Calculus 2

Answers to LN Exercises

1.8.1. By proving the triangular inequality for the Euclidean norm, we got the formula

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\sum_j x_j y_j.$$

From this,

$$\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|-\vec{y}\|^2 + 2\sum_j x_j(-y_j) = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\sum_j x_j y_j.$$

Therefore,

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\left(\|\vec{x}\|^2 + \|\vec{y}\|^2\right). \quad \square$$

1.8.2. We start with $\|\cdot\|_1$. We have to verify positivity, vanishing, homogeneity and the triangular inequality.

i) Positivity: $\|\vec{x}\|_1 = \sum_j |x_j| \geqslant 0$ because it is a sum of positive numbers.

ii) Vanishing: $\|\vec{x}\|_1 = 0$ iff $\sum_j |x_j| = 0$. Since this is the sum of positive numbers it can be $= 0$ iff $|x_j| = 0$ for every $j = 1, \dots, d$, that is iff $x_j = 0\ \forall j$, so iff $\vec{x} = \vec{0}$.

iii) Homogeneity: we have

$$\|\lambda\vec{x}\|_1 = \sum_j |\lambda x_j| = \sum_j |\lambda||x_j| = |\lambda|\sum_j |x_j| = |\lambda|\|\vec{x}\|_1.$$

(notice we used homogeneity of the modulus, that is $|ab| = |a||b|$)

iv) Triangular inequality: we have

$$\|\vec{x}+\vec{y}\|_1 = \sum_j |x_j + y_j|.$$

Now, because of the triangular inequality for the modulus, $|a+b| \leqslant |a| + |b|$ we have

$$|x_j + y_j| \leqslant |x_j| + |y_j|,\ \forall j.$$

So, summing up on j, we get

$$\|\vec{x}+\vec{y}\|_1 = \sum_j |x_j + y_j| \leqslant \sum_j \left(|x_j| + |y_j|\right) = \sum_j |x_j| + \sum_j |y_j| = \|\vec{x}\|_1 + \|\vec{y}\|_1.$$

Let now discuss the case of $\|\cdot\|_\infty$. We have to verify the same properties as done above.

i) Positivity: $\|\vec{x}\|_\infty = \max_j |x_j \geqslant 0$ (maximum of positive numbers is positive).

ii) Vanishing: $\|\vec{x}\|_\infty = 0$ iff $\max_j |x_j| = 0$. Since the maximum is made on positive numbers, it is clear that it can be $= 0$ iff $|x_j| = 0\ \forall j$, that is $x_j = 0\ \forall j$, so iff $\vec{x} = \vec{0}$.

iii) Homogeneity: we have

$$\|\lambda\vec{x}\|_\infty = \max_j |\lambda x_j| = \max_j |\lambda||x_j|.$$

Now, it is clear that $\max_j c|x_j| = c\max_j |x_j|$ for every $c \geqslant 0$. So $\|\lambda\vec{x}\|_\infty = |\lambda|\|\vec{x}\|_\infty$.

iv) Triangular inequality: we have

$$\|\vec{x}+\vec{y}\|_\infty = \max_j |x_j + y_j|.$$

Now, since

$$|x_j + y_j| \leqslant |x_j| + |y_j| \leqslant \max_k |x_k| + \max_k |y_k| = \|\vec{x}\|_\infty + \|\vec{y}\|_\infty,\ \forall j,$$

we have

$$\|\vec{x}+\vec{y}\|_\infty = \max_j |x_j + y_j| \leqslant \|\vec{x}\|_\infty + \|\vec{y}\|_\infty. \quad \square$$

1.8.3. i) Let $\vec{x}_n := (e^{-n}, 1)$. Since $e^{-n} \longrightarrow 0$ and $1 \longrightarrow 1$ we conclude that $\vec{x}_n \longrightarrow (0,1)$.
ii) Let $\vec{x}_n := \left(n, n^2\right)$. Here we notice that $\|\vec{x}_n\| = \sqrt{n^2+n^4} \longrightarrow +\infty$, so $\vec{x}_n \longrightarrow \infty_2$.
iii) Let $\vec{x}_n := \left(\frac{1}{n}, \frac{1}{n^2}, \sin\frac{1}{n}\right)$. Since $\frac{1}{n} \longrightarrow 0$, $\frac{1}{n^2} \longrightarrow 0$ and $\sin\frac{1}{n} \longrightarrow 0$, we conclude that $\vec{x}_n \longrightarrow (0,0,0)$.
iv) Let $\vec{x}_n := \left(1, 1+\frac{1}{n}, n\right)$. Here, $\|\vec{x}_n\| = \sqrt{1+\left(1+\frac{1}{n}\right)^2 + n^4} \geqslant \sqrt{n^4} = n^2 \longrightarrow +\infty$, so $\vec{x}_n \longrightarrow \infty_3$.
v) Let $\vec{x}_n := \left(\tanh n, \frac{\log n}{n}, \frac{\sin n}{n}\right)$. Notice that $\tanh n = \frac{\sinh n}{\cosh n} \sim_{n\to+\infty} \frac{e^n/2}{e^n/2} = 1$, so $\tanh n \longrightarrow 1$. Moreover, since $\log n =_{+\infty} o(n)$, $\frac{\log n}{n} \longrightarrow 0$ and clearly $\frac{\sin n}{n} \longrightarrow 0$. Therefore, $\vec{x}_n \longrightarrow (1,0,0)$.
vi) Let $\vec{x}_n := \left((-1)^n, (-1)^{n+1}\right)$. Here we have that $\vec{x}_{2k} \equiv (1,-1) \longrightarrow (1,-1)$ while $x_{2k+1} \equiv (-1,1) \longrightarrow (-1,1)$ so we conclude that there is no limit for $\vec{x}_n$ when $n \to +\infty$. $\square$

1.8.4. $\Longrightarrow$. Assumption: $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$. Thesis: $x_{n,k} \longrightarrow \ell_k$ for every $k = 1, \dots, d$. By definition

$$\forall \varepsilon > 0,\ \exists N\ :\ \|\vec{x}_n - \vec{\ell}\| \leqslant \varepsilon,\ \forall n \geqslant N.$$

Now,

$$\|\vec{x}_n - \vec{\ell}\| = \sqrt{\sum_{k=1}^{d} (x_{n,k} - \ell_k)^2} \geqslant \sqrt{(x_{n,k} - \ell_k)^2} = |x_{n,k} - \ell_k|,\ \forall k = 1, \dots, d.$$

Therefore,

$$|x_{n,k} - \ell_k| \leqslant \|\vec{x}_n - \vec{\ell}\| \leqslant \varepsilon,\ \forall n \geqslant N,\ \forall k = 1, \dots, d.$$

and this means that $x_{n,k} \longrightarrow \ell_k$ for every $k = 1, \dots, d$.

$\Longleftarrow$ Now the Assumption is: $x_{n,k} \longrightarrow \ell_k$ for every $k = 1, \dots, d$. The thesis is: $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$. From the assumption we can say that,

$$\forall k = 1, \dots, d,\ \forall \varepsilon > 0,\ \exists N_k\ :\ \|\vec{x}_{n_k} - \vec{\ell}_k\| \leqslant \varepsilon,\ \forall n \geqslant N_k.$$

Notice we wrote N_k because the initial N will depend on the sequence $(x_{n,k})$, so on k. Now, let $N := \max(N_1, \dots, N_d)$. Then if $n \geqslant N \geqslant N_k$ for every $k = 1, \dots, d$, so

$$|x_{n,k} - \ell_k| \leqslant \varepsilon,\ \forall k = 1, \dots, d.$$

Therefore,

$$\|\vec{x}_n - \vec{\ell}\| = \sqrt{\sum_{k=1}^{d} (x_{n,k} - \ell_k)^2} \leqslant \sqrt{\varepsilon^2 + \cdots + \varepsilon^2} = \sqrt{d\varepsilon^2} = \sqrt{d}\varepsilon,\ \forall n \geqslant N.$$

This is exactly the thesis (if you like you can replace ε by $\frac{\varepsilon}{\sqrt{d}}$). □

1.8.5. i) True, it follows by that proved in 1.8.4.
ii) False: for example $\vec{x}_n = (0, n) \longrightarrow \infty_2$ but $x_{n,1} \equiv 0 \longrightarrow 0$.
iii) True. Indeed, we may notice that

$$\|\vec{x}_n\| = \sqrt{\sum_{k=1}^{d} x_{n,k}^2} \geqslant \sqrt{x_{n,j}^2} = |x_{n,j}| \longrightarrow +\infty.$$

iv) False: take $\vec{x}_n = (0, (-1)^n)$, $\vec{x}_{2k} \equiv (0, 1) \longrightarrow (0, 1)$, $\vec{x}_{2k+1} \equiv (0, -1) \longrightarrow (0, -1)$, in particular $\lim_n \vec{x}_n$ cannot exists. However $x_{n,1} \equiv 0 \longrightarrow 0$. □

1.8.7. #1,2,4,6 done in class (see slides).
#3. Let $f(x, y) = \frac{y^2 - xy}{x^2 + y^2}$. We have $f(x, 0) \equiv 0 \longrightarrow 0$ when $x \to 0$, $f(0, y) = \frac{y^2}{y^2} \equiv 1 \longrightarrow 1$ when $y \to 0$.
#5. Let $f(x, y) = \frac{xy + \sqrt{y^2+1} - 1}{x^2 + y^2}$. We have $f(x, 0) = \frac{0}{x^2} \equiv 0 \longrightarrow 0$ when $x \to 0$, $f(0, y) = \frac{\sqrt{1+y^2} - 1}{y^2} \longrightarrow \frac{1}{2}$ when $y \to 0$, this because of the fundamental limit $\lim_{t \to 0} \frac{(1+t)^\alpha - 1}{t} = \alpha$ (here $\alpha = 1/2$).

1.8.8. #1,3,5 done in class (see slides).
#2 Let $f(x, y) := \frac{x^2 y^3}{(x^2 + y^2)^2}$. The limit is a $\frac{0}{0}$ indeterminate form. Introducing polar coordinates we have

$$f(x, y) = f(\rho \cos\theta, \rho \sin\theta) = \frac{\rho^5 \cos^2\theta \sin^3\theta}{(\rho^2)^2} = \rho \cos^2\theta \sin^3\theta,$$

so

$$|f(x, y) - 0| = \rho |\cos^2\theta| |\sin^3\theta| \leqslant \rho =: g(\rho) \longrightarrow 0, \quad \rho \to 0,$$

from which we conclude that $\exists \lim_{(x,y) \to \vec{0}} f = 0$.

#4. Let $f(x, y) := \frac{x\sqrt{|y|}}{\sqrt[3]{x^4 + y^4}}$. When $(x, y) \to \vec{0}$, the limit of f yields to an indeterminate form $\frac{0}{0}$. Let's write f in polar coords:

$$\begin{aligned} f(x, y) &= f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho\cos\theta\sqrt{\rho|\sin\theta|}}{\sqrt[3]{\rho^4(\cos^4\theta + \sin^4\theta)}} = \frac{\rho^{3/2}}{\rho^{4/3}} \frac{\cos\theta\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta + \sin^4\theta}} \\ &= \rho^{1/6} \frac{\cos\theta\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta + \sin^4\theta}}. \end{aligned}$$

Since $\rho^{1/6} \longrightarrow 0$ when $\rho \to 0$, we bet on the limit exists and it is equal to 0. To show this we notice that,

$$|f(x, y) - 0| = \rho^{1/6} \frac{|\cos\theta|\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta + \sin^4\theta}} \leqslant \rho^{1/6} \frac{1}{\sqrt[3]{\cos^4\theta + \sin^4\theta}}.$$

Let $K(\theta) := \sqrt[3]{\cos^4\theta + \sin^4\theta}$. This is a continuous function on $[0, 2\pi]$, so by Weierstrass' theorem there exists a minimum achieved at some $\theta_{min} \in [0, 2\pi]$, that is $K(\theta) \geqslant K(\theta_{min}) =: K_0 \geqslant 0$. Notice that $K_0 > 0$: if $K_0 = 0$ we would have $K(\theta_{min}) = 0$, so $\sqrt[3]{\cos^4\theta_{min} + \sin^4\theta_{min}} = 0$, that is $\cos^4\theta_{min} + \sin^4\theta_{min} = 0$.

Since both terms are positive, this is possible iff $\cos^4\theta_{min} = 0$ and (simultaneously) $\sin^4\theta_{min} = 0$, that is $\cos\theta_{min} = \sin\theta_{min} = 0$, which is impossible! Therefore, $K(\theta) \geqslant K_0 > 0$ for every $\theta \in [0, 2\pi]$, so

$$|f(x,y) - 0| \leqslant \rho^{1/6}\frac{1}{K_0} =: g(\rho) \longrightarrow 0,\ \rho \to 0.$$

From this we deduce that $\exists \lim_{(x,y)\to\vec{0}} f = 0$. □

1.8.9. #3. Let f be the function of which we aim to compute its limit at $\vec{0}$. The limit is an indeterminate form $\frac{0}{0}$. We notice that

$$f(x,0,0) = \frac{x^4}{\sqrt{x^4}} = \frac{x^4}{x^2} = x^2 \longrightarrow 0,\ x \to 0.$$

So, if a limit exists it must be $= 0$. Using spherical coordinates

$$x = \rho\sin\varphi\cos\theta,\quad y = \rho\sin\varphi\sin\theta,\quad z = \rho\cos\varphi,$$

we get

$$f(x,y,z) = \frac{(\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin\varphi\sin\theta\cos\varphi)^2}{\sqrt{(\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta))^2 + \rho^4\cos^4\varphi}} = \rho^2\frac{(\sin^2\varphi\cos^2\theta + \sin\varphi\sin\theta\cos\varphi)^2}{\sqrt{\sin^4\varphi + \cos^4\varphi}}.$$

From this we can prove that the limit exists and it is equal to 0: indeed,

$$|f(x,y,z) - 0| \leqslant \rho^2\frac{(1+1)^2}{\sqrt{\sin^4\varphi+\cos^4\varphi}}.$$

Let $F(\varphi) := \sqrt{\sin^4\varphi + \cos^4\varphi}$. Clearly $F \in \mathscr{C}([0,2\pi])$, so, by Weierstrass' theorem, there exists $K := \min F_{[0,2\pi]}$. Since the minimum is achieved at some φ_{min} if $K = F(\varphi_{min}) = 0$ we should have $\cos\varphi_{\min} = \sin\varphi_{min} = 0$, which is impossible. We conclude that $K > 0$ and

$$|f(x,y,z) - 0| \leqslant \rho^2\frac{(1+1)^2}{\sqrt{\sin^4\varphi+\cos^4\varphi}} \leqslant \rho^2\frac{4}{K} =: g(\rho) \longrightarrow 0,\ \rho \to 0. \quad \square$$

#5 We start noticing that $(x,y) \to (0,1)$ iff $(u,v) := (x, y-1) \to (0,0)$, so we are reduced to the limit

$$\lim_{(x,y)\to(0,1)} \frac{x^3\sinh(y-1)}{x^2+y^2-2y+1} = \lim_{(x,y)\to(0,1)} \frac{x^3\sinh(y-1)}{x^2+(y-1)^2} = \lim_{(u,v)\to(0,0)} \frac{u^3\sinh v}{u^2+v^2} =: \lim_{(u,v)\to(0,0)} \widetilde{f}(u,v).$$

The limit is an indeterminate form $\frac{0}{0}$. In polar coordinated for (u,v) we have

$$\widetilde{f}(u,v) = \frac{\rho^3\cos^3\theta\sinh(\rho\sin\theta)}{\rho^2} = \rho\cos^3\theta\sinh(\rho\sin\theta).$$

From this, we guess that the liumit exists and it is equal to 0. Indeed,

$$|\widetilde{f}(u,v) - 0| \leqslant \rho|\sinh(\rho\sin\theta)|.$$

Reminding of $\sinh t = t + o(t)$, we have

$$\sinh(\rho\sin\theta) = \rho\sin\theta + o(\rho\sin\theta),$$

so

$$|\sinh(\rho\sin\theta)| = |\rho\sin\theta + o(\rho)| \leqslant \rho + o(\rho),$$

from which

$$|\widetilde{f}(u,v) - 0| \leqslant \rho\,(\rho + o(\rho)) = \rho^2 + o(\rho^2) =: g(\rho) \longrightarrow 0,\ \rho \to 0.$$

We can now conclude that $\lim_{(u,v)\to\vec{0}} \widetilde{f}(u,v) = 0$, hence the same holds for the limit of f. □

#6 Since $(x,y) \to (1,1)$, $(u,v) := (x-1, y-1) \to (0,0)$, so

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(u,v)\to(0,0)} \frac{u^2v^7}{(u^2+v^2)^{5/2}} =: \lim_{(u,v)\to(0,0)} \widetilde{f}(u,v).$$

Clearly, the limit is an indeterminate form $\frac{0}{0}$. Let's pass to polar coordinates $u = \rho\cos\theta$, $v = \rho\sin\theta$

$$\widetilde{f}(u,v) = \frac{\rho^9\cos^2\theta\sin^7\theta}{(\rho^2)^{5/2}} = \rho^4\cos^2\theta\sin^7\theta,$$

so

$$|\widetilde{f}(u,v) - 0| \leqslant \rho^4 \longrightarrow 0,\ \rho \to 0,$$

and from this it follows that

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(u,v)\to(0,0)} \widetilde{f}(u,v) = 0. \quad \square$$

1.8.10 #2 The limit does not exist: indeed, $f(x,0) = x^4 - x^2 \longrightarrow +\infty$ is $|x| \to +\infty$. $f(x,x) \equiv 0 \longrightarrow 0$ when $|x| \to +\infty$.

