct link farm to get a PageRank multiplier effect

# Google bombs
## in 2004 US elections

Google

Web Images Groups News Froogle Local **more »**

miserable failure | Search | Advanced Search | Preferences

**Web** Results **1** - **10** of about **969,000** for **miserable failure**. (**0.06** seconds)

Biography of President George W. Bush
Biography of the president from the official White House web site.
www.whitehouse.gov/president/gwbbio.html - 29k - Cached - Similar pages
Past Presidents - Kids Only - Current News - President
More results from www.whitehouse.gov »

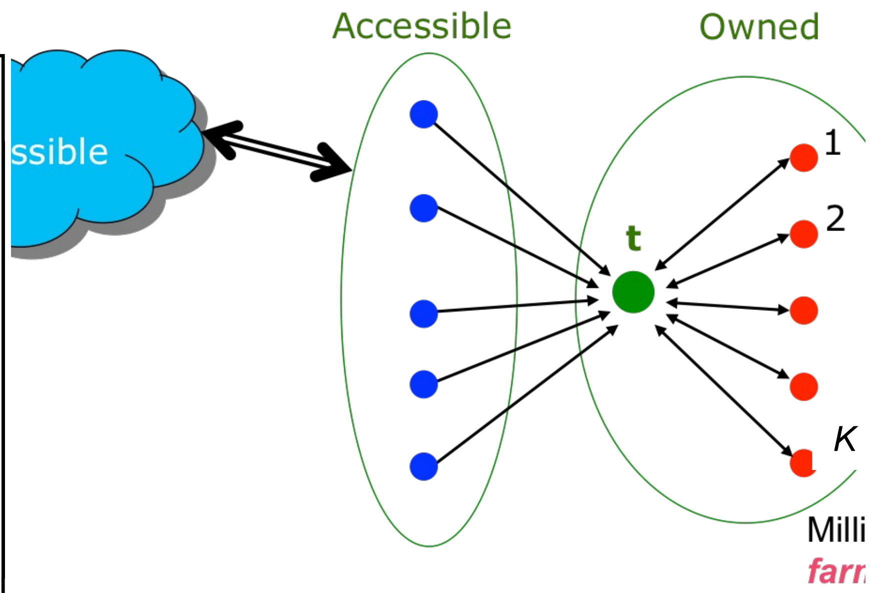
Welcome to MichaelMoore.com!
Official site of the gadfly of corporations, creator of the film Roger and Me and the television show The Awful Truth. Includes mailing list, message board, ...
www.michaelmoore.com/ - 35k - Sep 1, 2005 - Cached - Similar pages

BBC NEWS | Americas | '**Miserable failure**' links to Bush
Web users manipulate a popular search engine so an unflattering description leads to the president's page.
news.bbc.co.uk/2/hi/americas/3298443.stm - 31k - Cached - Similar pages

Google's (and Inktomi's) **Miserable Failure**
A search for **miserable failure** on Google brings up the official George W. Bush biography from the US White House web site. Dismissed by Google as not a ...
searchenginewatch.com/sereport/article.php/3296101 - 45k - Sep 1, 2005 - Cached - Similar pages

# PageRank analysis
## of spam farms

$$\begin{array}{rc} \begin{array}{r} \text{inacc.} \\ \\ \text{acc.} \\ \\ \text{t} \\ \\ \text{owned} \end{array} & M = \left[\begin{array}{c|c|c|c} ? & ? & \begin{matrix}0\\ \vdots \\ 0\end{matrix} & \mathbf{0} \\ \hline ? & ? & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \mathbf{0} \\ \hline 0 \cdots 0 & ? \cdots ? & 0 & 1 \cdots 1 \\ \hline \mathbf{0} & \mathbf{0} & \begin{matrix}\frac{1}{K}\\ \vdots \\ \frac{1}{K}\end{matrix} & \mathbf{0} \end{array}\right] \\ & \begin{array}{cccc} \text{inaccessible} & \text{accessible} & \text{t} & \text{owned} \end{array} \end{array}$$




ranking due to accessible pages

teleportation value to pages owned by the spammer

$$\boldsymbol{r} = c\,\boldsymbol{M}\,\boldsymbol{r} + (1-c)\,\boldsymbol{q} \quad \Rightarrow \quad \begin{aligned} r_t &= a + cKr_o + (1-c)q_o \\ r_o &= c\frac{1}{K}r_t + (1-c)q_o \end{aligned}$$

# PageRank outcome
## of spam farms

ranking due to accessible pages

teleportation value to pages owned by the spammer

$$r_t = \frac{a}{1 - c^2} + \frac{cK + 1}{1 + c} q_o$$

scaling factor (≃3.6)

spam factor (can be made as large as desired)

**solution** teleport only to trusted pages (i.e., set $q_o = 0$)
can also be used as a method to identify spam farms

# Row-normalized PageRank

For spreading information over the network

# Row-normalized PageRank
## an overview

PageRank equation $\boldsymbol{r} = c\,\boldsymbol{M}\,\boldsymbol{r} + (1\text{-}c)\,\boldsymbol{q}$

row-normalized

$\boldsymbol{M}\,\boldsymbol{1} = \boldsymbol{1}$

$\boldsymbol{M} = \mathrm{diag}^{-1}(\mathbf{d})\,\boldsymbol{A},\ \boldsymbol{d} = \boldsymbol{A}\,\boldsymbol{1}$

Markov chain

$$\boldsymbol{p}_{t+1} = c\,\boldsymbol{M}\,\boldsymbol{p}_t + (1\text{-}c)\,\boldsymbol{q}$$
$$\boldsymbol{p}_0 = \boldsymbol{q}$$

$$\boldsymbol{M}_1 = c\,\boldsymbol{M} + (1\text{-}c)\,\boldsymbol{q}\,\boldsymbol{v}^T$$
$$\boldsymbol{v}^T\boldsymbol{M} = \boldsymbol{v}^T$$
$$\boldsymbol{v}^T\boldsymbol{q} = 1$$

same properties of column-normalized PageRank:

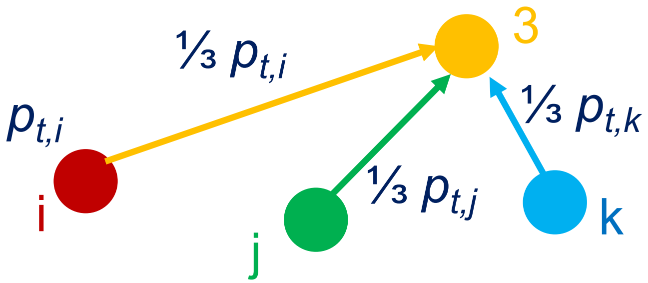
- $\boldsymbol{M}_1$ has one eigenvalue equal to 1
- The remaining eigenvalues satisfy $|\lambda| \le c$

# Row-normalized PageRank

## interpreting its action

A node gathers the average value of the neighbour nodes pointing to it

$$p_{t+1,3} = c \, \tfrac{1}{3} \, ( p_{t,i} + p_{t,j} + p_{t,k} ) + (1\text{-}c) \, q_3$$




It is a way of spreading the original information **q** over the network

UNIVERSITÀ
DEGLI STUDI
DI PADOVA

# Semantic network example

agency = action and goal orientation, sense of which is necessary for people to attempt social change

## q values of agency (in colour)


## r values after spreading

# Takeaways
## for PageRank centrality

- <u>This is the metric to be used</u> it for resizing nodes according to their importance
- Provides elaborate information on the relevance of nodes in the network
- For directed networks, it can be used in both its authority and hub forms
- Can also be put in the form of a PageRank distribution
- Can be used in different useful ways, e.g., to evaluate similarity or closeness, to spread information
- Exploit its potential at your best

# HITS centrality

a (less interesting) alternative to PageRank

## HITS – hubs and authorities

Kleinberg, J.M.
1999
«Authoritative sources in a hyperlinked environment»
*Journal of the ACM*
https://www.cs.cornell.edu/home/kleinber/auth.pdf

Conceptually similar to PageRank

Provides scores for authorities and hubs, separately, as PageRank can do

We deprecate its use

# HITS equations

## authorities score

$A_{2,4}$ = weight of connection 4 → 2

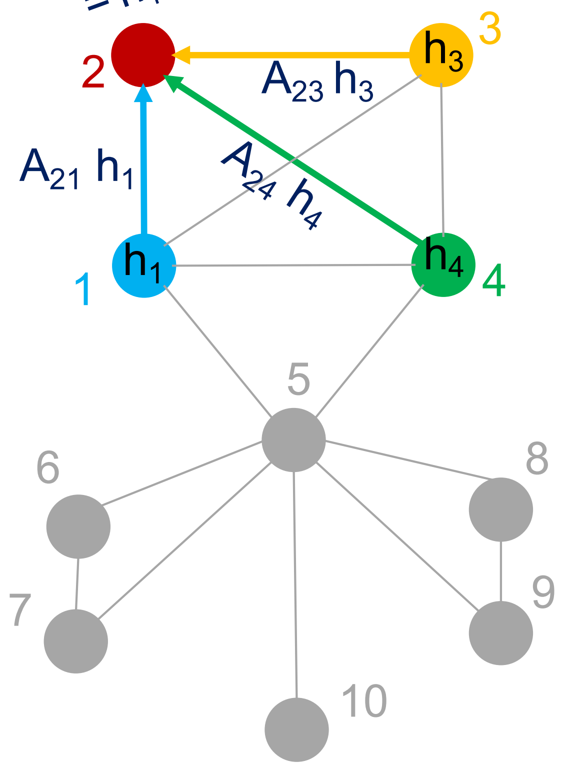
$$
\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\cdot
\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{bmatrix}
$$

$$\boldsymbol{a} = \boldsymbol{A}\boldsymbol{h}$$

authority scores

hub scores

$$a_2 = A_{21}\, h_1 + A_{23}\, h_3 + A_{24}\, h_4$$
$$= h_1 + h_3 + h_4$$



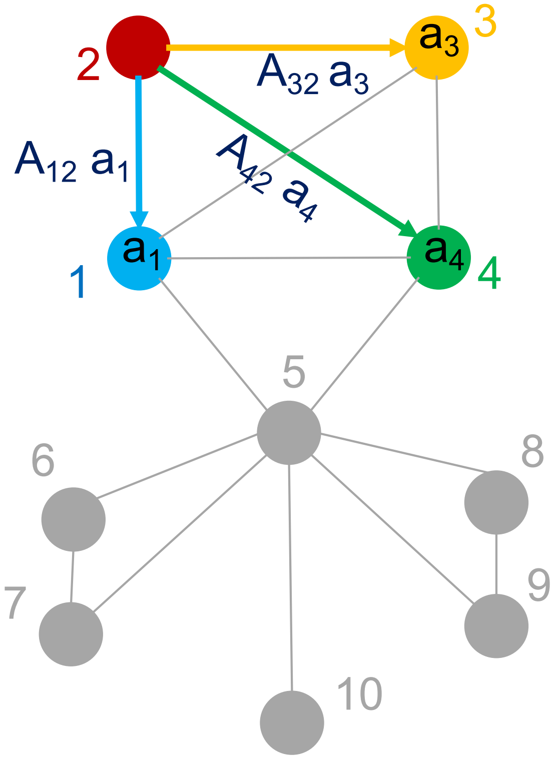
# HITS equations

## hubs score

$A_{3,2}$ = weight of connection 2 → 3

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \\ h_{10} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix}$$

$$h_2 = A_{12}\, a_1 + A_{32}\, a_3 + A_{42}\, a_4$$
$$= a_1 + a_3 + a_4$$




$$\boldsymbol{h} = \boldsymbol{A}^T \boldsymbol{a}$$

# HITS equations

## hubs and authorities

$$\boldsymbol{a} = c_a \cdot \boldsymbol{A}\boldsymbol{h}$$

$$\boldsymbol{h} = c_h \cdot \boldsymbol{A}^T\boldsymbol{a}$$

hubs

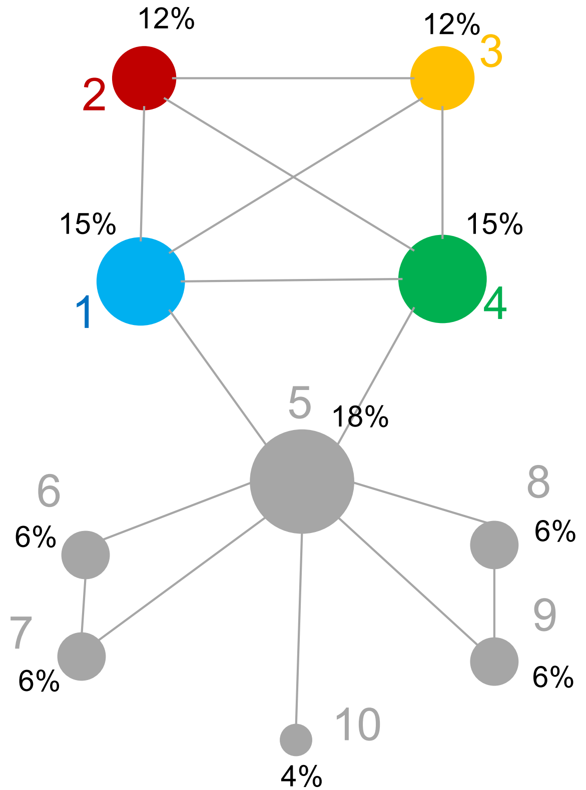
$$\boxed{\boldsymbol{h} = c\,\boldsymbol{M}\boldsymbol{h}}$$

$$\boldsymbol{M} = \boldsymbol{A}^T\boldsymbol{A}$$

$$c = c_a c_h$$

authorities

$$\boxed{\boldsymbol{a} = c_a \cdot \boldsymbol{A}\boldsymbol{h}}$$




- The formula says we are interested in the (principal) eigenvector of matrix $M = A^T \cdot A$
- Can be obtained by standard linear algebra algorithms

# Power iteration method

## for HITS

0. Start from an initial guess $\boxed{\boldsymbol{a}_0}$

1. Let the time go by $\boxed{\boldsymbol{a}_{t+1} = \boldsymbol{M}\,\boldsymbol{a}_t}$

product by a sparse matrix (twice) $\boldsymbol{M} = \boldsymbol{A}\,\boldsymbol{A}^T$

2. Keep normalizing (divide $\boldsymbol{a}_{t+1}$ by the sum of elements)

3. Stop when $\boldsymbol{a}$ converges (few iterations?)

# Convergence properties

## for HITS

- $\|\boldsymbol{a}_t - \boldsymbol{a}_\infty\|_2 \leq \sqrt{N} \cdot (\lambda_2/\lambda_1)^t$
- $\lambda_1$ largest eigenvalue of $\boldsymbol{M}$
- $\lambda_2$ second largest eigenvalue of $\boldsymbol{M}$
- Triang. inequality $\|\boldsymbol{a}_t - \boldsymbol{a}_{t+1}\|_2 \leq 2\sqrt{N} \cdot (\lambda_2/\lambda_1)^t$

Worst case result:

N = $10^9$ → 10.3

slow if $\lambda_2$ close to $\lambda_1$

- Precision $\varepsilon$ implies: $\|\boldsymbol{a}_t - \boldsymbol{a}_{t+1}\|_2 < \varepsilon$
- Iterations required: $\boxed{t = [\ln(2/\varepsilon) + ½\ln(N)] / \ln(\lambda_1/\lambda_2)}$

$10^{-3}$ precision → 7.6

# Eigenvector and Kats centralities

other (less interesting) alternatives to PageRank

# Eigenvector and Kats centralities

an overview

| | with constant term | without constant term |
|---|---|---|
| normalized | PageRank<br>$\mathbf{r} = c\,\mathbf{M}\,\mathbf{r} + (1-c)\,\mathbf{q}$ | Degree<br>$\mathbf{r} = \mathbf{M}\,\mathbf{r}$ |
| unnormalized | Katz<br>$\mathbf{r} = c\,\mathbf{A}\,\mathbf{r} + \mathbf{1}$ | Eigenvector<br>$\mathbf{r} = c\,\mathbf{A}\,\mathbf{r}$ |

$$\mathbf{r} = (\mathbf{I} - c\,\mathbf{A})^{-1}\,\mathbf{1}$$
$$= \Sigma\,(c\,\mathbf{A})^k\,\mathbf{1}$$