#3 We have $f(x,0) = x^2 \longrightarrow +\infty$ when $|x| \to +\infty$. Let's look at f under polar coordinates: we get

$$f(x,y) = \rho^4\cos^2\theta\sin^2\theta + \rho^2 - \rho^2\cos\theta\sin\theta.$$

Apparently, the first term is the strongest one. However, if one of the two coords vanishes, the first term is constantly $= 0$. This suggests that this term is not particularly determinant. Being also positive, we may notice that

$$f(x,y) \geqslant \rho^2 - \rho^2\cos\theta\sin\theta = \rho^2\left(1 - \frac{1}{2}\sin(2\theta)\right) \geqslant \frac{1}{2}\rho^2 \longrightarrow +\infty,\ \rho \to +\infty.$$

This is sufficient to establish that $\exists \lim_{(x,y)\to\infty_2} f(x,y) = +\infty$.

#4. We have $f(x,0,0) = x^4 \longrightarrow +\infty$ when $|x| \to +\infty$, so if a limit exists it must be $= +\infty$. To compute the limit, an idea could be to use spherical coordinates. This, however, does not simplify the term $x^4+y^4+z^4$. We might expect that this term is somehow correlated to $(x^2+y^2+z^2)^2$ and indeed

$$(x^2+y^2+z^2)^2 = x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2 \geqslant x^4+y^4+z^4.$$

This is an upper bound for $x^4+y^4+z^4$. To get a more useful lower bound we remind of the elementary inequality $2ab \leqslant a^2+b^2$, so

$$(x^2+y^2+z^2)^2 = x^4+y^4+z^4+\underbrace{2x^2y^2}_{\leqslant (x^2)^2+(y^2)^2 = x^4+y^4}+\underbrace{2x^2z^2}_{\leqslant x^4+z^4}+\underbrace{2y^2z^2}_{\leqslant y^4+z^4} \leqslant 3(x^4+y^4+z^4),$$

so

$$x^4+y^4+z^4 \geqslant \frac{1}{3}\left(x^2+y^2+z^2\right)^2.$$

Now, with this we can say that, in spherical coordinates

$$f(x,y,z) \geqslant \frac{1}{3}\left(x^2+y^2+z^2\right)^2 - xyz = \frac{1}{3}\rho^4 - \rho^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta \geqslant \frac{1}{3}\rho^4-\rho^3 \longrightarrow +\infty$$

when $\rho \to +\infty$. This shows that $\exists \lim_{(x,y,z)\to\infty_3} f = +\infty$. □

#5 Notice that $f(x,0,0) = x^2 \longrightarrow +\infty$ when $|x| \to +\infty$. So, if a limit exists, it must be equal to $+\infty$. Before applying the spherical coordinates, we write

$$f(x,y,z) = x^2+y^2+z^2-xz+z^4-z^2.$$

Let's focus on the first part $x^2+y^2+z^2-xz$. Using spherical coords, we have

$$x^2+y^2+z^2-xz = \rho^2-\rho^2\sin\varphi\cos\theta\cos\varphi = \rho^2\left(1-\frac{1}{2}\sin(2\varphi)\cos\theta\right) \geqslant \rho^2\frac{1}{2}.$$

On the other side, we notice that there exists a constant C such that $z^4-z^2 \geqslant C$ for every $z\in\mathbb{R}$. Indeed, if $h(z)=z^4-z^2$, for $z>0$ we have

$$h'(z)=4z^3-2z=2z(2z^2-1)\geqslant 0, \quad \Longleftrightarrow \quad 2z^2-1\geqslant 0, \quad \Longleftrightarrow \quad z\geqslant \frac{1}{\sqrt{2}}.$$

This means that $h \searrow$ on $[0,\frac{1}{\sqrt{2}}]$ and $h \nearrow$ on $[\frac{1}{\sqrt{2}},+\infty[$, so $z=\frac{1}{\sqrt{2}}$ is a minimum for h on $[0,+\infty[$. Since $h(-z)=h(z)$ we have that it is also a minimum for all $z\in\mathbb{R}$. Thus,

$$h(z)\geqslant h(\frac{1}{\sqrt{2}}) = \frac{1}{4}-\frac{1}{2}=-\frac{1}{4}, \quad \Longleftrightarrow \quad z^4-z^2\geqslant -\frac{1}{4}.$$

We can now put the pieces together. Combining the two bounds we have

$$f(x,y,z) = \underbrace{x^2+y^2+z^2-xz}_{\geqslant\frac{\rho^2}{2}} + \underbrace{z^4-z^2}_{\geqslant-\frac{1}{4}} \geqslant \frac{\rho^2}{2}-\frac{1}{4} =: g(\rho) \longrightarrow +\infty,\ \rho\to+\infty.$$

From this $\exists \lim_{(x,y,z)\to\infty_3} f = +\infty$. □

#6. Also here we have $f(0,0,z)=z^2-z \longrightarrow +\infty$ when $|z|\to+\infty$. So, the possible candidate limit is $+\infty$. Before applying spherical coords, we notice that

$$f(x,y,z)=\sqrt{x^2+y^2}+|z|+\left(z^2-z-|z|\right).$$

Now, let $h(z) := z^2-z-|z|$. As in the previous exercise, h is bounded from below. Indeed: for $z<0$, $h(z)=z^2$, so $h\geqslant 0$; for $z\geqslant 0$, $h(z)=z^2-2z$, $h'(z)=2z-2=2(z-1)$, so $z=1$ is a global minimum with $h(1)=-1$, so $h(z)\geqslant -1$ for every $z\in\mathbb{R}$. On the other hand, by using spherical coords,

$$\sqrt{x^2+y^2}+|z| = \sqrt{\rho^2\sin^2\varphi}+\rho|\cos\varphi| = \rho\left(|\sin\varphi|+|\cos\varphi|\right).$$

The coefficient $C(\varphi) := |\sin\varphi| + |\cos\varphi|$ is continuous for $[0, \pi]$, so by Weierstrass' theorem there exists the minimum of C. Since $C(\varphi) > 0$ for every φ, we deduce that $C(\varphi) \geqslant C(\varphi_{min}) =: K > 0$, Therefore, merging the two arguments,

$$f(x,y,z) = \underbrace{\sqrt{x^2+y^2}+|z|}_{\leqslant K\rho} + \underbrace{\left(z^2 - z - |z|\right)}_{\geqslant -1} \geqslant K\rho - 1 =: g(\rho) \longrightarrow +\infty,\ \rho \to +\infty.$$

We conclude that $\exists \lim_{(x,y,z)\to\infty_3} f = +\infty$. □

#7. Here we might have some suspect about the existence of the limit. The root term is positive and of size ρ^2, while the xyz is or order ρ^3 with variable sign. Take

$$f(x,x,x) = \sqrt{4x^4+x^4} - x^3 = \sqrt{5}x^2 - x^3 \longrightarrow \begin{cases} -\infty, & x \to +\infty, \\ \\ +\infty, & x \to -\infty. \end{cases}$$

We conclude that $\nexists \lim_{(x,y,z)\to\infty_3} f$. □

1.8.13. i) We have a double implication.

$\Longrightarrow$ Hypothesis: $\vec{x} \in \mathbb{R}^d \cap \mathrm{Acc}(D)$. Thesis: $\forall r > 0,\ (B(\vec{x}, r] \cap D)\backslash\{\vec{x}\} \neq \varnothing$. We argue by contradiction. Suppose the thesis is false:

$$(\star)\ \exists r > 0,\ (B(\vec{x}, r] \cap D)\backslash\{\vec{x}\} = \varnothing.$$

Now, since $\vec{x} \in \mathrm{Acc}(D)$, by definition there exists $(\vec{x}_n) \subset D\backslash\{\vec{x}\}$ such that $\vec{x}_n \longrightarrow \vec{x}$. In particular, $\|\vec{x}_n - \vec{x}\| \longrightarrow 0$, so $\exists N$ such that $\|\vec{x}_n - \vec{x}\| \leqslant r$ for all $n \geqslant N$. But then $\vec{x}_n \in (B(\vec{x}, r] \cap D)\backslash\{\vec{x}\}$ for $n \geqslant N$, and this is in contradiction with $(\star)$.

$\Longleftarrow$ Hypothesis: $\forall r > 0,\ (B(\vec{x}, r] \cap D)\backslash\{\vec{x}\} \neq \varnothing$. Thesis: $\vec{x} \in \mathbb{R}^d \cap \mathrm{Acc}(D)$. Take $r = \frac{1}{n}$:

$$\forall n \in \mathbb{N},\ n \geqslant 1,\ \exists \vec{x}_n \in (B(\vec{x}, 1/n] \cap D)\backslash\{\vec{x}\},$$

from which: $\vec{x}_n \in D$, $\vec{x}_n \neq \vec{x}$ and since

$$\|\vec{x}_n - \vec{x}\| \leqslant \frac{1}{n} \longrightarrow 0,\ \Longrightarrow\ \vec{x}_n \longrightarrow \vec{x}.$$

This shows that $\vec{x} \in \mathrm{Acc}(D)$.

ii) We have a double implication.

$\Longrightarrow$ Hypothesis: $\infty_d \in \mathrm{Acc}(D)$. Thesis: $\forall r > 0,\ B(\vec{x}, r]^c \cap D \neq \varnothing$. We argue by contradiction. Suppose the thesis is false:

$$(\star)\ \exists r > 0,\ B(\vec{x}, r]^c \cap D = \varnothing.$$

Now, since $\infty_d \in \mathrm{Acc}(D)$, by definition there exists $(\vec{x}_n) \subset D$ such that $\vec{x}_n \longrightarrow \infty_d$. In particular, $\|\vec{x}_n\| \longrightarrow +\infty$, so $\exists N$ such that $\|\vec{x}_n\| \geqslant r$ for all $n \geqslant N$. But then $\vec{x}_n \in B(\vec{x}, r]^c \cap D$ for $n \geqslant N$, and this is in contradiction with $(\star)$.

$\Longleftarrow$ Hypothesis: $\forall r > 0,\ B(\vec{x}, r]^c \cap D \neq \varnothing$. Thesis: $\infty_d \in \mathrm{Acc}(D)$. Take $r = n$:

$$\forall n \in \mathbb{N},\ \exists \vec{x}_n \in B(\vec{x}, n]^c \cap D,$$

from which: $\vec{x}_n \in D$, and since

$$\|\vec{x}_n\| \geqslant n \longrightarrow +\infty,\ \Longrightarrow\ \vec{x}_n \longrightarrow \infty_d.$$

This shows that $\infty_d \in \mathrm{Acc}(D)$. □

1.8.14. Let A_1, A_2 be open sets and let's prove that $A_1 \cup A_2$ and $A_1 \cap A_2$ are both open. Let $\vec{x} \in A_1 \cup A_2$. For example, $\vec{x} \in A_1$. Since A_1 is open,

$$\exists B(\vec{x}, r] \subset A_1 \subset A_1 \cup A_2, \implies \vec{x} \in \mathrm{Int}(A_1 \cup A_2).$$

So, $A_1 \cup A_2 \subset \mathrm{Int}(A_1 \cup A_2) \subset A_1 \cup A_2$ from which $\mathrm{Int}(A_1 \cup A_2) = A_1 \cup A_2$.

Let's now prove that $A_1 \cap A_2$ is open. Let $\vec{x} \in A_1 \cap A_2$. Since A_1, A_2 are open,

$$\exists B(\vec{x}, r_1] \subset A_1, \quad \exists B(\vec{x}, r_2] \subset A_2.$$

Let $r := \min(r_1, r_2)$. Then

$$B(\vec{x}, r] \subset B(\vec{x}, r_1] \subset A_1, \;\; B(\vec{x}, r] \subset B(\vec{x}, r_2] \subset A_2, \implies B(\vec{x}, r] \subset A_1 \cap A_2.$$

So, $\vec{x} \in \mathrm{Int}(A_1 \cap A_2)$, from which $A_1 \cap A_2 \subset \mathrm{Int}(A_1 \cap A_2) \subset A_1 \cap A_2$, from which $\mathrm{Int}(A_1 \cap A_2) = A_1 \cap A_2$.

Let now C_1, C_2 be closed, that is $A_1 := C_1^c$ and $A_2 := C_2^c$ are open. Then

$$(C_1 \cup C_2)^c = C_1^c \cap C_2^c = A_1 \cap A_2,$$

is open by the first part. Similarly,

$$(C_1 \cap C_2)^c = C_1^c \cup C_2^c = A_1 \cup A_2,$$

which is open by the first part. □

1.8.15. # 1,2,3,5,8 done in class.

#4. D is defined by large inequalities involving continuous functions, it is closed. It cannot be also open because it should be either $D = \emptyset$ (but, for example $(1,1) \in D$) or $D = \mathbb{R}^2$ (but $(0,0) \notin D$). Let's check if it is bounded. We first notice that if $(x,y) \in D$ then $x > 0$. If $x < 0$, then $2x < x$ so there would be no y such that $x \leqslant y \leqslant 2x$. Since $x > 0$, dividing by x the second constraint we get $1 \leqslant y \leqslant 2$, hence $0 < x \leqslant y \leqslant 2$, so both x, y are bounded, we conclude D is bounded. Being also closed, D is compact.

#6. D is defined by large inequalities involving continuous functions, it is closed. It cannot be also open because it should be either $D = \emptyset$ (but, for example $(0,0,0) \in D$) or $D = \mathbb{R}^3$ (but $(1,1,1) \notin D$). Let's check if it is bounded. From the second constraint we have that $x^2 + z^2 \leqslant 1$, so in particular $|x|, |z| \leqslant 1$. Plugging this into the first constraint, $y^2 \leqslant z - x^2 \leqslant z \leqslant 1$, so $x^2, y^2, z^2 \leqslant 1$ and $\|(x,y,z)\| = \sqrt{x^2 + y^2 + z^2} \leqslant \sqrt{3}$, so D is bounded, hence compact.