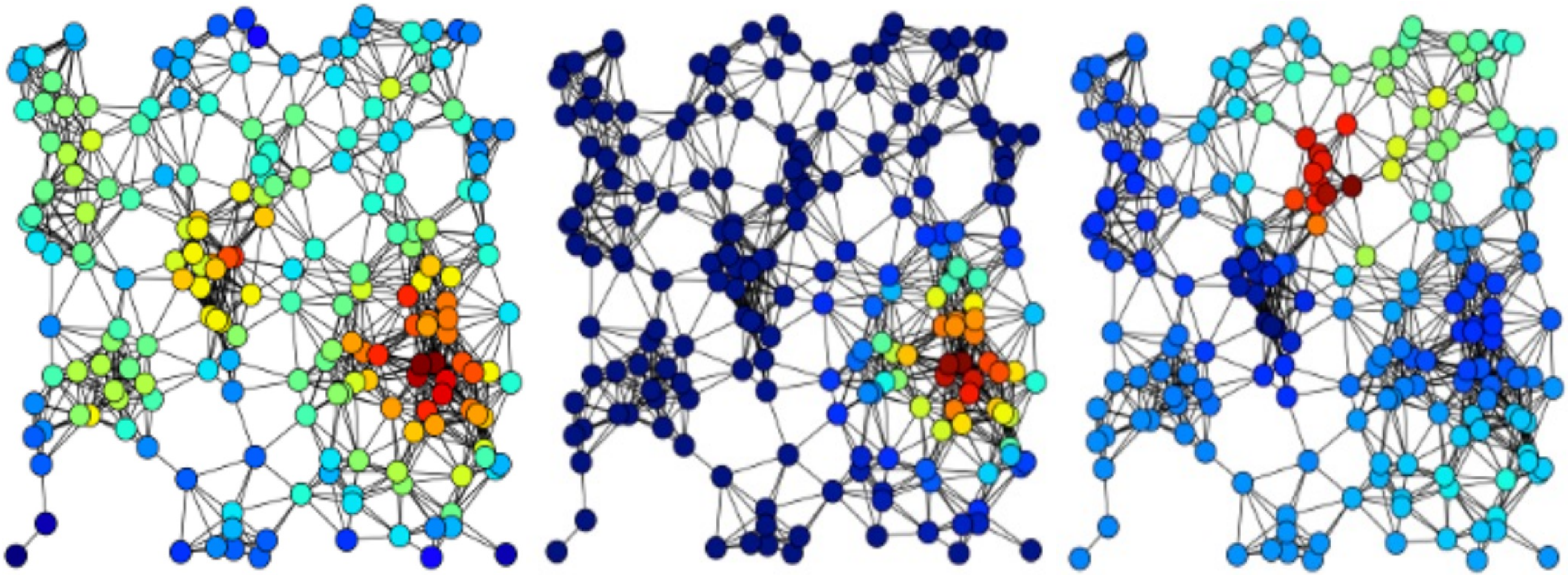
The absence of normalization makes them less robust and meaningful compared to PageRank

They are deprecated

# Eigenvector and Kats centralities

## their graphical interpretation

Degree Eigenvector Katz



# Closeness and Harmonic centralities

importance of nodes as spreaders of information

# Closeness centrality
a definition

## Closeness centrality

From Wikipedia, the free encyclopedia

In a connected graph, **closeness centrality** (or **closeness**) of a node is a measure of centrality in a network, calculated as the reciprocal of the sum of the length of the shortest paths between the node and all other nodes in the graph. Thus, the more central a node is, the *closer* it is to all other nodes.

Closeness was defined by Bavelas (1950) as the reciprocal of the **farness**,[1][2] that is:

$$C(x) = \frac{1}{\sum_y d(y,x)}.$$

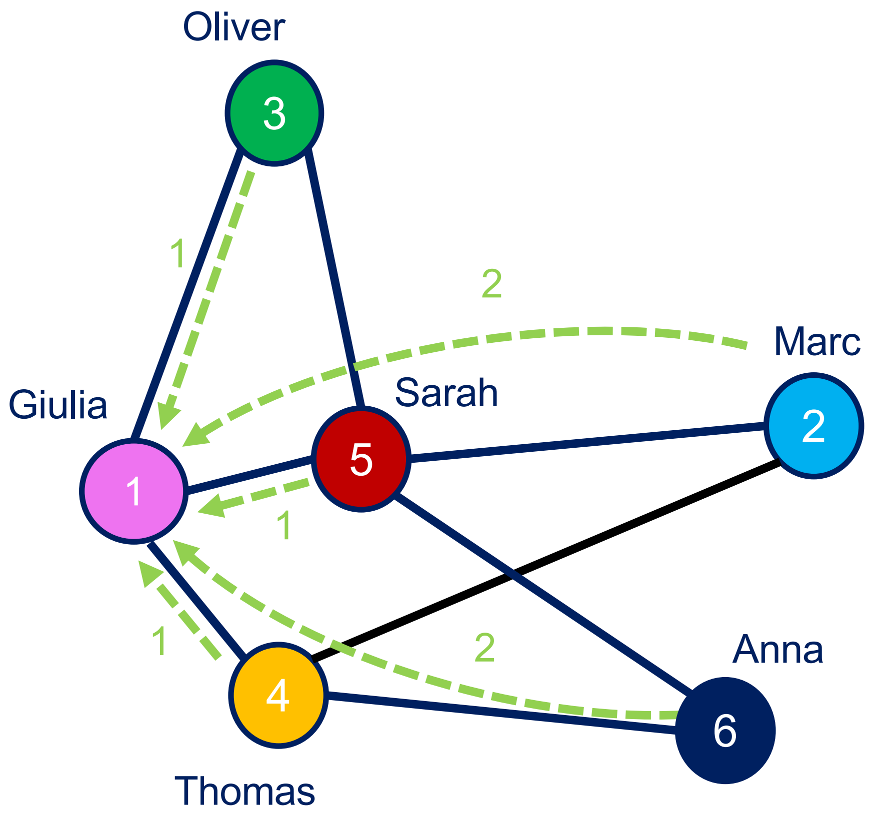
where $d(y,x)$ is the distance between vertices $x$ and $y$. When

Rationale: the node which is the easiest to reach, the one which is the best for spreading information

# An example

## on how to calculate closeness centrality

count the lengths of the shortest paths
leading to Giulia
1 + 2 + 1 + 2 + 1 = 7




Closeness

| | |
|---|---|
| 0.1429 | Giulia |
| 0.1250 | Marc |
| 0.1250 | Oliver |
| 0.1429 | Thomas |
| 0.1667 | Sarah |
| 0.1250 | Anna |

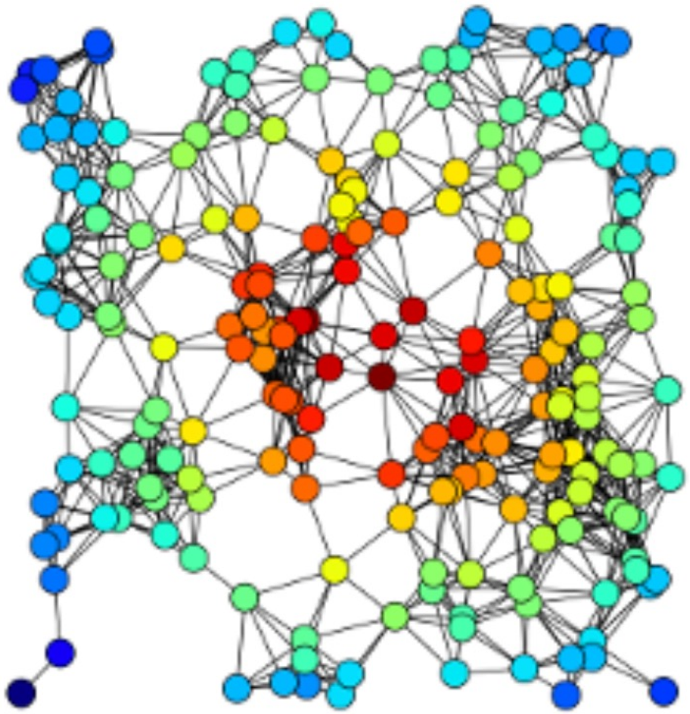
Sarah is the preferred node for spreading information

C(Giulia) = 1/7
= 0.1429

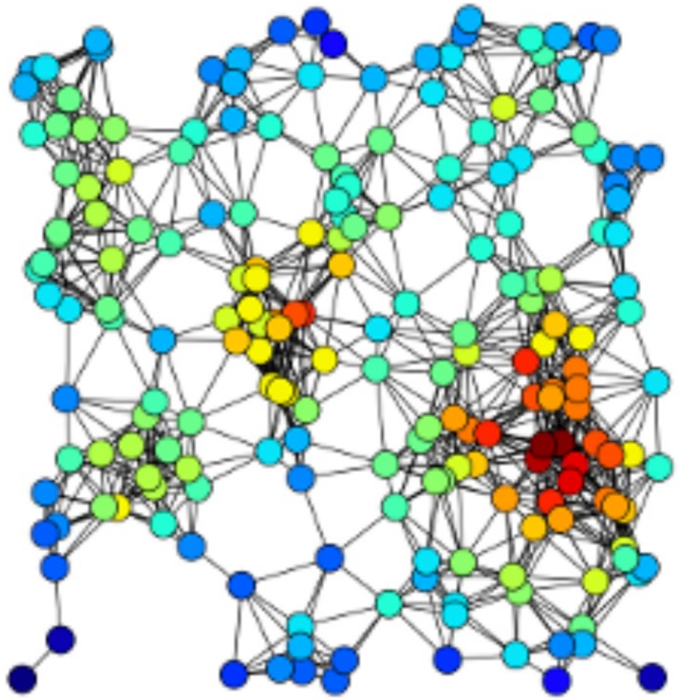
# Closeness versus degree centrality

## a graphical interpretation

Closeness



Degree



# Harmonic centrality

## a definition

### In disconnected graphs [ edit ]

When a graph is not strongly connected, a widespread idea is that of using the sum of reciprocal of distances, instead of the reciprocal of the sum of distances, with the convention $1/\infty = 0$:

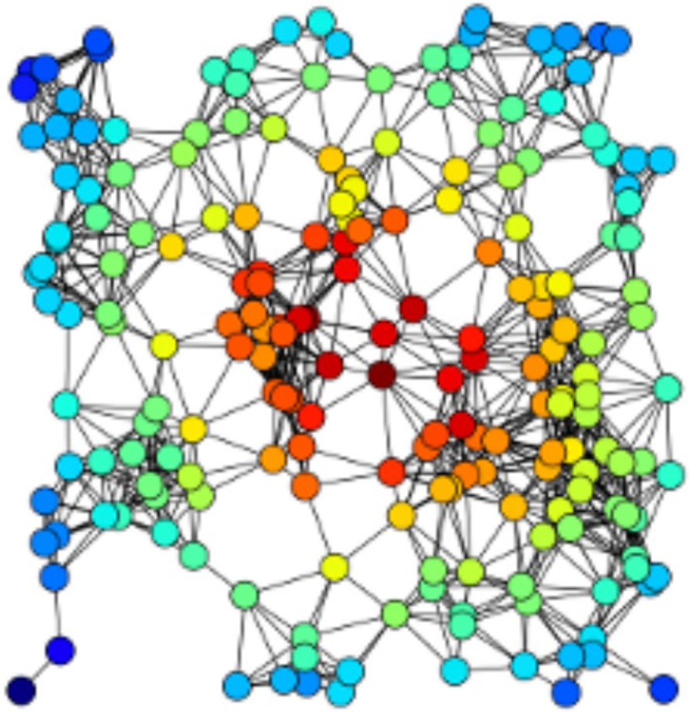
$$H(x) = \sum_{y \neq x} \frac{1}{d(y, x)}.$$

The most natural modification of Bavelas's definition of closeness is following the general principle proposed by Marchiori and Latora (2000)[3] that in graphs with infinite distances the harmonic mean behaves better than the arithmetic mean. Indeed, Bavelas's closeness can be described as the denormalized reciprocal of the arithmetic mean of distances, whereas harmonic centrality is the denormalized reciprocal of the harmonic mean of distances.

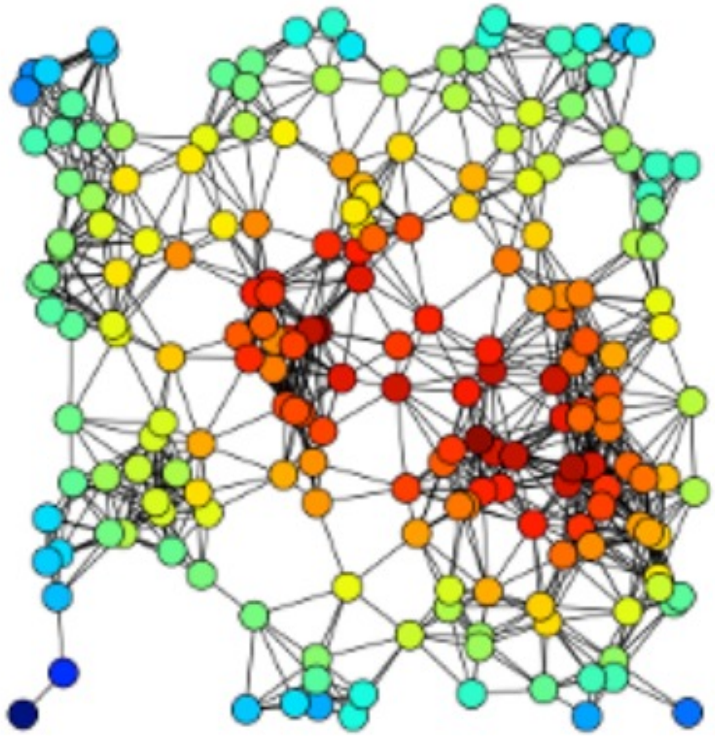
# Closeness versus harmonic centrality

a graphical interpretation

Closeness



Harmonic



# Betweenness centrality

importance of nodes as bridges or brokers

# Betweenness centrality
a definition

## Betweenness centrality

From Wikipedia, the free encyclopedia

In graph theory, **betweenness centrality** is a measure of centrality in a graph based on shortest paths. For every pair of vertices in a connected graph, there exists at least one shortest path between the vertices such that either the number of edges that the path passes through (for unweighted graphs) or the sum of the weights of the edges (for weighted graphs) is minimized. The betweenness centrality for each vertex is the number of these shortest paths that pass through the vertex.

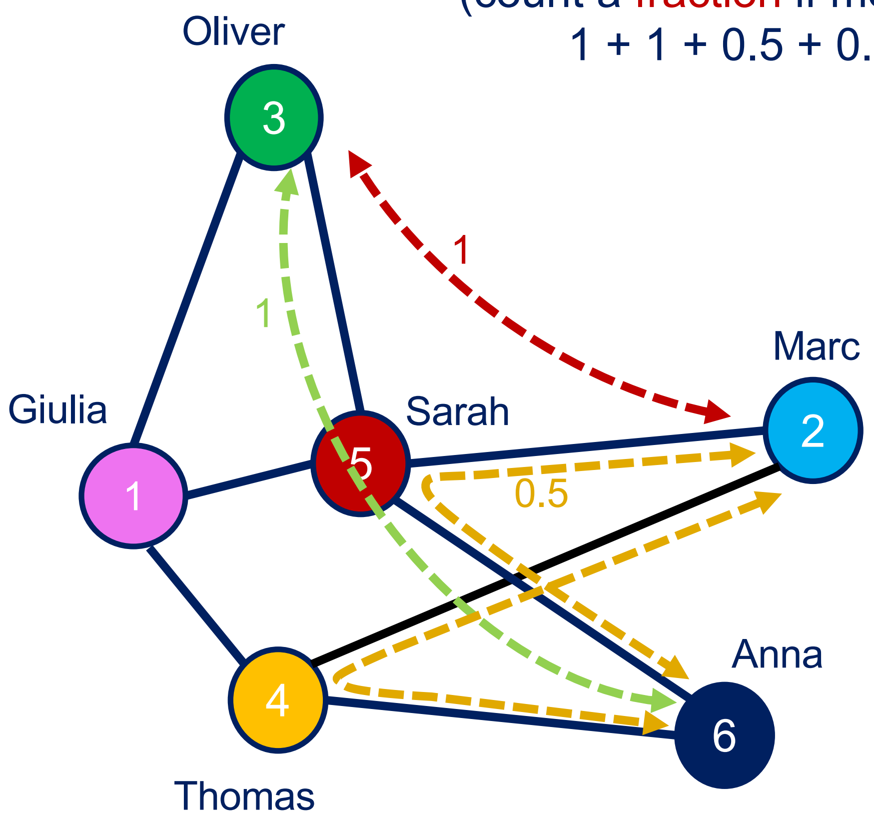
Betweenness centrality was devised as a general measure of centrality:[1] it applies to a wide range of problems in network theory, including problems related to social networks, biology, transport and scientific cooperation. Although earlier authors have intuitively described centrality as based on betweenness, Freeman (1977) gave the first formal definition of betweenness centrality.