#7. D is defined by large inequalities involving continuous functions, it is closed. It cannot be also open because it should be either $D = \emptyset$ (but, for example $(0,0,0) \in D$) or $D = \mathbb{R}^3$ (but $(0,0,2) \notin D$). Let's check if it is bounded. Since $x^2 - y^2 + z^2$ is not the sum of positive constants, we cannot conclude that each component must be bounded. We may imagine that, for example, if $x = y$, so if we consider a point (x,x,z), then $(x,x,z) \in D$ iff $z^2 \leqslant 1$. So, for example $(x,x,0) \in D$ for every $x \in \mathbb{R}$. Then, $(n,n,0) \in D$ for every $n \in \mathbb{N}$, and since $\|(n,n,0)\| = \sqrt{2}n \longrightarrow +\infty$ when $n \to +\infty$, we conclude that D is unbounded. In particular, it is not compact. □

1.8.16. The solution to this exercise requires to know properties of compact sets not introduced in the Course.

2.9.1. #1. Let $f(x,y) := \log(1+xy)$. We have

$$\begin{aligned}
\partial_{(\sqrt{3},1)} f(1,1) &= \lim_{t\to 0} \frac{f((1,1)+t(\sqrt{3},1)) - f(1,1)}{t} \\
&= \lim_{t\to 0} \frac{f(1+t\sqrt{3}, 1+t) - f(1,1)}{t} \\
&= \lim_{t\to 0} \frac{\log(1+(1+\sqrt{3}t)(1+t)) - \log 2}{t} \\
&= \lim_{t\to 0} \frac{\log(2+(\sqrt{3}+1)t+\sqrt{3}t^2) - \log 2}{t} \\
&= \lim_{t\to 0} \frac{\log(1+\frac{\sqrt{3}+1}{2}t+\frac{\sqrt{3}}{2}t^2)}{t} \\
&= \lim_{t\to 0} \frac{\frac{\sqrt{3}+1}{2}t+\frac{\sqrt{3}}{2}t^2}{t} \quad \left(\lim_{u\to 0} \frac{\log(1+u)}{u} = 1\right) \\
&= \lim_{t\to 0} \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}}{2}t = \frac{\sqrt{3}+1}{2}.
\end{aligned}$$

#3. Let $f(x,y) := \frac{x^2y}{|x|+y^2}$ for $(x,y) \neq (0,0)$, $f(0,0) = 0$. We have

$$\begin{aligned}
\partial_{(1,1)} f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(1,1)) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{\frac{t^2t}{|t|+t^2} - 0}{t} = \lim_{t\to 0} \frac{t^2}{|t|+t^2}.
\end{aligned}$$

Notice that $t^2 = |t|^2$ so

$$\lim_{t\to 0} \frac{t^2}{|t|+t^2} = \lim_{t\to 0} \frac{|t|^2}{|t|+|t|^2} = \lim_{t\to 0} \frac{|t|}{1+|t|} = 0.$$

Therefore $\partial_{(1,1)} f(0,0) = 0$.

#5. Let $f(x,y) := \frac{y(e^x-1)}{\sqrt{x^2+y^2}}$ for $(x,y) \neq (0,0)$, $f(0,0) = 0$. We have

$$\begin{aligned}\partial_{(-1,-2)} f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(-1,-2)) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(-t,-2t) - f(0,0)}{t} = \lim_{t\to 0} \frac{\frac{-t(e^{-2t}-1)}{\sqrt{t^2+4t^2}} - 0}{t} = \lim_{t\to 0} \frac{1-e^{-2t}}{\sqrt{5}|t|}\end{aligned}$$

Here we notice that since $\lim_{u\to 0} \frac{e^u-1}{u} = 1$, we have

$$\lim_{t\to 0+} \frac{1-e^{-2t}}{\sqrt{5}|t|} = \lim_{t\to 0+} \frac{1-e^{-2t}}{-2t} \frac{-2t}{\sqrt{5}t} = -\frac{2}{5},$$

while

$$\lim_{t\to 0-} \frac{1-e^{-2t}}{\sqrt{5}|t|} = \lim_{t\to 0-} \frac{1-e^{-2t}}{-2t} \frac{-2t}{-\sqrt{5}t} = \frac{2}{5}.$$

We conclude that the directional derivative does not exist. □

2.9.2. #2. Continuity at $(0,0)$: We have to check if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0), \quad \Longleftrightarrow \quad \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{|x|+|y|} = 0.$$

Using polar coordinates,

$$f(x,y) = \frac{\rho^2}{\rho\,(|\cos\theta| + |\sin\theta|)} = \rho \frac{1}{|\cos\theta| + |\sin\theta|}.$$

Now, since $g(\theta) := |\cos\theta| + |\sin\theta|$ is continuous on $[0,2\pi]$, by Weierstrass' thm it has a global minimum point $\theta_{min} \in [0,2\pi]$, so $g(\theta) \geqslant g(\theta_{min})$, for every $\theta \in [0,2\pi]$. We notice that $K := g(\theta_{min}) > 0$, otherwise, if $0 = g(\theta_{min}) = |\cos\theta_{min}| + |\sin\theta_{min}|$, necessarily $|\cos\theta_{min}| = |\sin\theta_{min}| = 0$, that is $\cos\theta_{min} = \sin\theta_{min} = 0$ which is impossible. Therefore

$$0 \leqslant f(x,y) \leqslant \frac{\rho}{K} \longrightarrow 0, \text{ when } \rho \to 0, \quad \Longleftrightarrow \quad (x,y) \to (0,0).$$

We conclude that $\exists \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, that is, f is continuous at $(0,0)$.

Partial derivatives: to compute $\partial_x f(0,0)$ and $\partial_y f(0,0)$ we need to invoke the definition of these partial derivatives as directional derivatives:

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0,$$

and similarly $\partial_y f(0,0) = 0$.

Differentiability: f is differentiable at $(0,0)$ iff

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0})\vec{h}}{\|\vec{h}\|} = 0.$$

Since $\nabla f(\vec{0}) = (\partial_x f(0,0), \partial_y f(0,0)) = (0,0)$, and $f(0,0) = 0$, setting $\vec{h} = (u,v)$ the previous limit boils down to

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \frac{u^2+v^2}{(|u|+|v|)\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \underbrace{\frac{\sqrt{u^2+v^2}}{|u|+|v|}}_{:=g(u,v)}.$$

This is suspicious and indeed

$$g(u,0) = \frac{\sqrt{u^2}}{|u|} = \frac{|u|}{|u|} \equiv 1 \longrightarrow 1, \quad g(u,u) = \frac{\sqrt{2}|u|}{2|u|} \equiv \frac{1}{\sqrt{2}} \longrightarrow \frac{1}{\sqrt{2}}$$

when $u \to 0$. We conclude that $\lim_{(u,v)\to(0,0)} g(u,v)$ does not exist, so f is not differentiable at $(0,0)$.

#3. Continuity at $(0,0)$: We have to check if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0), \quad \Longleftrightarrow \quad \lim_{(x,y)\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^2} = 0.$$

Using polar coordinates,

$$f(x,y) = \frac{\rho^5 \cos^2\theta \sin^3\theta}{\rho^4} = \rho\cos^2\theta\sin^3\theta,$$

so

$$|f(x,y) - 0| = \rho|\cos^2\theta\sin^3\theta| \leqslant \rho \longrightarrow 0, \ \rho \to 0.$$

We conclude that $\exists \lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, that is, f is continuous at $(0,0)$.

Partial derivatives: to compute $\partial_x f(0,0)$ and $\partial_y f(0,0)$ we need to invoke the definition of these partial derivatives as directional derivatives:

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to0} \frac{0}{t} = 0,$$

and similarly $\partial_y f(0,0) = 0$.

Differentiability: f is differentiable at $(0,0)$ iff

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0})\vec{h}}{\|\vec{h}\|} = 0.$$

Since $\nabla f(\vec{0}) = (\partial_x f(0,0), \partial_y f(0,0)) = (0,0)$, and $f(0,0) = 0$, setting $\vec{h} = (u,v)$ the previous limit boils down to

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}} = \lim_{(u,v)\to(0,0)} \frac{u^2v^3}{(u^2+v^2)^{2+1/2}} = \lim_{(u,v)\to(0,0)} \underbrace{\frac{u^2v^3}{(u^2+v^2)^{5/2}}}_{:=g(u,v)}.$$

This is suspicious and indeed

$$g(u,u) = \frac{u^5}{2^{5/2}|u|^5} \equiv \frac{1}{2^{5/2}}\operatorname{sgn}(u)$$

which has not a limit when $u \to 0$. We conclude that $\lim_{(u,v)\to(0,0)} g(u,v)$ does not exist, so f is not differentiable at $(0,0)$. $\square$

2.9.4. We start computing the partial derivatives at $(0,0)$. Because of the definition of f, we must proceed with the definition:

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0,$$

and

$$\partial_y f(0,0) = \partial_{(0,1)} f(0,0) = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

We notice that, for $y \neq 0$,

$$\partial_y f(x,y) = -\frac{x^2}{y^2} \cos \frac{1}{y},$$

so, in particular, $\partial_y f(y,y) = -\cos\frac{1}{y}$, and since $\lim_{y\to 0} \partial_y f(y,y)$ does not exist, we conclude that $\partial_y f$ is not continuous at $(0,0)$. This means that differentiability test does not apply to this case. To check differentiability we must proceed with the definition: f is differentiable at $(0,0)$ iff

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0})\vec{h}}{\|\vec{h}\|} = 0.$$

Introducing coordinates $\vec{h} = (u,v)$, this limit becomes

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}}.$$

We notice that, if $v = 0$,

$$\frac{f(u,v)}{\sqrt{u^2+v^2}} = \frac{0}{\sqrt{u^2+v^2}} \equiv 0,$$

while, if $v \neq 0$,

$$\frac{f(u,v)}{\sqrt{u^2+v^2}} = \frac{u^2\sin(1/v)}{\sqrt{u^2+v^2}} \overset{pol\ coords}{=} \frac{\rho^2\cos\theta\sin\left(\frac{1}{\rho\sin\theta}\right)}{\rho} = \rho\cos\theta\sin\left(\frac{1}{\rho\sin\theta}\right).$$

Therefore,

$$\left|\frac{f(u,v)}{\sqrt{u^2+v^2}} - 0\right| = \rho\left|\cos\theta\sin\left(\frac{1}{\rho\sin\theta}\right)\right| \leqslant \rho \longrightarrow 0,\ \rho\to 0.$$

From this we conclude that $\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\sqrt{u^2+v^2}} = 0$m whence f is differentiable at $(0,0)$. □

2.9.6. iv) Let $f(x,y) = xe^y + ye^x$. We have

$$\partial_x f(x,y) = e^y + ye^x, \quad \partial_y f(x,y) = xe^y + e^x,$$

from which $\partial_x f, \partial_y f \in \mathsf{C}(\mathbb{R}^2)$ so, by the differentiability test, f is differentiable on $\mathbb{R}^2$. Now, (x,y) is a stationary point for f iff

$$\nabla f(x,y) = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} e^y + ye^x = 0, \\ xe^y + e^x = 0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y = -e^{y-x}, \\ x = -e^{x-y}. \end{cases}$$

In particular, both $x, y \neq 0$ and $y = -e^{y-x} = -\frac{1}{e^{x-y}} = \frac{1}{x}$. Therefore, x must verify

$$x = -e^{x-1/x}, \quad \Longleftrightarrow \quad e^{x-1/x} + x = 0.$$

This is a non trivial equation. We may notice that there is no solution for $x \geqslant 0$ ($e^{\cdots} > 0$). For $x < 0$ let $g(x) := e^{x-1/x} + x$. We notice that $g(x) \longrightarrow -\infty$ for $x \to -\infty$ while $g(x) \longrightarrow +\infty$ for $x \to 0-$. Since

$$g'(x) = e^{x-1/x}\left(1 + \frac{1}{x^2}\right) + 1 > 0,$$

we have that g is strictly increasing, and because of previous limits, $g(x) = 0$ for a unique value of x. Finally, since $g(-1) = e^{-1+1} + (-1) = e^0 - 1 = 0$, the unique solution is $x = -1$. Therefore, the unique solution of $\nabla f(x, y) = \vec{0}$ is $(-1, \frac{1}{-1}) = (-1, -1)$. □

2.9.7. We have

$$\partial_x f(x, y) = 2x - 4, \quad \partial_y f(x, y) = 2\lambda y + 2.$$

Clearly, $\partial_x f, \partial_y f \in \mathsf{C}(\mathbb{R}^2)$, so f is differentiable on $\mathbb{R}^2$. A point (x, y) is a stationary point for f iff

$$\nabla f(x, y) = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} x - 2 = 0, \\ \\ \lambda y + 1 = 0. \end{cases}$$

In order $(2, -1)$ be a solution, we need $\lambda(-1) + 1 = 0$ that is $\lambda = 1$. In this case

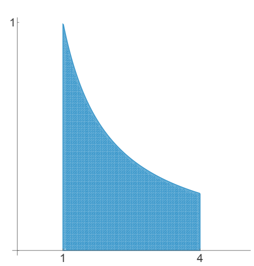
$$f(x, y) = x^2 + y^2 - 4x + 2y = (x - 2)^2 + (y + 1)^2 - 5,$$

from which it is evident that $(2, -1)$ is a global minimum point for f. □

2.9.9. Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ 1 \leqslant x \leqslant 4, \ |xy| \leqslant 1\}$. We notice that D is closed (being defined by large inequalities). Since $1 \leqslant x \leqslant 4$, x is bounded and being

$$|xy| \leqslant 1, \quad \Longleftrightarrow \quad |x||y| \leqslant 1, \quad \overset{1\leqslant x\leqslant 4}{\Longleftrightarrow} \quad |y| \leqslant \frac{1}{|x|} \leqslant 1,$$

we get that also y is bounded. Thus, D is compact.



Since $f \in \mathsf{C}(D)$, by Weierstrass' theorem f has both global min/max pts on D. Let $(x, y) \in D$ be any min/max point. We have the following alternative:

- $(x,y)\in \text{Int}(D)$: since
$$\partial_x f = e^{-xy}(y-(xy)^2),\quad \partial_y f = e^{-xy}(x-(xy)^2),$$
we have $\partial_x f, \partial_y f \in \mathsf{C}(D)$, so f is differentiable on D (differentiability test). Therefore, Fermat's thm applies and
$$\nabla f(x,y)=\vec{0}, \iff \begin{cases} e^{-xy}(y-(xy)^2)=0, \\ \\ e^{-xy}(x-(xy)^2)=0, \end{cases} \iff \begin{cases} y(1-yx^2)=0, \\ \\ x(1-xy^2)=0, \end{cases}$$
The first equation yields the alternatives
$$\begin{cases} y=0, \\ \\ x=0, \end{cases} \vee \begin{cases} x^2y=1, \\ \\ x(1-xy^2)=0. \end{cases} \iff \begin{cases} x^2y=1, \\ \\ xy^2=1, \end{cases}$$
In the last system, $x,y\neq 0$ (otherwise it is impossible), so $xy=\frac{1}{x}=\frac{1}{y}$ from which $x=y$, and plugging back into the equations we get $x^3=1$, $y^3=1$, that is $(x,y)=(1,1)$. Conclusion: the stationary points are $(0,0)\notin \text{Int}(D)$ and $(1,1)\notin \text{Int}(D)$.
- $(x,y)\in\partial D$: We have
$$\partial D = S_1\cup S_2\cup S_3\cup S_4,$$
where
$$S_1 := \{(1,y)\ :\ 0\leqslant y\leqslant 1\}\quad S_2 := \{(x,0)\ :\ 1\leqslant x\leqslant 4\}$$
$$S_3 := \{(4,y)\ :\ 0\leqslant y\leqslant \tfrac{1}{4}\}\quad S_4 := \left\{\left(x,\tfrac{1}{x}\right)\ :\ 1\leqslant x\leqslant 4\right\}$$
On S_1: $f(x,y)=f(1,y)=ye^{-y}=:g(y)$ for $y\in[0,1]$. We have $g(0)=0$, $g(1)=e^{-1}$, $g'(y)=e^{-y}(1-y)\geqslant 0$ iff $y\leqslant 1$. So $g\nearrow$ on $[0,1]$, the min point is $y=0$, the max point $y=1$. From this, the min point for f on S_1 is $(1,0)$, the max point is $(1,1)$.
  On S_2: $f(x,y)=f(x,0)\equiv 0$. Each point $(x,0)$ is min and max point for f on S_2.
  On S_3: $f(x,y)=f(4,y)=4ye^{-4y}=g(4y)$ for $y\in[0,\frac{1}{4}]$ with g the same function introduced above. So, the min (max) are attained at $y=0$ ($y=\frac{1}{4}$), thus for f at point $(4,0)$ ($(4,\frac{1}{4})$).
  On S_4: $f(x,y)=f(x,\frac{1}{x})\equiv e^{-1}$, every point at same time min and max point for f on S_4.