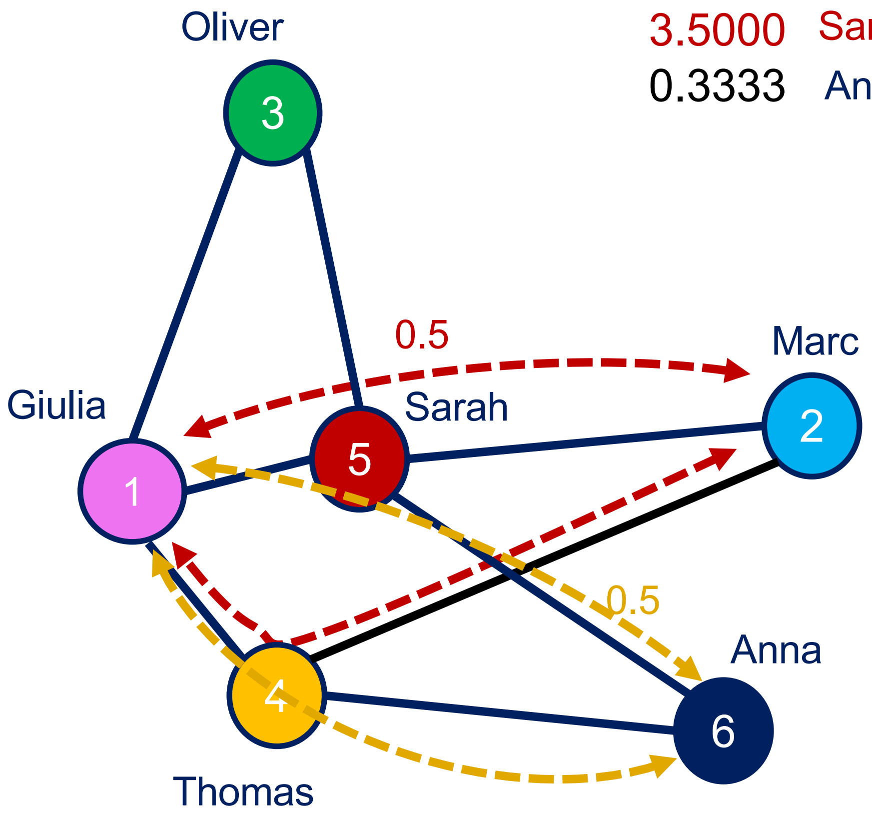
Rationale: the node which takes you elsewhere (bridge, broker)

# An example

## on how to calculate betweenness centrality

count the # of shortest paths
passing through Sarah
(count a fraction if more than one path)
1 + 1 + 0.5 + 0.5 + 0.5 = 3.5




**Betweenness**

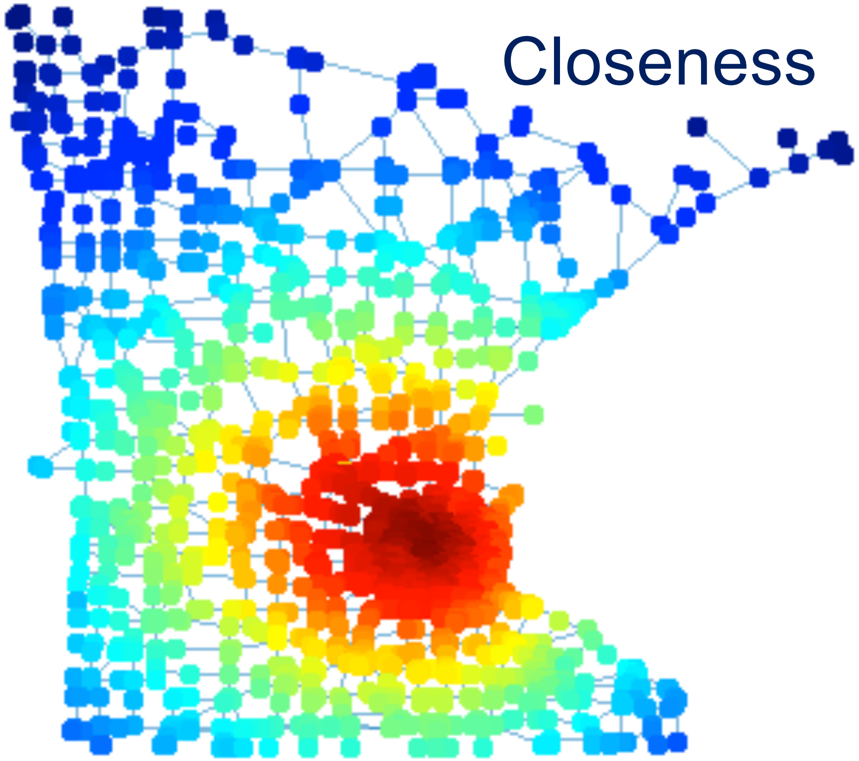
| | |
|---|---|
| 1.3333 | Giulia |
| 0.3333 | Marc |
| 0 | Oliver |
| 1.5000 | Thomas |
| 3.5000 | Sarah |
| 0.3333 | Anna |