We are now ready to conclude.

- Candidates minimum point are $(1,0)$ with $f=0$, $(x,0)$ ($x\in[1,4]$) with $f=0$, $(4,0)$ with $f=0$, $(x,\frac{1}{x})$ ($x\in[1,4]$) with $f=e^{-1}$. The minimum point of f on D are $(x,0)$ with $x\in[1,4]$.
- Candidates maxmum point are $(1,1)$ with $f=e^{-1}$, $(4,\frac{1}{4})$ ($x\in[1,4]$) with $f=e^{-1}$, $(x,\frac{1}{x})$ ($x\in[1,4]$) with $f=e^{-1}$. The maxmum point of f on D are $(x,\frac{1}{x})$ with $x\in[1,4]$. □

2.9.10. i) $f\in\mathsf{C}(\mathbb{R}^2)$. We check that
$$\lim_{(x,y)\to\infty_2} f(x,y)=+\infty.$$
In polar coordinates,
$$f(x,y)=\rho^4\left(\cos^4\theta+\sin^\theta\right)-\rho^2\cos\theta\sin\theta.$$

Let $C(\theta) := \cos^4\theta + \sin^4\theta \geqslant 0$. We notice that $C \in \mathsf{C}([0, 2\pi])$. By Weierstrass' theorem there exists θ^* such that $C(\theta) \geqslant C(\theta^*)$ for every $\theta \in [02, \pi]$. If $C(\theta^*) = 0$ then $\cos^4\theta^* = \sin^4\theta^* = 0$, from which $\cos\theta^* = \sin\theta^* = 0$, which is impossible. So, if $K := C(\theta^*) > 0$, we have

$$f(x, y) \geqslant K\rho^4 - \rho^2 \longrightarrow +\infty, \text{ if } \rho \to +\infty.$$

From this the conclusion follows. Since the domain $\mathbb{R}^2$ is closed and unbounded, there exists $\min_{\mathbb{R}^2} f$ (but not $\max_{\mathbb{R}^2} f$).

Let (x, y) be any min point. Since $\mathbb{R}^2$ is open, $(x, y) \in \mathrm{Int}(\mathbb{R}^2)$. Moreover, since

$$\partial_x f = 4x^3 - y, \quad \partial_y f = 4y^3 - x,$$

we have $\partial_x f, \partial_y f \in \mathsf{C}(\mathbb{R}^2)$, so f is differentiable on $\mathbb{R}^2$. By Fermat's thm, at min point (x, y) we must have $\nabla f(x, y) = \vec{0}$. Now,

$$\nabla f(x, y) = \vec{0}, \iff \begin{cases} 4x^3 - y = 0, \\ \\ 4y^3 - x = 0. \end{cases} \iff \begin{cases} y = 4x^3, \\ \\ 4^4x^9 - x = 0. \end{cases}$$

We notice that $4^4x^9 - x = x(4^4x^8 - 1) = 0$ iff $x = 0$ or $x^8 = \frac{1}{4^4} = \frac{1}{2^8}$, from which $x = \pm\frac{1}{2}$. In the first case $y = 4x^3 = 0$ while in the second $y = 4\left(\pm\frac{1}{2}\right)^3 = \pm\frac{4}{8} = \pm\frac{1}{2}$. The stationary points are $(0, 0)$ and $\pm\left(\frac{1}{2}, \frac{1}{2}\right)$. Since $f(0, 0) = 0$ and $f\left(\pm\left(\frac{1}{2}, \frac{1}{2}\right)\right) = \frac{1}{16} + \frac{1}{16} - \frac{1}{4} = \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$ we conclude that the min points of f are $\pm\left(\frac{1}{2}, \frac{1}{2}\right)$.

ii) The domain D is open and unbounded. We notice that $f(x, 0) = x(\log x)^2 \longrightarrow +\infty$ when $(x, 0) \to \infty_2$. Therefore, $\nexists \max_D f$. We also notice that $f \geqslant 0$ on D. Moreover $f(1, 0) = 0$ so, $(1, 0)$ is definitely a min point for f on D, so $\exists \min_D f$. Is there any other min point (x, y)? If yes, being D open, $(x, y) \in \mathrm{Int}(D)$. Now, since

$$\partial_x f = (\log x)^2 + y^2 + 2x\frac{\log x}{x} = (\log x)^2 + y^2 + 2\log x, \quad \partial_y f = 2xy,$$

we have $\partial_x f, \partial_y f \in \mathsf{C}(\mathbb{R}^2)$, so f is differentiable. Then, for any min point (x, y), by Fermat's thm $\nabla f(x, y) = \vec{0}$, that is

$$\begin{cases} (\log x)^2 + y^2 + 2\log x = 0, \\ \\ 2xy = 0, \end{cases} \overset{(x,y)\in D =]0,+\infty[\times\mathbb{R}}{\iff} \begin{cases} y = 0, \\ \\ (\log x)^2 + 2\log x = 0. \end{cases}$$

Now,

$$(\log x)^2 + 2\log x = 0, \iff (\log x)(\log x + 2) = 0, \iff x = 1, e^{-2}.$$

Therefore, $(x, y) = (1, 0), (e^{-2}, 0)$. Now, $f(1, 0) = 0$ and $f(e^{-2}, 0) = 4e^{-2}$, so we conclude that the unique min point is $(1, 0)$.

iii) $f(x, x) = 2x^3 \longrightarrow \pm\infty$, according to $x \to \pm\infty$. Therefore $\nexists \min_{\mathbb{R}^2} f, \max_{\mathbb{R}^2} f$.

iv) The domain $D = \mathbb{R}^3$ is closed and unbounded. Notice that

$$f(x, y, z) = (x - y)^2 + z^2 + 2xz$$

so $f(x,x,z) = z^2 + 2xz$. In particular

$$f(x,x,x) = x^2 + 2x^2 = 3x^2 \longrightarrow +\infty, \text{ if } |x| \to +\infty,$$

while

$$f(x,x,-x) = x^2 - 2x^2 = -x^2 \longrightarrow -\infty, \text{ if } |x| \to +\infty.$$

From this we see that $\nexists \lim_{(x,y,z)\to\infty_3} f$ and, at the same time, $\sup_{\mathbb{R}^3} f = +\infty$ and $\inf_{\mathbb{R}^3} f = -\infty$, thus $\nexists \max_{\mathbb{R}^3} f, \min_{\mathbb{R}^3} f$.

v) The domain $D = \mathbb{R}^3$ is closed and unbounded. We compute

$$\lim_{(x,y,z)\to\infty_3} f(x,y,z).$$

In spherical coordinates

$$f(x,y,z) = \rho^4 \underbrace{\left(\cos^4\theta\sin^4\varphi + \sin^4\theta\sin^4\varphi + \cos^4\varphi\right)}_{=:C(\theta,\varphi)} - \rho^3\cos\theta\sin\theta\sin^2\varphi\cos\varphi.$$

We notice that $C(\theta,\varphi) \geqslant 0$, C is a continuous function of $(\theta,\varphi) \in [0,2\pi]\times[0,\pi]$, which is a compact set of $\mathbb{R}^2$. By Weierstrass' thm, C has a minimum on such domain, that is $C(\theta,\varphi) \geqslant C(\theta^*,\varphi^*)$. We claim $K = C(\theta^*,\varphi^*) > 0$. If $= 0$, then being C sum of positive quantities, necessarily,

$$\begin{cases} \cos^4\theta^*\sin^4\varphi^* = 0, \\ \sin^4\theta^*\sin^4\varphi^* = 0, \\ \cos^4\varphi^* = 0, \iff \varphi^* = \frac{\pi}{2}, \end{cases} \iff \begin{cases} \cos^4\theta^* = 0, \\ \sin^4\theta^* = 0, \\ \varphi^* = \frac{\pi}{2}, \end{cases}$$

which is impossible. Therefore $K > 0$ and

$$f(x,y,z) \geqslant K\rho^4 - \rho^3 \longrightarrow +\infty, \text{ if } \rho \to +\infty.$$

From this we conclude that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. Being $f \in \mathsf{C}(\mathbb{R}^3)$, we deduce that $\exists \min_{\mathbb{R}^3} f$ but $\nexists \max_{\mathbb{R}^3} f$.

To determine the min points we apply Fermat's thm. We start noticing that

$$\partial_x f = 4x^3 - yz, \quad \partial_y f = 4y^3 - xz, \quad \partial_z f = 4z^3 - xy,$$

from which $\partial_x f, \partial_y f, \partial_z f \in \mathsf{C}(\mathbb{R}^3)$, thus f is differentiable because of the differentiability test. So, if $(x,y,z) \in \mathbb{R}^3$ any min point, noticed that $\mathbb{R}^3$ is also open (thus $(x,y,z) \in \text{Int}(\mathbb{R}^3) = \mathbb{R}^3$), we necessarily must have $\nabla f(x,y,z) = \vec{0}$ (Fermat's thm), that is

$$\begin{cases} 4x^3 - yz = 0, \\ 4y^3 - xz = 0, \\ 4z^3 - xy = 0. \end{cases}$$

To solve this system we argue as follows. If one of the coords is $= 0$, say x for example, the system boils down to

$$\begin{cases} yz = 0, \\ 4y^3 = 0, \\ 4z^3 = 0, \end{cases} \iff (x,y,z) = (0,0,0).$$

The same happens if $y = 0$ or $z = 0$. We can therefore assume that $x, y, z \neq 0$. Then, by multiplying each of the equations by x, y, z respectively, the system is equivalent to

$$\begin{cases} xyz = 4x^4, \\ xyz = 4y^4, \\ xyz = 4z^4, \end{cases} \Longrightarrow x^4 = y^4 = z^4.$$

From this we get $y = \pm x$, $z = \pm x$ with all possible combinations of signs. Plugging these into the first equation we have, for $y = z = x$ $y = z = -x$

$$x^3 = 4x^4, \quad \overset{x\neq 0}{\Longleftrightarrow} \quad 4x = 1, \quad \Longleftrightarrow \quad x = \frac{1}{4}.$$

This yields points $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ and $\left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right)$. For $y = x$, $z = -x$ or $y = -x$ and $z = -x$ we get

$$-x^3 = 4x^4, \quad \overset{x\neq 0}{\Longleftrightarrow} \quad 4x = -1, \quad \Longleftrightarrow \quad x = -\frac{1}{4},$$

this yielding points $\left(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right)$ and $\left(-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right)$. Now, since

$$f\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) = f\left(\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right) = f\left(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}\right) = f\left(-\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right) = 3\frac{1}{4^4} - \frac{1}{4^3} = -\frac{1}{4^4},$$

we conclude that all the stationary points are global min pts for f. □

2.9.13. i) We have $f(x, x-1) \equiv 0 \longrightarrow 0$ when $(x, x-1) \to \infty_2$, that is $|x| \to +\infty$. On theother hand $f(x, 0) = x^2(x-1)^2 \longrightarrow -\infty$ when $(x, 0) \to \infty_2$, that is $|x| \to +\infty$. We conclude that the limit of f at ∞_2 does not exist.

ii) We have

$$\partial_x f = 2x(y^2 - (x-1)^2) + x^2(-2(x-1)), \quad \partial_y f = 2yx^2,$$

from which $\partial_x, \partial_y f \in C(\mathbb{R}^2)$, so f is differentiable on $\mathbb{R}^2$. So, (x, y) is a stationary point for f iff $\nabla f(x, y) = \vec{0}$, that is

$$\begin{cases} 2x(y^2 - (x-1)^2) + x^2(-2(x-1)) = 0, \\ \\ 2yx^2 = 0, \end{cases} \Longleftrightarrow \begin{cases} x = 0, \\ \\ 0 = 0, \end{cases} \vee \begin{cases} y = 0, \\ \\ x(x-1)(2x-1) = 0. \end{cases}$$

The second equation of last system yields $x = 0, 1, \frac{1}{2}$. We conclude that the stationary points of f are $(0, y)$, $y \in \mathbb{R}$, $(1, 0)$ and $\left(\frac{1}{2}, 0\right)$.

To understand if they are local extrema, let's compute the Hessian matrix. We have

$$\begin{aligned} &\partial_{xx} f = 2(y^2 - (x-1)^2) - 4x(x-1) - x^2, \\ &\partial_{xy} f = \partial_{yx} f = 4xy, \\ &\partial_{yy} f = 2x^2. \end{aligned}$$

From this, $\partial_{xx} f, \partial_{xy} f, \partial_{yy} f \in C(\mathbb{R}^2)$, so f is twice differentiable. We have

$$\nabla^2 f(0, y) = \begin{bmatrix} 2(y^2 - 1) & 0 \\ 0 & 0 \end{bmatrix}$$

so $\nabla^2 f(0,y)$ is not strictly positive/negative and the Hessian matrix test fails. To determine the nature or $(0,y)$ we need to proceed with a direct inspection. Since y is fixed, let call it y_0. We have $f(0,y_0)=0$. Since, for $x\approx 0$, $f(x,y_0)\approx x^2(y_0^2-1)$, this suggests to distinguish cases $y_0\neq\pm1$ from $y_0=\pm1$.

- If $y_0\neq\pm1$, then since $f(x,y)\sim_{(x,y)\to(0,y_0)} x^2(y_0^2-1)$ we guess that $(0,y_0)$ is a local minimum if $y_0^2-1>0$, a local maximum if $y_0^2-1<0$. Here is a precise argument for $y_0^2-1>0$ (the other case is similar). Since for $(x,y)\to(0,y_0)$ we have $y^2-(x-1)^2\longrightarrow y_0^2-1>0$, by the permanence of sign we can say that in a sufficiently small ball $B((0,y_0),r]$, we have
$$y^2-(x-1)^2\geqslant\frac{y_0^2-1}{2}.$$
Then
$$f(x,y)=x^2(y^2-(x-1)^2)\geqslant x^2\frac{y_0^2-1}{2}\geqslant 0=f(0,y_0),\ \forall(x,y)\in B((0,y_0),r].$$
This shows that f has a local minimum at $(0,y_0)$.
- If $y_0^2=1$, that is for $y_0=\pm1$, the previous argument does not hold. In this case, we may suspect that $(0,\pm1)$ is not a local extrema. For example,
$$f(x,\pm1)=x^2(1-(x-1)^2)=x^2(-x^2+2x)=-x^4+2x^3=x^3\,(2+x)\,.$$
Recalling that $f(0,\pm1)=0$, we see that, for $x>0$ we have $f(x,\pm1)=x^3(2+x)>0=f(0,\pm1)$ while for $-2<x<0$ we have $f(x)<0=f(0,\pm1)$. We conclude that $(0,\pm1)$ is not a local extrema.

Next,
$$\nabla^2 f(1,0)=\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}.$$
In this case, $\nabla^2 f$ is already diagonal with eigenvalues -1 and 2. This means that $(1,0)$ is a saddle point. Finally,
$$\nabla^2 f(1/2,0)=\begin{bmatrix}1/4 & 0\\ 0 & 1/2\end{bmatrix}$$
so $\nabla^2 f$ is already diagonal with eigenvalues $1/4$ and $1/2$. This means that $(1/2,0)$ is a local minimum point.

iii) We already noticed that $\inf_{\mathbb{R}^2} f=-\infty$, so $\nexists\min_{\mathbb{R}^2} f$. Noticed also that $f(1,y)=y^2\longrightarrow+\infty$ when $|y|\to+\infty$, we deduce that $\sup_{\mathbb{R}^2} f=+\infty$, so $\nexists\max_{\mathbb{R}^2} f$. Thus, even if Fermat's thm applies, the stationary pts are not global extrema. About $f(\mathbb{R}^2)=]-\infty,+\infty[$.

iv) Let now $D:=\{(x,y)\in\mathbb{R}^2\ :\ y\leqslant 0,\ 0\leqslant x\leqslant y+1\}$. Since D is defined by large inequalities with continuous constraints, it is closed. Plugging the first constraint into the second one, we get $0\leqslant x\leqslant y+1\leqslant 1$, so x is bounded, and since $y\geqslant x-1\geqslant -1$, we have $-1\leqslant y\leqslant 0$, so also y is bounded. We deduce that D is compact, and since $f\in \mathsf{C}(D)$ Weierstrass' thm applies and f has both global min/max on D.

Since f is differentiable, Fermat's thm applies, so to determine these points we have the following alternative:

- $(x, y) \in \text{Int}(D)$: then (x, y) must be a stationary point for f. However, since none of the stationary points of f lies in $\text{Int}(D)$, this case simply does not happen.
- $(x, y) \in \partial D$: in this case we notice that

  $$\partial D = \{(x,0)\ :\ 0 \leqslant x \leqslant 1\} \cup \{(0,y)\ :\ -1 \leqslant y \leqslant 0\} \cup \{(x, x-1)\ :\ 0 \leqslant x \leqslant 1\}.$$

  We have

  $$f(x,0) = x^2(-(x-1)^2) = -\left(x(x-1)\right)^2.$$

  Since $x(x-1)$ is minimum at $x = 0, 1$ and maximum at $x = \frac{1}{2}$, we deduce that f in minimum at point $(1/2, 0)$, maximum at $(0,0), (1,0)$.

  Next, $f(0,y) \equiv 0$, so all points $(0,y)$ $y \in [-1,0]$ are min/max points for f on this part of boundary.

  Finally, $f(x, x-1) \equiv 0$ so, again, so all points $(x, x-1)$ $x \in [0,1]$ are min/max points for f on this part of boundary.

Conclusion. Candidates to be min points for f on D are $(1/2, 0)$, $(0, y)$ and $(x, x-1)$. Since $f(1/2, 0) = -\frac{1}{16} < 0$ and $f(0,y) = f(x, x-1) \equiv 0$, we conclude that the min point of f on D is $(1/2, 0)$ with min value $= -1/16$.

Candidates to be max points for f on D are $(0, y)$ and $(x, x-1)$. Since $f(0,y) = f(x, x-1) \equiv 0$, we conclude that all pts $(0, y)$, $(x, x-1)$ are max pts for f on D, with max value $= -1/16$.

Finally, being D a connected set, $f(D) = [-1/16, 0]$. □

2.9.14. i) $f(x,0) = x^4 - 8x^2 \longrightarrow +\infty$ if $(x,0) \to \infty_2$. So, if a limit exists it must be $= +\infty$. In polar coords,

$$f(x,y) = \rho^4 \left(\cos^4\theta + \sin^4\theta\right) - 8\rho^2.$$

As discussed in many othe exercises, $\cos^4\theta + \sin^4\theta$ is a continuous function of $\theta \in [0, 2\pi]$, so, by Weierstrass' thm, it has a minimum value. Being the function $\geqslant 0$, the minimum value will be $\geqslant 0$. If $= 0$, there should be a θ such that $\cos^4\theta + \sin^4\theta = 0$, but this is impossible. We conclude that $\cos^4\theta + \sin^4\theta \geqslant K > 0$ for every $\theta \in [0, 2\pi]$. Therefore

$$f(x,y) \geqslant K\rho^4 - 8\rho^2 \longrightarrow +\infty,\ \rho \to +\infty,\ \iff\ (x,y) \to \infty_2.$$

Conclusion: $\exists \lim_{(x,y)\to\infty_2} f = +\infty$.

ii) We have $\partial_x f = 4x^3 - 16x$, $\partial_y f = 4y^3 - 16y$, so $\partial_x f, \partial_y f \in \mathsf{C}(\mathbb{R}^2)$, whence f is differentiable on $\mathbb{R}^2$. Point (x, y) is a stationary point for f iff

$$\nabla f(x,y) = \vec{0}, \iff \begin{cases} 4x^3 - 16x = 0, \\ \\ 4y^3 - 16y = 0. \end{cases} \iff \begin{cases} x(x^2 - 4) = 0, \\ \\ y(y^2-4) = 0. \end{cases}$$

Both equations yield $x, y = 0, \pm 2$, and since the two equations are independent, the solutions of the system are $(0,0), (0, \pm 2), (\pm 2, 0), (\pm 2, \pm 2)$ with all possible combinations of sign (9 points in total). To classify these points we compute first the hessian matrix. We have $\partial^2_{xx} f = 12x^2 - 16$, $\partial^2_{yx} f = 0 = \partial^2_{xy} f$, $\partial^2_{yy} f = 12y^2 - 16$. From this we see that $\partial^2_{xx} f, \partial^2_{xy} f, \partial^2_{yy} f \in \mathsf{C}(\mathbb{R}^2)$, so f is twice differentiable and the hessian matrix is

$$\nabla^2 f(x,y) = \begin{bmatrix} 12x^2 - 16 & 0 \\ 0 & 12y^2 - 16 \end{bmatrix}.$$

We have

- $\nabla^2 f(0,0) = \begin{bmatrix} -16 & 0 \\ 0 & -16 \end{bmatrix}$: point $(0,0)$ is a local maximum point.
- $\nabla^2 f(\pm 2,0) = \begin{bmatrix} 32 & 0 \\ 0 & -16 \end{bmatrix}$: points $(\pm 2,0)$ are saddle points. The same conclusion for points $(0,\pm 2)$.
- $\nabla^2 f(\pm 2,\pm 2) = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$: points $(\pm 2,\pm 2)$ are all local minimum points.

Let's discuss global min/max for f. We notice that $f \in \mathsf{C}(\mathbb{R}^2)$, $\mathbb{R}^2$ is closed and unbounded: by i) and a known fact, f has global minimum on $\mathbb{R}^2$ but not global maximum. Let $(x,y) \in \mathbb{R}^2$ be any minimum point. Since $\mathbb{R}^2$ is open, $(x,y) \in \mathrm{Int}(\mathbb{R}^2)$. Being f differentiable, Fermat's thm applies: necessarily, (x,y) must be a stationary point for f. Since (x,y) is a global minimum it is automatically a local minimum so (x,y) is one (or more) of $(\pm 2,\pm 2)$. Since f takes the same value at these four points, we conclude that they are global minimums for f on $\mathbb{R}^2$. Finally, $f(\mathbb{R}^2) = [f(\pm 2,\pm 2), +\infty[= [-32,+\infty[$.

iii) The domain $D := \{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leqslant 9\}$ is clearly closed and bounded. Since f is continuous on D, by Weierstrass thm f has both global min/max on D. Let's determine these points. Let $(x,y) \in D$ be any min/max point. We have the following alternative:

- either $(x,y) \in \mathrm{Int}(D) = \{x^2+y^2 < 9\}$: in this case (x,y) is an interior point and Fermat's thm applies, so (x,y) must be a stationary point for f. Since all stationary points of f are in $\mathrm{Int}(D)$, (x,y) is one or more of $(0,0).(\pm 2,0), (0,\pm 2), (\pm 2,\pm 2)$. We notice that since $(\pm 2,\pm 2)$ are global min points for f on $\mathbb{R}^2$, they are global min points for f on D. So we can say that the search is limited to global max point for f on D.
- or $(x,y) \in \partial D = \{x^2+y^2 = 9\}$. Points of the circle can be represented as $(x,y) = 3(\cos\theta, \sin\theta)$ with $\theta \in [0,2\pi]$. On these points,
$$f(x,y) = 3^4\left(\cos^4\theta + \sin^4\theta\right) - 8\cdot 9 = 81\left(\cos^4\theta + \sin^4\theta\right) - 72.$$
So, f is maximum iff $g(\theta) := \cos^4\theta + \sin^4\theta$ it is. We have
$$g'(\theta) = -4\cos^3\theta\sin\theta + 4\sin^3\theta\cos\theta = 4\sin\theta\cos\theta\left(\sin^2\theta - \cos^2\theta\right).$$
We notice that $g'(\theta) = 0$ iff $\theta = 0, \pi, 2\pi, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. We see that $g(0) = g(\pi) = g(2\pi) = g(\pi/2) = g(3\pi/2) = g(5\pi/2) = 1$, while $g(\pi/4) = g(3\pi/4) = g(5\pi/4) = g(7\pi/4) = 1/2$.

Candidates max points on D are then: $(0,0)$, $(\pm 3,0), (0,\pm 3)$ and since $f(0,0) = 0$ and $f(\pm 3,0) = f(0,\pm 3) = 9$ we deduce that max points for f on D are $(\pm 3,0), (0,\pm 3)$. □

2.9.15. #1 See slides.

#2 See slides.

#3 <u>Existence.</u> Let $D := \{(x,y) \ : \ x^2+y^2+xy-1 = 0\} = \{g(x,y) = 0\}$. Being g continuous, D is closed. Is it also bounded? We notice that, if $\rho = \|(x,y)\|$,
$$(x,y) \in D, \iff \rho^2 + \rho^2\cos\theta\sin\theta - 1 = 0, \iff \rho^2(1+\cos\theta\sin\theta) = 1,$$

that is

$$\rho^2 = \frac{1}{1+\cos\theta\sin\theta}.$$

Since

$$1+\cos\theta\sin\theta = 1+\frac{1}{2}\sin(2\theta) \geqslant 1-\frac{1}{2} = \frac{1}{2},$$

we get

$$\rho^2 \leqslant \frac{1}{1/2} = 2,$$

that is $\rho = \|(x,y)\|$ is bounded for $(x,y) \in D$. This says that D is bounded, hence compact. Since $f(x,y) = xy$ is continuous on D compact, there exist both min/max of f on D (Weierstrass' thm).

Determination. We apply Lagrange's thm. Since $D = \{g = 0\}$, let's check first if g is a submersion on D. This happens iff $\nabla g \neq \vec{0}$ on D. We gave $\nabla g = (2x+y, 2y+x)$ from which we see that $\partial_x g, \partial_y g \in \mathscr{C}$, so g is differentiable. Moreover,

$$\nabla g = \vec{0}, \quad \Longleftrightarrow \quad \begin{cases} 2x+y=0, \\ \\ 2y+x=0, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} y=-2x, \\ \\ -3x=0, \end{cases} \quad \begin{cases} x=0, \\ \\ y=0, \end{cases}$$

Therefore, there is a unique point where $\nabla g = \vec{0}$, that is $(0,0)$. However, $(0,0) \notin D$ because $0^2+0^2+0\cdot 0-1 = -1 \neq 0$. We can then conclude that $\nabla g \neq \vec{0}$ on D, that is g is a submersion on D. Since clearly also f is differentiable, Lagrange's thm applies: at any min/max point for f on D we have

$$\nabla f = \lambda \nabla g, \quad \Longleftrightarrow \quad \text{rank}\begin{bmatrix} y & x \\ 2x+y & 2y+x \end{bmatrix} < 2, \quad \Longleftrightarrow \quad \det\begin{bmatrix} y & x \\ 2x+y & 2y+x \end{bmatrix} = 0.$$

This yields the equation

$$y(2y+x) - x(2x+y) = 0, \quad \Longleftrightarrow \quad y^2 - x^2 = 0, \quad \Longleftrightarrow \quad y = x, \ \vee\, y = -x.$$

Thus we get points (x,x) and $(x,-x)$ with $x \in \mathbb{R}$. Now $(x,x) \in D$ iff $x^2+x^2+xx-1 = 0$, that is $3x^2 = 1$, from which $x = \pm\frac{1}{\sqrt{3}}$, so we get pts $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. On these pts $f = \frac{1}{3}$. Similarly, $(x,-x) \in D$ iff $x^2+x^2-xx-1 = 0$, from which $x^2 = 1$, or $x = \pm 1$. We get points $\pm(1,-1)$, on which $f = -1$. We conclude that $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are max pts, $\pm(1,-1)$ are min pts.

#4. Existence. Let $D = \{x^2+4y^2 = 4\}$. D is clearly closed and bounded, hence compact. Since $f \in \mathscr{C}$, both min and max of f on D exist because of Weierstrass' thm.

Determination. We apply Lagrange's multiplier thm. Let's start checking that $D = \{g(x,y) = 0\}$ with g a submersion on D. Clearly, g is differentiable and since $\nabla g = (2x, 8y) = \vec{0}$ iff $(x,y) = \vec{0} \notin D$, we have that $\nabla g \neq \vec{0}$, so g is a submersion on D. Therefore, at min/max points we have

$$\nabla f = \lambda\nabla g, \quad \Longleftrightarrow \quad \text{rank}\begin{bmatrix} 2x-\frac{y}{2} & 10y-\frac{x}{2} \\ 2x & 8y \end{bmatrix} < 2, \quad \Longleftrightarrow \quad \det\begin{bmatrix} 2x-\frac{y}{2} & 10y-\frac{x}{2} \\ 2x & 8y \end{bmatrix} = 0.$$

This yields the equation

$$8y(2x-\frac{y}{2}) - 2x(10y-\frac{x}{2}) = 0, \quad \Longleftrightarrow \quad x^2-4y^2-4xy = 0.$$

This last can be also written as

$$x^2-4xy+4y^2=8y^2, \iff (x-2y)^2=8y^2, \iff x-2y=\pm 2\sqrt{2}y, \iff x=2(1\pm\sqrt{2})y.$$

We get points $(2(1\pm\sqrt{2})y, y)$ with $y\in\mathbb{R}$. Now, such point belongs to D iff

$$\left(2(1\pm\sqrt{2})y\right)^2+4y^2=4, \iff (1+(1\pm\sqrt{2})^2)y^2=1, \ \ y=\pm\frac{1}{\sqrt{1+(1\pm\sqrt{2})^2}}.$$

Thus min/max points are among

$$\pm\frac{1}{\sqrt{1+(1\pm\sqrt{2})^2}}\left(2(1\pm\sqrt{2}),1\right).$$

Notice that $\pm$ in front of the two $\sqrt{2}$ is the same sign (both + or both −), while the $\pm$ in front of everything is independent of the first one. So we have in total four points. Now, since f depends on x^2, y^2 and xy, the value of f is independent of the front $\pm$, and

$$\begin{aligned} f\left(\frac{1}{\sqrt{1+(1+\sqrt{2})^2}}\left(2(1+\sqrt{2}),1\right)\right) &= \frac{4(1+\sqrt{2})^2}{1+(1+\sqrt{2})^2}+\frac{5}{1+(1+\sqrt{2})^2}-\frac{1}{2}\frac{2(1+\sqrt{2})}{1+(1+\sqrt{2})^2} \\ &= \frac{5+3(1+\sqrt{2})^2}{1+(1+\sqrt{2})^2}=\frac{7+3\sqrt{2}}{2+\sqrt{2}}=4-\frac{\sqrt{2}}{2} \end{aligned}$$

while

$$\begin{aligned} f\left(\frac{1}{\sqrt{1+(1-\sqrt{2})^2}}\left(2(1-\sqrt{2}),1\right)\right) &= \frac{4(1-\sqrt{2})^2}{1+(1-\sqrt{2})^2}+\frac{5}{1+(1-\sqrt{2})^2}-\frac{1}{2}\frac{2(1-\sqrt{2})}{1+(1-\sqrt{2})^2} \\ &= \frac{5+3(1-\sqrt{2})^2}{1+(1-\sqrt{2})^2}=\frac{7-3\sqrt{2}}{2-\sqrt{2}}=4+\frac{\sqrt{2}}{2} \end{aligned}$$

from which we conclude that

- $\pm\frac{1}{\sqrt{1+(1+\sqrt{2})^2}}\left(2(1+\sqrt{2}),1\right)$ are min points for f on D,
- $\pm\frac{1}{\sqrt{1-(1-\sqrt{2})^2}}\left(2(1-\sqrt{2}),1\right)$ are max points for f on D. □

2.9.16. #1 See slides.

#2 Existence. Clearly, D is closed and bounded, f is continuous, therefore $\exists \min_D f, \max_D f$.