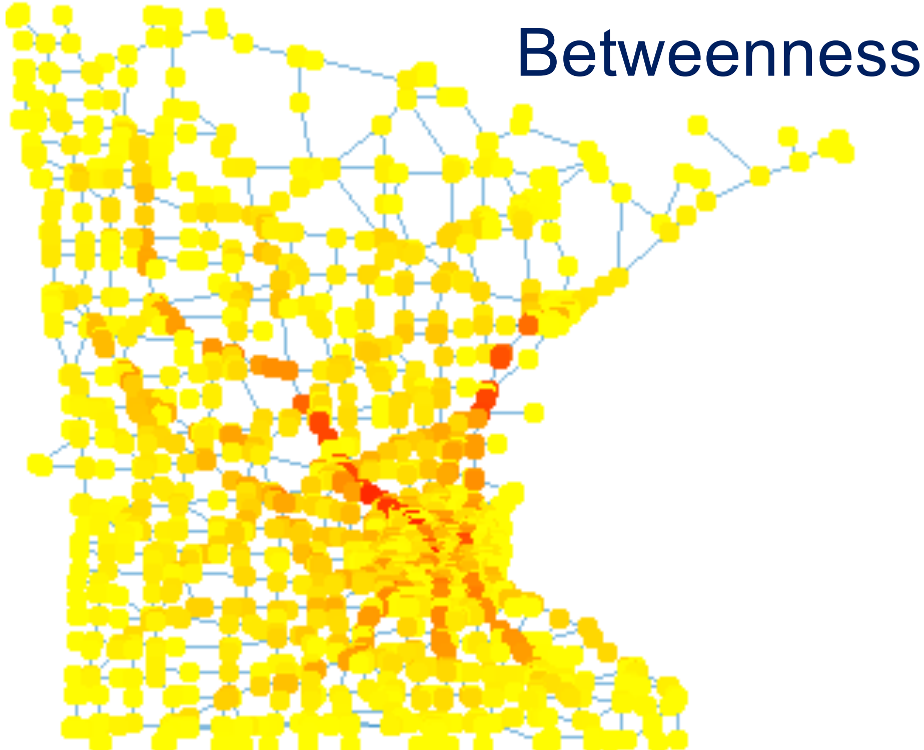
UNIVERSITÀ DEGLI STUDI DI PADOVA

# Closeness vs betweenness centrality

## a graphical interpretation

Minnesota road network

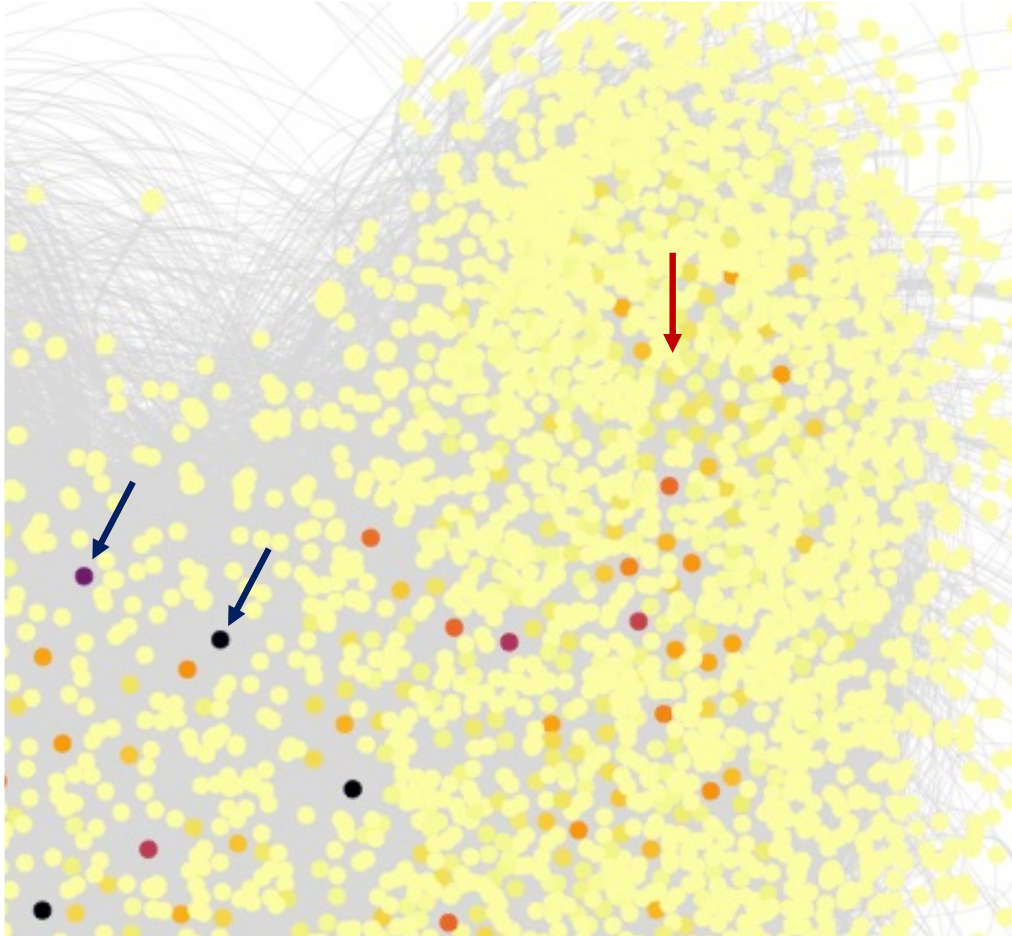

Closeness is a measure of center of gravity (best node to spread info)

Betweenness is a measure of brokerage (i.e., being a bridge)

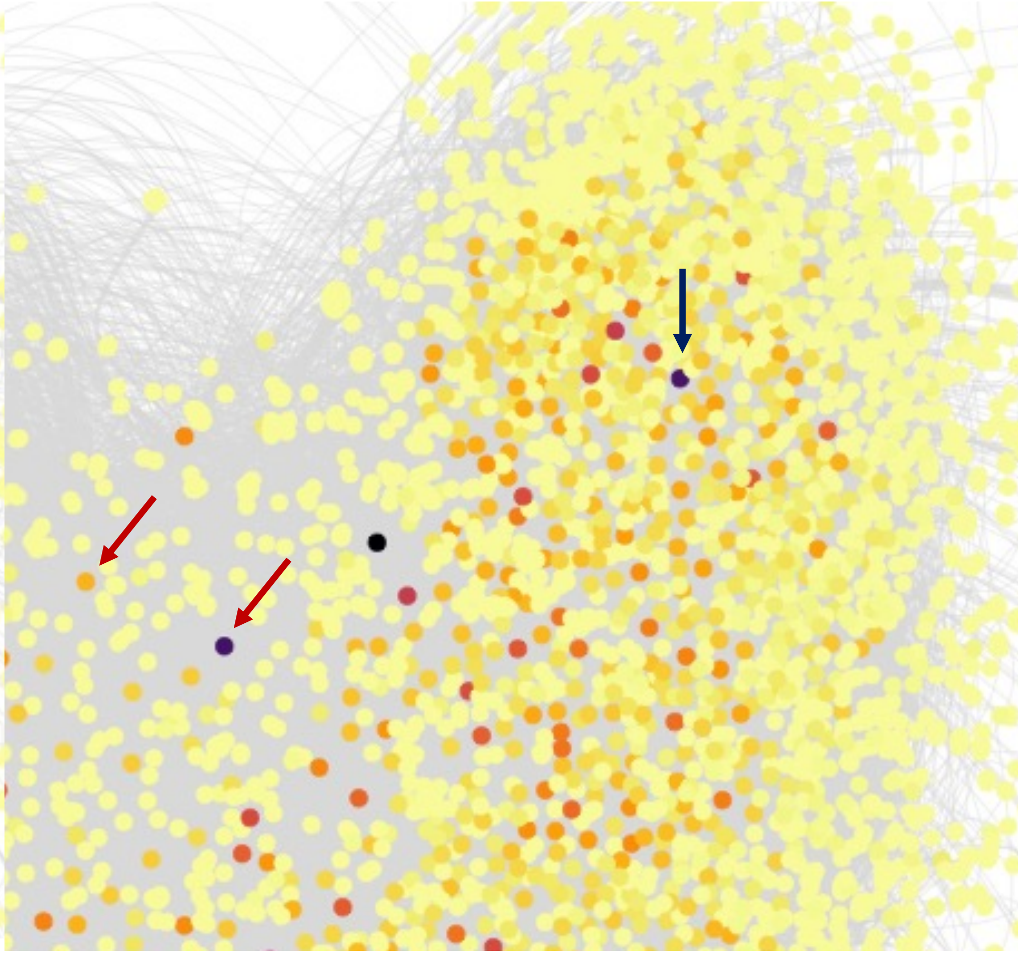
# Betweenness vs PageRank centrality

wiki vote network

Betweenness

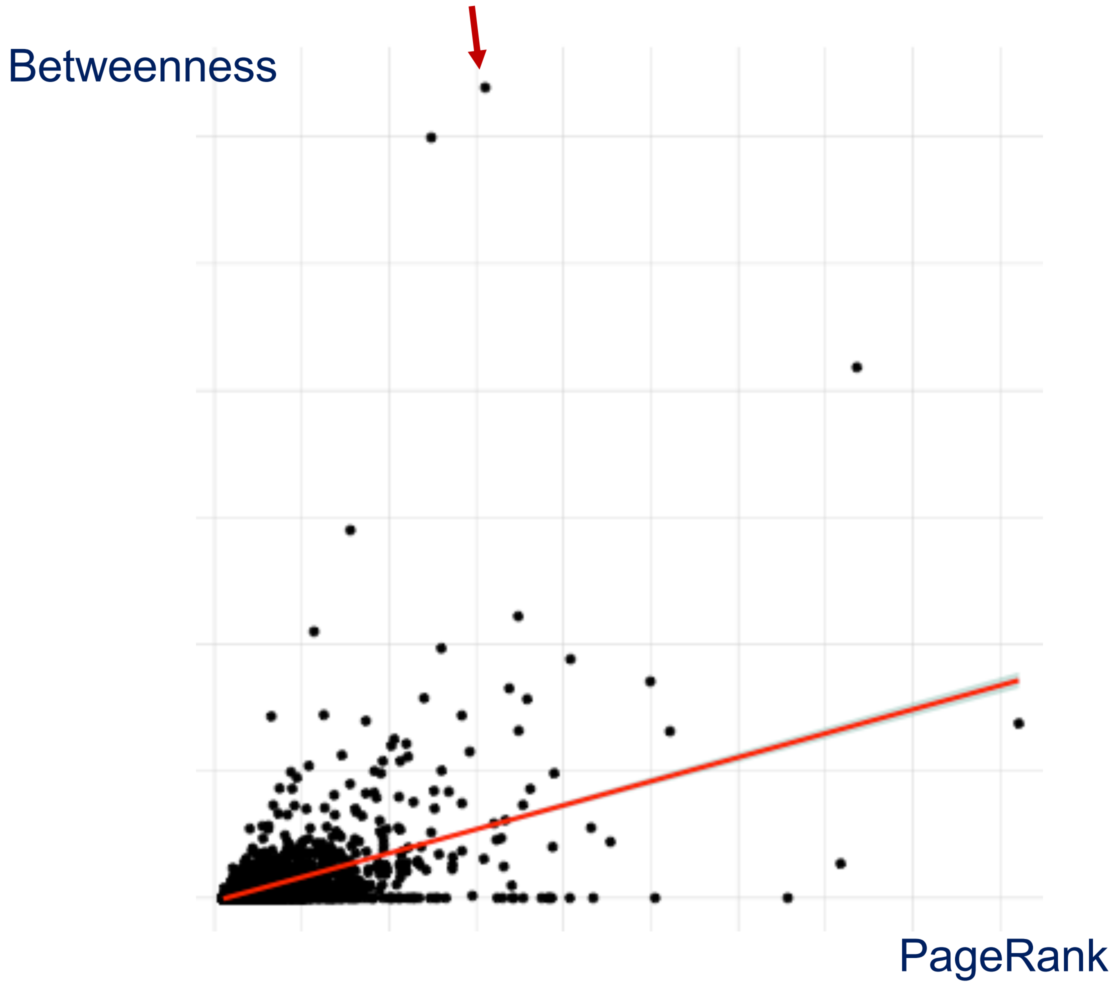


PageRank



# Betweenness vs PageRank centrality

## a correlation view



# Clustering coefficient

how tightly linked is the network locally

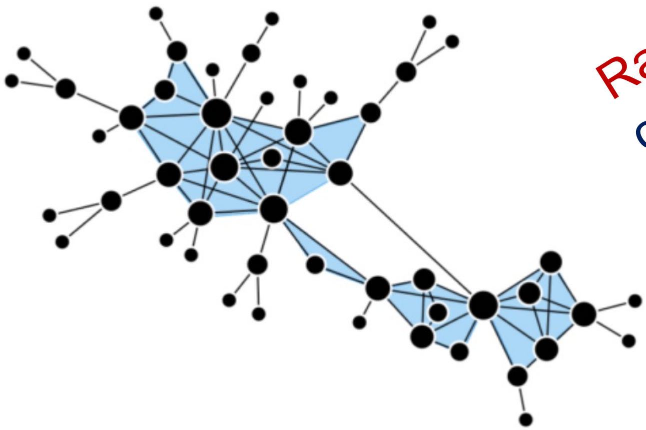
# Clustering coefficient

a definition

## Local clustering coefficient [ edit ]

The **local clustering coefficient** of a vertex (node) in a graph quantifies how close its neighbours are to being a clique (complete graph). Duncan J. Watts and Steven Strogatz introduced the measure in 1998 to determine whether a graph is a small-world network.