Determination. We apply Lagrange's thm. Since $D=\{g(x,y,z)=0\}$ where $g=x^2+y^2+z^2-1$, we start checking that g is a submersion on D. Clearly g is differentiable. We have $\nabla g=(2x,2y,2z)=\vec{0}$ iff $(x,y,z)=\vec{0}\notin D$, so g is a submersion on D.

Therefore, at any (x,y,z) min/max point for f on D we have

$$\nabla f=\lambda\nabla g, \iff \operatorname{rk}\begin{bmatrix} yz^2e^{xy} & xz^2e^{xy} & 2ze^{xy} \\ 2x & 2y & 2z \end{bmatrix}<2.$$

Now, this happens iff all 2×2 sub=determinants of this matrix vanish: we get the system

$$\begin{cases} \det \begin{bmatrix} yz^2e^{xy} & xz^2e^{xy} \\ 2x & 2y \end{bmatrix} = 0, \\ \\ \det \begin{bmatrix} yz^2e^{xy} & 2ze^{xy} \\ 2x & 2z \end{bmatrix} = 0, \\ \\ \det \begin{bmatrix} xz^2e^{xy} & 2ze^{xy} \\ 2y & 2z \end{bmatrix} = 0 \end{cases} \iff \begin{cases} 2e^{xy}z^2(y^2 - x^2) = 0, \\ \\ 2e^{xy}z(yz^2 - x) = 0, \\ \\ 2e^{xy}x(xz^2 - y) = 0. \end{cases}$$

Let's work on the first equation. We have

$$2e^{xy}z^2(y^2 - x^2) = 0, \quad \iff \quad z = 0, \ \vee \ y = x, \ \vee \ y = -x.$$

Case $z = 0$: the system reduces to

$$xy = 0, \quad \iff \quad x = 0, \ \vee \ y = 0.$$

So, we get points $(x, 0, 0)$, $(0, y, 0)$, with $x, y \in \mathbb{R}$. These points belong to D iff $x^2 = y^2 = 1$, so $x = y = \pm 1$, we get points $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$.

Case $y = x$. We can also consider $z \neq 0$ (otherwise we are in previous case). The system reduces to

$$\begin{cases} x(z^2 - 1) = 0, \\ \\ x^2(z^2 - 1) = 0, \end{cases} \iff \quad x = 0, \ \vee \ z = \pm 1.$$

We get points $(0, 0, z)$ with $z \in \mathbb{R}$ and $(x, x, \pm 1)$, $x \in \mathbb{R}$). Now, $(0, 0, z) \in D$ iff $z^2 = 1$, that is $z = \pm 1$, so points $(0, 0, \pm 1)$. We have $(x, x, \pm 1) \in D$ iff $x^2 + x^2 + 1 = 1$, that is $2x^2 = 0$, so $x = 0$, and we get $(0, 0, \pm 1)$ again.

Finally, case $y = -x$. As above, we may assume also $z \neq 0$. The system boils down to

$$\begin{cases} x(z^2 + 1) = 0, \\ \\ x^2(z^2 + 1) = 0, \end{cases} \iff \quad x = 0.$$

So we get points $(0, 0, z)$ and as above this yields points $(0, 0, \pm 1)$.

We can now conclude. The unique candidate min/max points are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$. We have

$$f(\pm 1, 0, 0) = 0 = f(0, \pm 1, 0), \ f(0, 0, \pm 1) = 1,$$

so $(\pm 1, 0, 0)$ and $(0, \pm 1, 0)$ are minimum points, $(0, 0, \pm 1)$ are maximum points. □

2.9.18. See slides.

2.9.19. i) We have $(x, 0, 0) \in D$ iff $x^4 = 1$, so $(\pm 1, 0, 0) \in D$ and $D \neq \emptyset$. We also have $D = \{g(x, y, z) = 0\}$ where $g(x, y, z) = (x^2 + y^2)z^2)^2 - xyz - 1$. Clearly g is differentiable and

$$\nabla g(x, y, z) = \left(4x(x^2 + y^2 + z^2) - yz, 4y(x^2 + y^2 + z^2) - xz, 4z(x^2 + y^2 + z^2) - xy\right).$$

To check if g is a submersion on $\{g = 0\}$. we have to determine if $\nabla g \neq \vec{0}$ on $\{g = 0\}$. Let's start determining points where $\nabla g = \vec{0}$: we have

$$\nabla g(x, y, z) = \vec{0}, \quad \iff \quad \begin{cases} 4x(x^2 + y^2 + z^2) - yz = 0, \\ 4y(x^2 + y^2 + z^2) - xz = 0, \\ 4z(x^2 + y^2 + z^2) - xy = 0. \end{cases}$$

To solve this system we may argue as follows: either $x \neq 0$, or $x = 0$. Let's discuss first $x = 0$: the system reduces to

$$\begin{cases} yz = 0, \\ 4y(y^2 + z^2) = 0, \\ 4z(y^2 + z^2) = 0. \end{cases} \quad \iff \quad (x, y, z) = (0, 0, 0).$$

In the case $x \neq 0$, the system becomes

$$\begin{cases} 4(x^2 + y^2 + z^2) = \frac{yz}{x}, \\ \\ y\frac{yz}{x} - xz = 0, \quad \iff \quad z(y^2 - x^2) = 0, \quad \iff \quad z = 0, \ \vee\ y = x, \ \vee\ y = -x \\ \\ z\frac{yz}{x} - xy = 0 \quad \iff \quad y(z^2 - x^2) = 0. \end{cases}$$

Now $z(y^2 - x^2) = 0$, iff $z = 0$ or $y = x$ or $y = -x$. In the first case the system reduces to

$$\begin{cases} z = 0, \\ x^2 + y^2 = 0, \\ xy = 0, \end{cases} \quad \iff \quad (x, y, z) = (0, 0, 0)$$

In the case $y = x$ we have

$$\begin{cases} y = x, \\ 4(2x^2 + z^2) = z, \\ x(z^2 - x^2) = 0. \end{cases} \quad \iff \quad \begin{aligned} &\begin{cases} y = x = 0, \\ 4z^2 = z, \end{cases} \quad \iff \quad (x, y, z) = (0, 0, 0), (0, 0, 1/4) \\ &\begin{cases} y = x \neq 0, \\ z = x, \\ 12x^2 = x, \end{cases} \quad \iff \quad (x, y, z) = (1/12, 1/12, 1/12) \\ &\begin{cases} y = x \neq 0, \\ z = -x, \\ 12x^2 = -x, \end{cases} \quad \iff \quad (x, y, z) = (-1/12, -1/12, 1/12) \end{aligned}$$

In the case $y = -x$ we have

$$\begin{cases} y = -x = 0, \\ 4z^2 = -z, \end{cases} \iff (x, y, z) = (0, 0, 0), (0, 0, -1/4)$$

$$\begin{cases} y = -x, \\ 4(2x^2 + z^2) = -z, \\ -x(z^2 - x^2) = 0. \end{cases} \iff \begin{cases} y = -x \neq 0, \\ z = x, \\ 12x^2 = -x, \end{cases} \iff (x, y, z) = (-1/12, 1/12, -1/12)$$

$$\begin{cases} y = -x \neq 0, \\ z = -x, \\ 12x^2 = x, \end{cases} \iff (x, y, z) = (1/12, -1/12, -1/12)$$

Let's now check which of these points are in D:

- $(0, 0, 0) \in D$ iff $g(0, 0, 0) = 0$. However, $g(0, 0, 0) = -1 \neq 0$, so $(0, 0, 0) \notin D$.
- $g(0, 0, \pm\frac{1}{4}) = \frac{9}{16^2} - 1 < 0$, so $(0, 0, 1/4) \notin D$.
- $g(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}) = g(-\frac{1}{12}, -\frac{1}{12}, \frac{1}{12}) = g(-\frac{1}{12}, \frac{1}{12}, -\frac{1}{12}) = g(\frac{1}{12}, -\frac{1}{12}, -\frac{1}{12}) = \frac{9}{12^4} - \frac{1}{12^3} - 1 < 0$, so none of these points belong to $\{g = 0\}$.

Conclusion: none of points where $\nabla g = \vec{0}$ belongs to $\{g = 0\}$, so g is a submersion on $\{g = 0\}$.

ii) D is defined by an equation $g = 0$ with $g \in \mathscr{C}$, therefore it is closed. Let's check that it is also bounded: indeed, if $\rho := \|(x, y, z)\|$ we have that

$$(x, y, z) \in D, \iff \rho^4 - \rho^3 \cos\theta \sin\theta \sin^2\varphi \cos\varphi = 1,$$

so

$$\rho^4 = 1 + \rho^3 \cos\theta \sin\theta \sin^2\varphi \cos\varphi \leqslant 1 + \rho^3.$$

It is now clear that ρ must be bounded. Indeed, if ρ is unbounded it means there are points in D with ρ arbitrarily large. However, when $\rho \to +\infty$, the inequality $\rho^4 \leqslant 1 + \rho^3$ is impossible because $1 + \rho^3 = o(\rho^4)$. We conclude that ρ must be bounded, whence D is bounded, and being also closed, it is compact.

iii) We have to determine

$$\min_D / \max_D (x^2 + y^2 + z^2).$$

Let $f = x^2 + y^2 + z^2$. Clearly f is continuous, D is compact and Weierstrass' thm applies: this ensures existence of both min/max of f on D.

To determine these points we apply Lagrange's multiplier theorem. We already proved that g is a submersion on D. Since f is clearly differentiable, at any min/max point it must hold

$$\nabla f = \lambda \nabla g, \iff \operatorname{rk}\begin{bmatrix} \nabla f \\ \\ \nabla g \end{bmatrix} = \operatorname{rk}\begin{bmatrix} 2x & 2y & 2z \\ 4x\rho^2 - yz & 4y\rho^2 - xz & 4z\rho^2 - xy \end{bmatrix} < 2.$$

(we wrote $\rho^2 = x^2 + y^2 + z^2$). From this we get the system:

$$\begin{cases} 2x(4y\rho^2 - xz) - 2y(4x\rho^2 - yz) = 0, \\ 2x(4z\rho^2 - xy) - 2z(4x\rho^2 - yz) = 0, \\ 2y(4z\rho^2 - xy) - 2z(4y\rho^2 - xz) = 0. \end{cases} \iff \begin{cases} z(y^2 - x^2) = 0, \\ y(x^2 - z^2) = 0, \\ x(y^2 - z^2) = 0, \end{cases}$$

Working on the first equation, we get the alternatives

$$\begin{cases} z = 0, \\ yx^2 = 0, \\ xy^2 = 0, \end{cases} \iff (x, 0, 0), (0, y, 0),\ x, y \in \mathbb{R},$$

or

$$\begin{cases} y = x, \\ x(x^2 - z^2) = 0, \end{cases} \iff (0, 0, z), (x, x, x), (x, x, -x),\ z, x \in \mathbb{R},$$

or, again

$$\begin{cases} y = -x, \\ x(x^2 - z^2) = 0, \end{cases} \iff (0, 0, z), (x, -x, x), (x, -x, -x),\ z, x \in \mathbb{R},$$

We have now to determine which of these points are in D:

- $(x, 0, 0) \in D$ iff $x^4 - x^3 = 1$ which has two real solutions $-0, 81, +1, 38$.
- $(0, y, 0) \in D$ iff $y^4 - y^3 = 1$ which has two real solutions $-0, 81, +1, 38$.
- $(0, 0, z) \in D$ iff $z^4 - z^3 = 1$ which has two real solutions $-0, 81, +1, 38$.
- $(x, x, x), (x, -x, -x) \in D$ iff $9x^4 - x^3 = 1$, which has real solutions $-0, 55, +0, 60$
- $(x, x, -x), (x, -x, x) \in D$ iff $9x^4 + x^3 = 1$, which has real solutions $+0, 55, -0, 60$

Computing distances of these points to $\vec{0}$, we easily see that $(1.38, 0, 0), (0, 1.38, 0), (0, 0, 1.38)$ are points of D at maximum distance to $\vec{0}$, while points $(-.81, 0, 0), (0, -.81, 0), (0, 0, -.81)$ are those at min distance to $\vec{0}$. □

2.9.22. See slides.

2.9.25. Let $D := \{(x, y, z) \in \mathbb{R}^3 \ :\ x^2 + y^2 + z^2 = 1,\ x^2 + y^2 = x\}$. We have to determine

$$\min_D/\max_D \|(x, y, z) - (0, 1, 0)\| = \min_D/\max_D \sqrt{x^2 + (y - 1)^2 + z^2}.$$

Since min/max points of $\sqrt{x^2 + (y - 1)^2 + z^2}$ are the same of $x^2 + (y - 1)^2 + z^2$, we solve for

$$\min_D/\max_D \underbrace{(x^2 + (y - 1)^2 + z^2)}_{:=f(x,y,z)}.$$

We start with the existence. We notice that D is closed being $D\{g_1 = 0,\ g_2 = 0\}$ with $g_1 = x^2 + y^2 + z^2 - 1$ and $g_2 = x^2 + y^2 - x$ continuous functions. Moreover, D is bounded, because if $(x, y, z) \in D$, in particular, $x^2 + y^2 + z^2 = 1$, that is $\|(x, y, z)\| = 1$. Therefore, D is compact, and since f is clearly continuous, existence of $\min_D f$ and $\max_D f$ is ensured by Weierstrass' thm.

We now move to the search of extrema. To this aim, we wish to apply Lagrange's multipliers thm. We start checking that (g_1, g_2) is a submersion on D. We notice that g_1, g_2 are differentiable and

$$\nabla g_1 = (2x, 2y, 2x),\ \ \nabla g_2 = (2x - 1, 2y, 0).$$

The map (g_1, g_2) is not a submersion at (x, y, z) iff

$$\mathrm{rk}\begin{bmatrix} \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \mathrm{rk}\begin{bmatrix} 2x & 2y & 2z \\ 2x-1 & 2y & 0 \end{bmatrix} < 2, \iff \begin{cases} 4xy - 2y(2x-1) = 0, \\ 2z(2x-1) = 0, \\ 4yz = 0. \end{cases} \iff \begin{cases} y = 0, \\ \\ z(2x-1) = 0. \end{cases}$$

We get points $(x, 0, 0)$ and $(1/2, 0, z)$. Now, we have to check whether these points are in D or not. We have

$$(x, 0, 0) \in D \iff \begin{cases} x^2 = 1, \\ \\ x^2 = x, \end{cases} \iff (\pm 1, 0, 0),$$

and

$$(1/2, 0, z) \in D \iff \begin{cases} \frac{1}{4} + z^2 = 1, \\ \\ \frac{1}{4} = \frac{1}{2}, \end{cases} \text{impossible!}$$

From this we get that the unique points of D where (g_1, g_2) is not a submersion are $(\pm 1, 0, 0)$. So, if $\widetilde{D} := D \backslash \{(\pm 1, 0, 0)\}$, (g_1, g_2) is a submersion on $\widetilde{D}$ and Lagrange's thm applies on $\widetilde{D}$.

So let $(x, y, z) \in D$ be any min/max point. We have the following alternative:

- either $(x, y, z) = (\pm 1, 0, 0)$;
- or $(x, y, z) \neq (\pm 1, 0, 0)$, in particular $(x, y, z) \in \widetilde{D}$. Therefore, Lagrange's thm applies and

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \iff \mathrm{rk}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} < 3, \iff \det\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = 0.$$

Now,

$$\begin{aligned} \det\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} &= \det\begin{bmatrix} 2x & 2(y-1) & 2z \\ 2x & 2y & 2z \\ 2x-1 & 2y & 0 \end{bmatrix} \\ &= (2x-1)(4z(y-1) - 4zy) - 2y(4xz - 4xz) + 0 \cdots \\ &= -4(2x-1) = 0, \iff x = \tfrac{1}{2}. \end{aligned}$$

Therefore $(x, y, z) = (\frac{1}{2}, y, z)$ which belongs to D iff

$$\begin{cases} \frac{1}{4} + y^2 + z^2 = 1, \\ \\ \frac{1}{4} + y^2 = \frac{1}{2} \end{cases} \iff \begin{cases} y = \pm\frac{1}{2}, \\ \\ z^2 = \frac{1}{2}, \end{cases} \iff \left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$$

(four points).