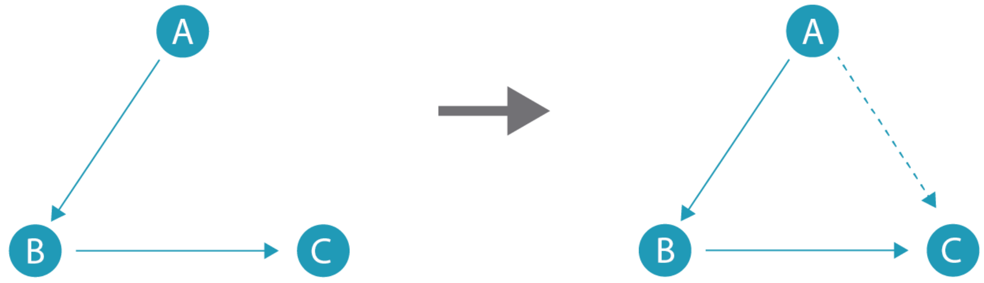
Rationale: how strongly connected is the network locally / general indication of the graph's tendency to be organized into clusters



# Triadic closure
in social networks

Forbidden triad

Triadic closure
(A and C are likely to be friends)



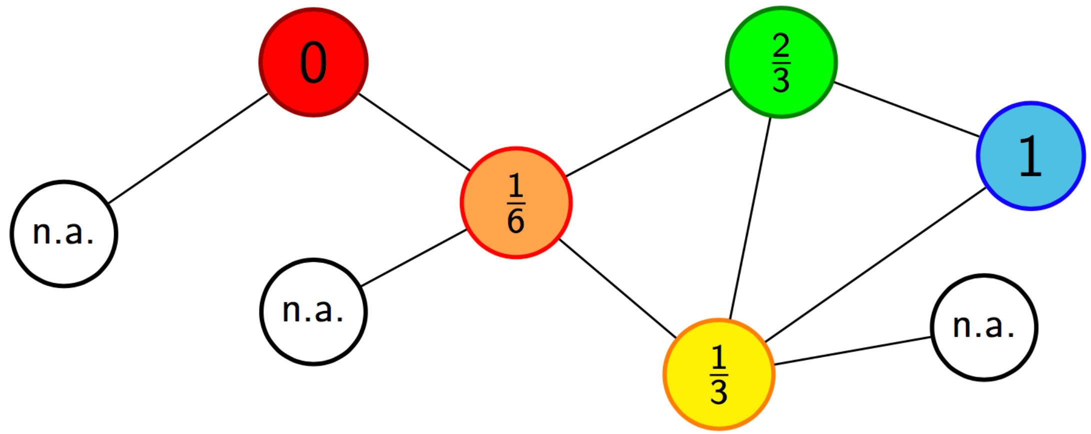

## Triadic closure

- A and C are likely to have the opportunity to meet because they have a common friend B
- The fact that A and C is friends with B gives them the basis of trusting each other
- B may have the incentive to bring A and C together, as it may be hard for B to maintain disjoint relationships

Granovetter, The strength of weak ties [1973]
https://www.jstor.org/stable/pdf/2776392.pdf

# Local clustering coefficient

## a measure of triadic closures




Local Clustering coefficient $C_i$ counts the fraction of pairs of neighbours $N_i$ which form a triadic closure with node $i$

$$C_i = \frac{1}{|\mathcal{N}_i|(|\mathcal{N}_i|-1)} \sum_{\substack{(j,k)\in\mathcal{N}_i^2 \\ j\neq k}} \mathrm{tc}_{i,j,k}$$

equal to diag($\mathbf{A}^3$)

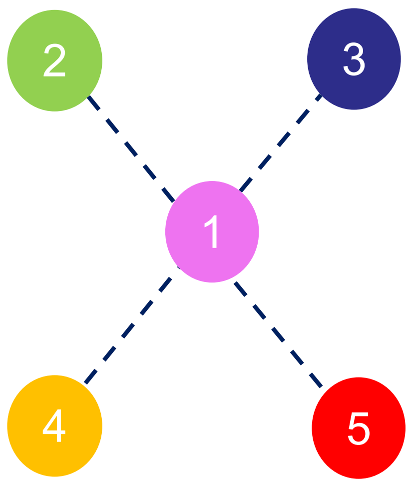
where $tc_{ijk} = 1$ if the triplet $(i,j,k)$ forms a triadic closure, and zero otherwise