Conclusion: the candidates min/max points are $(\pm 1, 0, 0)$ and $\left(\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$. Since

$$f(\pm 1, 0, 0) = 2, \quad f\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1, \quad f\left(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = \frac{1}{4} + \frac{9}{4} + \frac{1}{2} = \frac{12}{4} = 3$$

we conclude that $\left(\frac{1}{2}, +\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ are points of D at min distance to $(0,1,0)$, while $\left(\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right)$ are those at maximum distance. □

2.9.26. Existence. Let

$$D = \{(x,y,z) \in \mathbb{R}^3 : \; z = x^2 + y^2, \;\; x + y + z = 0\} = \{g_1 = 0, \; g_2 = 0\},$$

where

$$g_1(x,y,z) := z - x^2 - y^2, \qquad g_2(x,y,z) := x + y + z.$$

Clearly, D is closed (defined by equalities involving the continuous functions g_1, g_2. Let's discuss if D is also bounded. We notice that, if $(x,y,z) \in D$, then

$$z = x^2 + y^2, \implies 0 = x + y + x^2 + y^2, \iff \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4},$$

from which $\left|x + \frac{1}{2}\right| \leqslant \frac{1}{2}$, that is $-\frac{1}{2} \leqslant x + \frac{1}{2} \leqslant \frac{1}{2}$, or $-1 \leqslant x \leqslant 0$ and, similarly, $-1 \leqslant y \leqslant 0$. So x, y are bounded, and since $z = x^2 + y^2$ we also have $0 \leqslant z \leqslant 1 + 1 = 2$, thus also z is bounded. We conclude that D is bounded, whence compact, and since $f(x,y,z) = z$ is continuous, by Weierstrass' thm f takes both min and max on D.

Search. We apply Lagrange's multipliers theorem. To this aim, let's first check that (g_1, g_2) is a submersion. Clearly, g_1, g_2 are both differentiable. Notice that

$$\operatorname{rank}\begin{bmatrix} \nabla g_1 \\ \\ \nabla g_2 \end{bmatrix} = \begin{bmatrix} -2x & -2y & 1 \\ \\ 1 & 1 & 1 \end{bmatrix} < 2, \iff \begin{cases} 2(y-x) = 0, \\ 2x + 1 = 0, \\ 2y + 1 = 0, \end{cases} \iff \left(\frac{1}{2}, \frac{1}{2}, z\right).$$

Now,

$$\left(\frac{1}{2}, \frac{1}{2}, z\right) \in D, \iff \begin{cases} z = \frac{1}{2}, \\ \\ \frac{1}{2} + \frac{1}{2} + z = 0, \end{cases}, \; \nexists z.$$

We conclude that (g_1, g_2) is submersive on D. According to Lagrange's thm, at min/max points we have

$$\nabla f(x,y,z) = \lambda \nabla g_1(x,y,z) + \mu \nabla g_2(x,y,z), \iff \operatorname{rank}\begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 1 \\ -2x & -2y & 1 \\ 1 & 1 & 1 \end{bmatrix} < 3,$$

that is, iff

$$1 \cdot \det\begin{bmatrix} -2x & -2y \\ 1 & 1 \end{bmatrix} = 2(y - x) = 0, \iff y = x.$$

This yields points (x,x,z). We have

$$(x,x,z) \in D, \iff \begin{cases} z = 2x^2, \\ \\ z = -2x, \end{cases} \iff \begin{cases} x^2 + x = 0, \iff x(x+1) = 0, \\ \\ z = 2x^2, \end{cases}$$

which yields $x = z = 0$, or $x = -1$, $z = 2$. We obtain points $(0,0,0)$ and $(-1 - 1, 2)$. Since $f(0,0,0) = 0$ and $f - 1 - 1, 2) = 2$ we see that $(0,0,0)$ is the min point while $-1 - 1, 2)$ is the max point. □

2.9.27. i) Define

$$G : \mathbb{R}^3 \to \mathbb{R}^2, \qquad G(x,y,z) = \big(g_1(x,y,z),\ g_2(x,y,z)\big) := \big(x^2+y^2+z^2-1,\ 2z-3x\big).$$

To show that G is a submersion on D it suffices to show that the Jacobian matrix of G has rank 2 for every $(x,y,z) \in D$. We have

$$\operatorname{rank} G' = \operatorname{rank} \begin{bmatrix} 2x & 2y & 2z \\ \\ -3 & 0 & 2 \end{bmatrix} < 2, \iff \begin{cases} 6y = 0, \\ 4x + 6z = 0, \\ 4y = 0, \end{cases} \iff (x, 0, -\frac{2}{3}x).$$

Now,

$$(x, 0, -\frac{2}{3}x) \in D, \iff \begin{cases} x^2 + \frac{4}{9}x^2 = 1, \\ \\ -\frac{4}{3}x = 3x \end{cases} \iff \begin{cases} x = 0, \\ \\ 0 = 1, \end{cases} \text{impossible!}$$

Therefore, $\operatorname{rank} G' = 2$ at every point of D, thus G is a submersion on D. Finally, to see $D \neq \varnothing$ note e.g. that the point $(x,y,z) = (0,1,0)$ satisfies $x^2+y^2+z^2 = 1$ and $2z - 3x = 0$. Hence D is nonempty.

ii) The set D is closed and contained in the unit sphere $S^2 = \{x^2+y^2+z^2 = 1\}$, so it is bounded, thus D is compact.

iii) Existence: $f(x,y,z) = xz$ is clearly continuous on D, D is compact, the existence of min/max points for f on D follows from Weierstrass' thm.

Search. We apply Lagrange's multipliers thm. By i), G is a submersion on D. Therefore, at any min/max point (x,y,z) we must have

$$\operatorname{rank} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \nabla g_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} z & 0 & x \\ 2x & 2y & 2z \\ -3 & 0 & 2 \end{bmatrix} < 3, \iff \det \begin{bmatrix} z & 0 & x \\ 2x & 2y & 2z \\ -3 & 0 & 2 \end{bmatrix} = z4y + x6y = 0,$$

that is $y(2z+3x) = 0$. This yields $y = 0$ or $z = -\frac{3}{2}x$, that is points $(x,0,z)$ and $(x,y,-\frac{3}{2}x)$. Now,

$$(x,0,z) \in D, \iff \begin{cases} x^2 + z^2 = 1, \\ \\ 2z - 3x = 0, \end{cases} \iff \begin{cases} x^2 + \frac{9}{4}x^2 = 1, \iff x^2 = \frac{4}{13}, \iff x = \pm\frac{2}{\sqrt{13}}. \\ \\ z = \frac{3}{2}x, \end{cases}$$

We get points $\pm\left(\frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}}\right)$.

$$(x, y, -\frac{3}{2}x) \in D, \begin{cases} x^2 + y^2 + \frac{9}{4}x^2 = 1, \\ \\ -3x - 3x = 0, \iff x = 0, \end{cases} \iff \begin{cases} y^2 = 1, \\ \\ x = 0, \end{cases} \iff (0, \pm 1, 0).$$

Conclusion. The function f attains its minimum value 0 at $(0,\pm 1,0)$, and its maximum value $6/13$ at the two points $\big(\pm 2/\sqrt{13}, 0, \pm 3/\sqrt{13}\big)$.

2.9.31. Since minimizing $\|\vec{x}\|$ is the same of minimizing $\|\vec{x}\|^2 = x_1^2 + \cdots + x_d^2 =: f(x_1,\ldots,x_d)$, we discuss the problem

$$\min_D f,$$

where $D = \{\vec{x} \in \mathbb{R}^d \ \ \vec{a} \cdot \vec{x} = 1\}$. Clearly, D is closed and unbounded, and since

$$\lim_{\vec{x} \to \infty_d} f(\vec{x}) = +\infty,$$

we conclude that f has global minimum on D.

To determine the minimum, we apply Lagrange's thm. Since $D = \{g(\vec{x}) = 0\}$, where $g(\vec{x}) = \vec{a} \cdot \vec{x} - 1$, we start checking that g is a submersion on D. This happens iff $\nabla g \neq \vec{0}$ on D. Since $\nabla g = \vec{a} \neq 0$ by hypothesis, we conclude that g is a submersion. Therefore, if $\vec{x}$ is any minimum point for f, it must be

$$\nabla f = \lambda \nabla g, \quad \Longleftrightarrow \quad 2\vec{x} = \lambda \vec{a}, \quad \Longleftrightarrow \quad \vec{x}^* = \lambda \vec{a}.$$

Now, $\vec{x}^* \in D$ iff $\lambda \vec{a} \cdot \vec{a} = 1$, that is $\lambda = \frac{1}{\|\vec{a}\|^2}$. Therefore $\vec{x}^* = \frac{\vec{a}}{\|\vec{a}\|^2}$. □

3.4.1. #1 $\vec{F} = (x, y-1)$ is irrotational iff $\partial_y x \equiv \partial_x (y-1)$, that is $0 \equiv 0$, which is true. It is conservative iff $\vec{F} = \nabla f$, that is, iff there exists f such that

$$\begin{cases} \partial_x f = x, \\ \\ \partial_y f = y - 1. \end{cases}$$

We have

$$\partial_x f(x, y) = x, \implies f(x, y) = \int x \, dx + c(y) = \frac{x^2}{2} + c(y).$$

Imposing the second equation

$$\partial_y f = y - 1, \implies c'(y) = y - 1, \iff c(y) = \frac{y^2}{2} - y + c,$$

from which we obtain

$$f(x, y) = \frac{x^2 + y^2}{2} - y + c, \ c \in \mathbb{R}.$$

It is now easy to checl that this f is a potential for $\vec{F}$.

3.4.2. The field $\vec{F}(x, y) = ax^3 + by + 3x^2y^2, cx^4 + 2x^3y + 1)$ is irrotational on $\mathbb{R}^2$ iff

$$\partial_y (ax^3 + by + 3x^2y^2) \equiv \partial_x (cx^4 + 2x^3y + 1), \iff b + 6x^2y \equiv 4cx^3 + 6x^2y, \iff 4cx^3 - b \equiv 0.$$

This is possible iff $c = b = 0$, so $\vec{F} = (ax^3 + 3x^2y^2, 2x^3y + 1)$. This field is conservative iff $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = ax^3 + 3x^2y^2, \\ \\ \partial_y f = 2x^3y + 1. \end{cases}$$

From the first equation we have

$$f(x, y) = \int \left(ax^3 + 3x^2y^2 \right) \, dx = \frac{a}{4}x^4 + x^3y^2 + c(y).$$

Imposing the second equation we get

$$2x^3y + c'(y) = 2x^3y + 1, \iff c'(y) = 1, \iff c(y) = y + k, \ k \in \mathbb{R},$$

from which we see that, if a potential f exists, then

$$f(x,y) = \frac{a}{4}x^4 + x^3y^2 + y + k.$$

It is now a straightforward check to verifdy that this f is a potential of $\vec{F}$. $\square$

3.4.6. Let $\vec{F}(x,y,z) = (a(x,y,z), x^2+2yz, y^2-z^2)$ for $(x,y,z) \in \mathbb{R}^3$. The field $\vec{F}$ is irrotational on $\mathbb{R}$ iff

$$\begin{cases} \partial_y a = \partial_x(x^2+2yz) = 2x, \\ \partial_z a = \partial_x(y^2-z^2) = 0, \\ \partial_z(y^2+2yz) = \partial_y(y^2-z^2), \end{cases} \iff \begin{cases} \partial_y a = 2x, \\ \partial_z a = 0, \\ 2y = 2y, \ \checkmark \end{cases}$$

From $\partial_y a = 2x$ we get

$$a(x,y,z) = \int 2x \ dy + c(x,z) = 2xy + c(x,z).$$

Imposing $\partial_z a = 0$ we get $\partial_z c = 0$, that is $c(x,z) = c(x)$, so

$$a(x,y,z) = 2xy + c(x).$$

This is the form of a in order $\vec{F}$ be irrotational. If we add the extra condition $a(x,0,0) = 0$ we get $c(x) \equiv 0$, so $a(x,y,z) = 2xy$, which is unique.

For such a the field is $\vec{F} = (2xy, x^2+2yz, y^2-z^2)$. This field is conservative iff $\exists f$ such that $\vec{F} = \nabla f$, that is

$$\begin{cases} \partial_x f = 2xy, \\ \partial_y f = x^2 + 2yz, \\ \partial_z f = y^2 - z^2. \end{cases}$$

The first equation yields,

$$f(x,y,z) = \int 2xy \ dx + c(y,z) = x^2y + c(y,z).$$

Imposing the second equation we get

$$x^2 + \partial_y c(y,z) = x^2 + 2yz, \iff \partial_y c(y,z) = 2yz, \iff c(y,z) = \int 2yz \ dy + c(z) = y^2z + c(z),$$

so

$$f(x,y,z) = x^2y + y^2z + c(z).$$

Finally, imposing also the third equation from the system $\vec{F} = \nabla f$, we get

$$y^2 + c'(z) = y^2 - z^2, \iff c'(z) = -z^2, \iff c(z) = -\frac{z^3}{3} + k,$$

where now $k \in \mathbb{R}$ is just a constant. We conclude that the potentials of $\vec{F}$ are

$$f(x,y,z) = x^2y + y^2z - \frac{z^3}{3} + k, \ k \in \mathbb{R}. \quad \square$$

3.4.8. i) $\vec{F}$ is irrotational iff

$$\partial_y \frac{ax^2+by^2}{(x^2+y^2)^2} \equiv \partial_x \frac{cxy}{(x^2+y^2)^2}, \text{ on } D = \mathbb{R}^2 \backslash \{\vec{0}\},$$

that is, iff

$$\frac{(2by)(x^2+y^2)-4(ax^2+by^2)y}{(x^2+y^2)^3} \equiv \frac{cy(x^2+y^2)-4cx^2y}{(x^2+y^2)^3},\ \forall (x,y)\neq \vec{0},$$

which is equivalent to

$$(2b-4a)x^2y - 2by^3 = -3cx^2y + cy^3,\ \ \forall (x,y)\in\mathbb{R}^2.$$

This is possible iff

$$\begin{cases} 2b-4a=-3c, \\ \\ -2b=c, \end{cases} \iff \begin{cases} 4a = 2b+3c = 2b-6b = -4b, & \iff \quad a=-b \\ \\ c=-2b. \end{cases}$$

Therefore, $\vec{F}$ is irrotational iff

$$\vec{F}(x,y) = b\left(\frac{y^2-x^2}{(x^2+y^2)^2}, \frac{-2xy}{(x^2+y^2)^2}\right).\ (\star)$$

ii) To be conservative, $\vec{F}$ must be irrotational, so $\vec{F}$ is given by $(\star)$. Now, this field is conservative iff there exists f such that

$$\begin{cases} \partial_x f = b\frac{y^2-x^2}{(x^2+y^2)^2}, \\ \\ \partial_y f = b\frac{-2xy}{(x^2+y^2)^2}. \end{cases}$$