# Local clustering coefficient

## examples

not connected neighbourhood

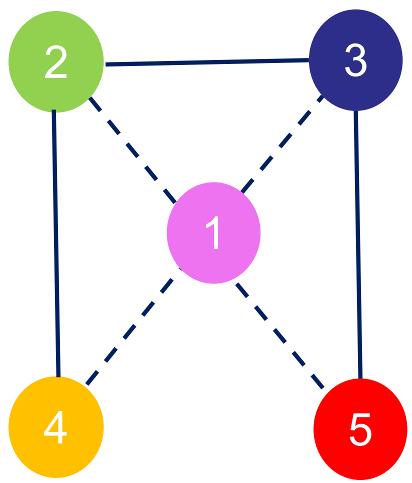
$<C> = 0$




$C_1 = 0$

weakly connected neighbourhood

$<C> = 0.766$



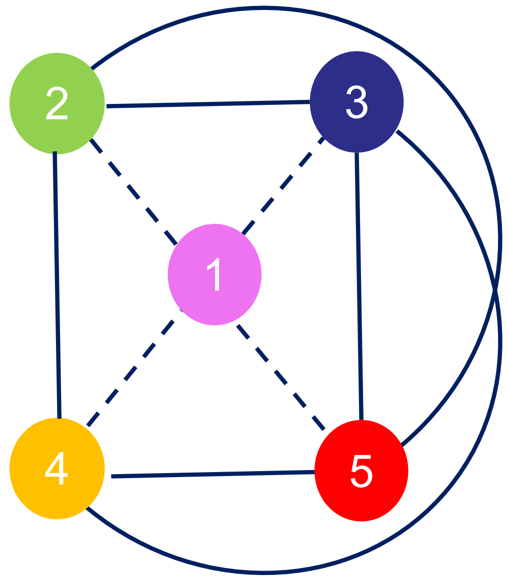

$C_1 = ½ = 3 / (4x3/2)$

$C_2 = C_3 = ⅔$

$C_4 = C_5 = 1$

strongly connected neighbourhood

$<C> = 1$



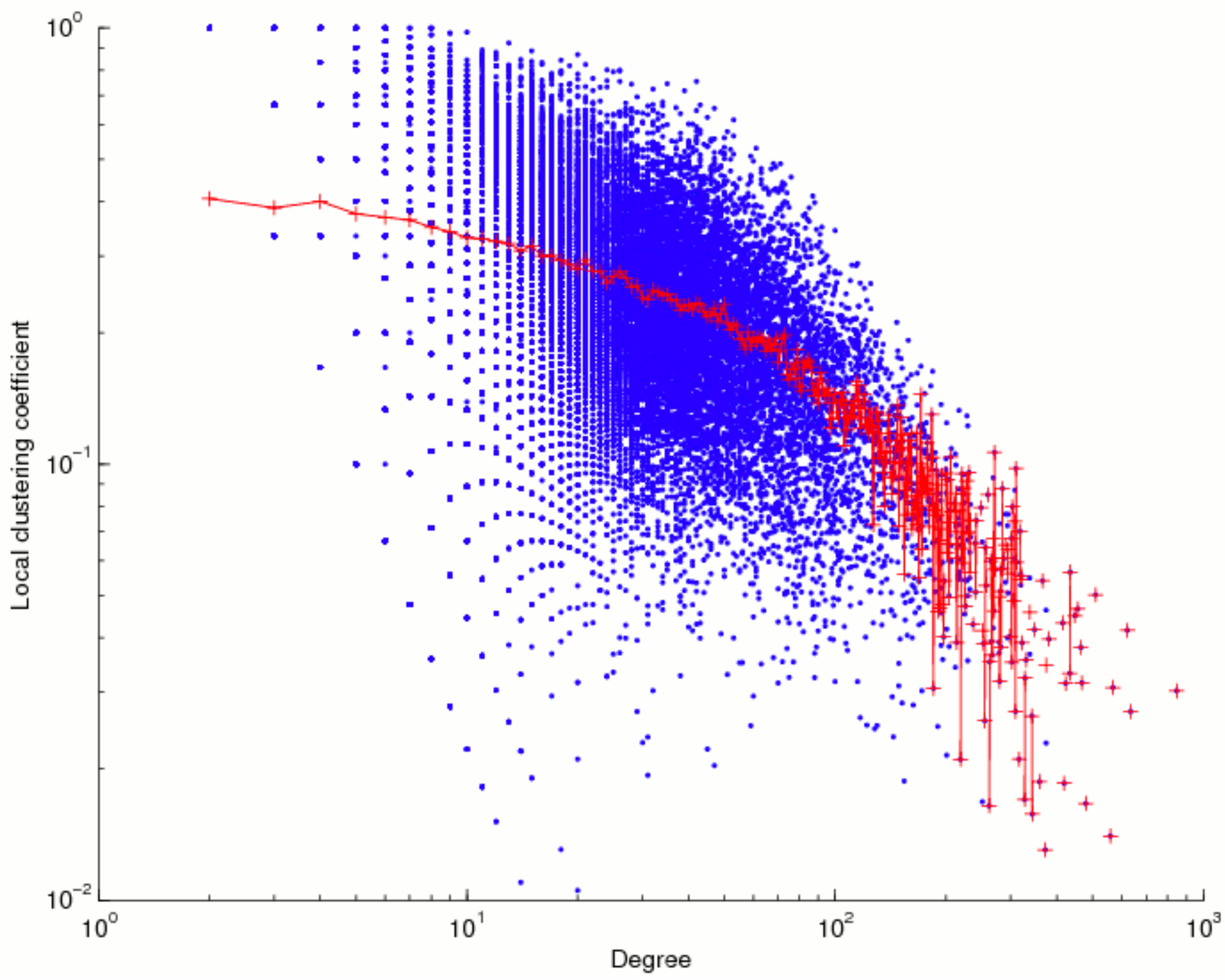

$C_1 = 1 = 6 / (4x3/2)$

Università degli Studi di Padova

# Clustering coeff. vs degree centrality

## a correlation view

citation network from arXiv’s High Energy Physics / Phenomenology section

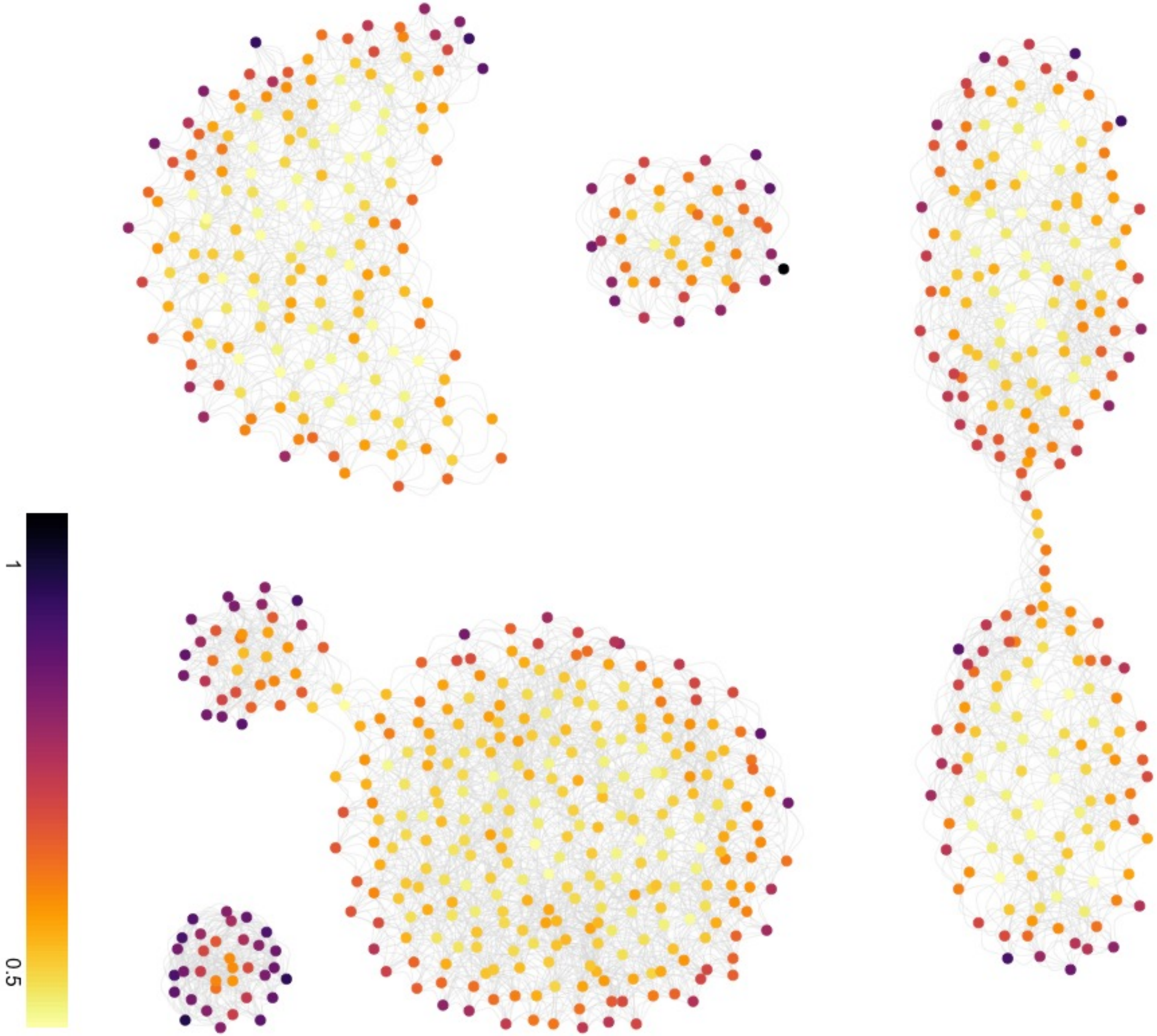



when person has many friends, these friends have less edges among them, which is to be expected since a person with many friends is likely to have friends from more diverse communities, and a paper getting cited many times is likely to be cited by papers from more diverse areas

# Warning

But clustering coefficient is generally hard to see and visual interpretation is considered unreliable

# Visual example



# Takeaways

## for Closeness, Betwenness and Clustering coefficient

- Closeness, betweenness and clustering coefficient are alternative centrality measures that have a different view wrt PageRank
- They provide useful insights especially in social networks, as they are linked to sociology concepts
- Closeness and betweenness are based on distances, that require algorithms that are less scalable than PageRank
- Exploit their potential at your best

# Wrap-up

on centrality measures

# Takeaways
## on centrality measures

| Centrality measure | Technical property | Meaning |
|---|---|---|
| Degree (in/out) | Measures number (and quality) of direct connections | Cohesion<br>Entrepreneurship |
| Attractiveness | Measures the speed of growing of a node's degree | Dinamicity<br>Enterprising |
| PageRank (authorities/hubs) | Measures number (and quality) of direct and indirect connections | Cohesion<br>Entrepreneurship<br>Similarity/Friendship with a direction → Dependence |
| Closeness | Measures length of shortest paths | Visual centrality<br>Significant spreading points<br>Outliers/Ostracism |
| Betweenness | Measures number of shortest paths | Brokerage<br>Structural holes |
| Clustering coeff. | Measures number of triadic closures | Centrality in a community<br>Cohesion of the neighbourhood |

UNIVERSITÀ DEGLI STUDI DI PADOVA

# More on the meaning

https://reticular.hypotheses.org/1745