From the second equation

$$f(x,y) = \int b\frac{-2xy}{(x^2+y^2)^2}\,dy + c(x)$$

and noticed that

$$\partial_y \frac{1}{x^2+y^2} = \partial_y (x^2+y^2)^{-1} = -(x^2+y^2)^{-2}(2y) = -\frac{2y}{(x^2+y^2)^2},$$

we have

$$f(x,y) = \frac{bx}{x^2+y^2} + c(x).$$

Imposing the first equation,

$$\partial_x\left(\frac{bx}{x^2+y^2}+c(x)\right) = b\frac{y^2-x^2}{(x^2+y^2)^2}, \quad \iff \quad \frac{b(x^2+y^2)-2bx^2}{(x^2+y^2)^2} + c'(x) = b\frac{y^2-x^2}{(x^2+y^2)^2},$$

that is $c'(x)=0$, from which $c(x)\equiv k\in\mathbb{R}$. Therefore, in order f be a potential we have

$$f(x,y) = \frac{bx}{x^2+y^2} + k,\ k\in\mathbb{R}.$$

It is now easy to check that these are potentials of $\vec{F}$. □

3.4.10. #1 We can describe $\vec{\gamma}$ by $\vec{\gamma}(t) = (t, t^2)$, $t \in [0, 1]$. Therefore

$$\begin{aligned}
\int_\gamma \vec{F} &= \int_0^1 \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt = \int_0^1 \vec{F}(t, t^2) \cdot (1, 2t)\, dt \\
&= \int_0^1 (t^6 + t, -\sqrt{t}) \cdot (1, 2t)\, dt \\
&= \int_0^1 t^6 + t - 2t^{3/2}\, dt = \left[\frac{t^7}{7}\right]_{t=0}^{t=1} + \left[\frac{t^2}{2}\right]_{t=0}^{t=1} - 2\left[\frac{t^{5/2}}{5/2}\right]_{t=0}^{t=1} = \frac{1}{7} + \frac{1}{2} - \frac{4}{5}.
\end{aligned}$$

#2 We describe $\vec{\gamma}$ by $\vec{\gamma}(t) = (t, \sqrt{|t-1|})$, $t \in [0, 2]$. Therefore

$$\begin{aligned}
\int_\gamma \vec{F} &= \int_0^2 \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt = \int_0^2 \vec{F}(t, \sqrt{|t-1|}) \cdot \left(1, \frac{\operatorname{sgn}(t-1)}{2\sqrt{|t-1|}}\right) dt \\
&= \int_0^2 \left(|t-1|, 2t\sqrt{|t-1|} + 1\right) \cdot \left(1, \frac{\operatorname{sgn}(t-1)}{2\sqrt{|t-1|}}\right) dt \\
&= \int_0^2 |t-1| + \left(2t\sqrt{|t-1|} + 1\right) \frac{\operatorname{sgn}(t-1)}{2\sqrt{|t-1|}}\, dt. \\
&\overset{t-1=u}{=} \int_{-1}^1 |u| + \left(2(u+1)\sqrt{|u|} + 1\right) \frac{\operatorname{sgn}(u)}{2\sqrt{|u|}}\, du.
\end{aligned}$$

We notice that $\int_{-1}^1 |u|\, du = 2\int_0^1 |u|\, du = 2\int_0^1 u\, du = 2\left[\frac{u^2}{2}\right]_{u=0}^{u=1} = 1$. Next,

$$\int_{-1}^1 2u\sqrt{|u|}\frac{\operatorname{sgn}(u)}{2\sqrt{|u|}}\, du = \int_{-1}^1 |u|\, du = 1,$$

while

$$\int_{-1}^1 2\sqrt{|u|}\frac{\operatorname{sgn}(u)}{2\sqrt{|u|}}\, du = \int_{-1}^1 \operatorname{sgn}(u)\, du = 0,$$

and

$$\int_{-1}^1 \frac{\operatorname{sgn}(u)}{2\sqrt{|u|}}\, du = 0,$$

so

$$\int_{\vec{\gamma}} \vec{F} = 2.$$

#6 We have

$$\begin{aligned}\int_{\vec\gamma}\vec F &= \int_0^{2\pi} \vec F(r\cos t, r\sin t, kt)\cdot(-r\sin t, r\cos t, k)\ dt\\ &= \int_0^{2\pi} (r\sin t + kt, r\cos t + kt, r(\cos t+\sin t))\cdot(-r\sin t, r\cos t, k)\ dt\\ &= \int_0^{2\pi} -r^2\sin^2 t - krt\sin t + r^2\cos^2 t + krt\cos t + kr(\cos t+\sin t)\ dt.\end{aligned}$$

Now,

$$\int_0^{2\pi}\cos t\ dt = \int_0^{2\pi}\sin t\ dt = 0.$$

Moreover,

$$\begin{aligned}\int_0^{2\pi} -\sin^2 t\ dt &= \int_0^{2\pi}\sin t\underbrace{(-\sin t)}_{=(\cos t)'}\ dt = [\sin t\cos t]_{t=0}^{t=2\pi} - \int_0^{2\pi}\cos^2 t\ dt\\ &= -\int_0^{2\pi} 1-\sin^2 t\ dt = -2\pi + \int_0^{2\pi}\sin^2 t\ dt\end{aligned}$$

from which $-\int_0^{2\pi}\sin^2 t\ dt = -\pi$. Similarly, $\int_0^{2\pi}\cos^2 t\ dt = \pi$. Finally,

$$\int_0^{2\pi} t\sin t\ dt = [-t\cos t]_{t=0}^{t=2\pi} - \int_0^{2\pi} -\cos t\ dt = -2\pi,$$

and

$$\int_0^{2\pi} t\cos t\ dt = [t\sin t]_{t=0}^{t=2\pi} - \int_0^{2\pi}\sin t\ dt = 2\pi.$$

Merging pieces we finally have

$$\int_{\vec\gamma}\vec F = -r^2\pi - kr(-2\pi) + r^2\pi + kr2\pi = 4\pi kr. \quad\square$$

3.4.14. i) In order $\vec F = (F_1, F_2, F_3)$ be irrotational, we must check the crossed derivatives:

$$\begin{cases}\partial_y F_1 \equiv \partial_x F_2,\\ \partial_z F_1 \equiv \partial_x F_3,\\ \partial_z F_2 \equiv \partial_y F_3.\end{cases}$$

that is

$$\begin{cases}\partial_y\left(\frac{1}{x}+\frac{y^\alpha}{1+x^2y^2}\right)\equiv\partial_x\left(\frac{1}{y}+\frac{x}{1+x^2y^2}\right),\\ \partial_z\left(\frac{1}{x}+\frac{y^\alpha}{1+x^2y^2}\right)\equiv\partial_x\frac{1}{z},\\ \partial_z\left(\frac{1}{y}+\frac{x}{1+x^2y^2}\right)\equiv\partial_y\frac{1}{z}\end{cases} \iff \begin{cases}\frac{\alpha y^{\alpha-1}(1+x^2y^2)-2y^{\alpha+1}x^2}{(1+x^2y^2)^2}\equiv\frac{(1+x^2y^2)-2x^2y^2}{(1+x^2y^2)^2},\\ 0\equiv 0,\\ 0\equiv 0.\end{cases}$$

The first condition is equivalent to

$$\alpha y^{\alpha-1}(1+x^2y^2) - 2y^{\alpha+1}x^2 \equiv (1+x^2y^2) - 2x^2y^2, \quad \Longleftrightarrow \quad \alpha = 1.$$

ii) For $\alpha = 1$ the field $\vec{F}$ is conservative iff there exists $f = f(x, y, z)$ such that

$$\begin{cases} \partial_x f = \frac{1}{x} + \frac{y}{1+x^2y^2}, \\ \partial_y f = \frac{1}{y} + \frac{x}{1+x^2y^2}, \\ \partial_z f = \frac{1}{z}. \end{cases}$$

The third equation yields, for $z > 0$

$$f(x, y, z) = \log z + c(x, y).$$

Imposing the first equation we get

$$\partial_x c = \frac{1}{x} + \frac{y}{1+(xy)^2},$$

from which

$$c(x, y) = \int \frac{1}{x} + \frac{y}{1+(xy)^2}\, dx + c(y) \overset{x,y>0}{=} \log x + \arctan(xy) + c(y).$$

Thus,

$$f(x, y, z) = \log z + \log x + \arctan(xy) + c(y).$$

Plugging this into the second equation we get

$$\frac{x}{1+(xy)^2} + c'(y) = \frac{x}{1+x^2y^2}, \quad \Longleftrightarrow \quad c'(y) = 0, \quad \Longleftrightarrow \quad c(y) \equiv k \in \mathbb{R}.$$

We conclude that, if f is a potential then, necessarily,

$$f(x, y, z) = \log z + \log x + \arctan(xy) + k.\ (k \in \mathbb{R})$$

It is now easy to check that thes f are potentials of $\vec{F}$. □

5.7.1. #1,2,3,5,6,7 see notes

#4 Let $f(x, y) := \frac{1}{(x-y)^2}$. Clearly, $f \in \mathscr{C}(\mathbb{R}^2 \backslash \{y = x\})$ and since $D = [0, 1] \times [2, 4]$ is closed and bounded and contained in $\mathbb{R}^2 \backslash \{y = x\})$ we have that $f \in \mathscr{C}(D)$, so f it is integrable on D. To compute the integral we apply the reduction formula

$$\int_D f \;\; = \int_0^1 \int_2^4 \underbrace{\frac{1}{(x-y)^2}}_{\partial_y (x-y)^{-1}}\, dy\, dx = \int_0^1 \left[\frac{1}{x-y}\right]_{y=2}^{y=4} dx = \int_0^1 \left(\frac{1}{x-4} - \frac{1}{x-2}\right) dx$$

$$= [\log|x-4| - \log|x-2|]_{x=0}^{x=1} = \log 3 - (\log 4 - \log 2) = \log 3 - \log 2 = \log \frac{3}{2}.$$

#8 Let $f(x, y) = x\sqrt{y^2 - x^2}$. This function is well defined and continuous on its natural domain,

$$\{(x, y) \in \mathbb{R}^2 \;:\; y^2 - x^2 \geqslant 0\} = \{(x, y) \in \mathbb{R}^2 \;:\; x^2 \leqslant y^2\} = \{(x, y) \;:\; |x| \leqslant |y|\}.$$

Since the integration domain $D = \{(x, y) \ :\ 0 \leqslant x \leqslant y \leqslant 1\}$ is contained in the natural domain of f, we have that $f \in \mathscr{C}(D)$. Furthermore, D is clearly closed and bounded, therefore f is integrable on D. To compute the integral of f we apply the reduction formula:

$$\int_D f = \int_0^1 \int_0^y x\sqrt{y^2 - x^2}\, dx\, dy = \int_0^1 \int_x^1 x\sqrt{y^2 - x^2}\, dy\, dx$$

It seems easier to integrate first in x: we notice that

$$\partial_x \left(y^2 = x^2\right)^{3/2} = \frac{3}{2}(y^2 - x^2)^{1/2}(-2x) = -3x\sqrt{y^2 - x^2},$$

from which

$$\int_0^y x\sqrt{y^2 - x^2}\, dx = \left[-\frac{1}{3}(y^2 - x^2)^{3/2}\right]_{x=0}^{x=y} = -\frac{1}{3}\left[0 - (y^2)^{3/2}\right] = \frac{1}{3}|y|^3 \stackrel{y \geqslant 0}{=} \frac{1}{3}y^3.$$

Therefore,

$$\int_D f = \int_0^1 \frac{1}{3}y^3\, dy \frac{1}{3}\left[\frac{y^4}{4}\right]_{y=0}^{y=1} = \frac{1}{12}.$$

#9. Let $f(x, y) = \frac{x^2 e^{-x^2}}{1+(xy)^2}$. Clearly f is well defined and continuous on $\mathbb{R}^2$. About the integration domain $D = \{(x, y) \ :\ |xy| \leqslant 1\}$ we notice that it is closed but unbounded (for example: $(0, y) \in D$ for every $y \in \mathbb{R}$). We need to check integrability. We apply Tonelli's thm: we notice that since $f \geqslant 0$, $|f| = f$ so

$$\int_{D_x \neq \varnothing} \int_{D_x} |f|\, dy\, dx = \int_{D_x \neq \varnothing} \int_{D_x} f\, dy\, dx.$$

Now, notice that $(x, y) \in D$ iff $|x||y| \leqslant 1$. For $x = 0$ this means $y \in \mathbb{R}$, for $x \neq 0$ this means $|y| \leqslant \frac{1}{|x|}$, that is $-\frac{1}{|x|} \leqslant y \leqslant \frac{1}{|x|}$. Accepting that $\frac{1}{|0|} = +\infty$ we have

$$\int_{D_x \neq \varnothing} \int_{D_x} f\, dy\, dx = \int_{-\infty}^{+\infty} \int_{-1/|x|}^{1/|x|} \frac{x^2 e^{-x^2}}{1 + (xy)^2}\, dy\, dx.$$

Now,

$$\int_{-1/|x|}^{1/|x|} \frac{x^2 e^{-x^2}}{1 + (xy)^2}\, dy = xe^{-x^2} \int_{-1/|x|}^{1/|x|} \underbrace{\frac{x}{1 + (xy)^2}}_{\partial_y \arctan(xy)}\, dy = xe^{-x^2} \left[\arctan(xy)\right]_{y=-1/|x|}^{y=1/|x|}.$$

We have

$$\left[\arctan(xy)\right]_{y=-1/|x|}^{y=1/|x|} = \arctan(\frac{x}{|x|}) - \arctan(-\frac{x}{|x|}) = 2\arctan\frac{x}{|x|} = \begin{cases} 2\arctan 1 = 2\frac{\pi}{4} = \frac{\pi}{2}, & x > 0, \\ 2\arctan(-1) = -2\frac{\pi}{4} = -\frac{\pi}{2}, & x < 0. \end{cases}$$

From this we obtain

$$\int_{-1/|x|}^{1/|x|} \frac{x^2 e^{-x^2}}{1 + (xy)^2}\, dy = \begin{cases} \frac{\pi}{2}xe^{-x^2}, & x > 0, \\ -\frac{\pi}{2}xe^{-x^2}, & x < 0 \end{cases} = \frac{\pi}{2}|x|e^{-x^2}.$$

Going back to the iterated integral of $|f| = f$, we have

$$\int_{D_x \neq \emptyset} \int_{D_x} f \, dy \, dx = \int_{-\infty}^{+\infty} \frac{\pi}{2} |x| e^{-x^2} \, dx = \pi \int_0^{+\infty} \underbrace{x e^{-x^2}}_{=\partial_x - \frac{1}{2} e^{-x^2}} \, dx = \pi \left[-\frac{1}{2} e^{-x^2} \right]_{x=0}^{x=+\infty} = \frac{\pi}{2}.$$

From this, at once, we get that f is integrable on D and $\int_D f = \frac{\pi}{2}$. □

5.7.2. #1. See notes.

#2. (**Warning: this exercise contains a typo in the integration domain**) Let $f(x, y, z) := xyz$. Clearly f is continuous on $\mathbb{R}^3$. The integration domain $D = \{x \geqslant 0, y \geqslant 0, z \geqslant 0, x + y + z \leqslant 1\}$ is clearly closed. It is also bounded because, being $x, y, z \geqslant 0$ from $x + y + z \leqslant 1$ we must have $x, y, z \leqslant 1$. Indeed, if one of $x, y, z > 1$, say $x > 1$, we would have $1 \geqslant x + y + z \geqslant x > 1$, which is impossible. Therefore, the integration domain D is compact, so f is integrable on D. To compute the value of the integral, let's apply the reduction formula:

$$\int_D f = \int_{D_{x,y} \neq \emptyset} \int_{D_{x,y}} xyz \, dz \, dxdy.$$

To describe the proper parametrization of the integral we notice that

$$(x, y, z) \in D, \iff 0 \leqslant z \leqslant 1 - (x + y), \; x \geqslant 0, y \geqslant 0.$$

Here we may notice that these inequalities hide a condition on x, y: indeed, in order there can be a z such that $0 \leqslant z \leqslant 1 - (x + y)$, it must be $1 - (x + y) \geqslant 0$, that is $x + y \leqslant 1$. So,

$$\int_D f = \int_{x \geqslant 0, y \geqslant 0, x+y \leqslant 1} \int_{0 \leqslant z \leqslant 1-(x+y)} xyz \, dz \, dxdy = \int_{x \geqslant 0, y \geqslant 0, x+y \leqslant 1} \int_0^{1-(x+y)} xyz \, dz \, dxdy.$$

Now,

$$\int_0^{1-(x+y)} xyz \, dz = xy \int_0^{1-(x+y)} z \, dz = xy \left[\frac{z^2}{2} \right]_{z=0}^{z=1-(x+y)} = \frac{1}{2} xy \, (1 - (x + y))^2 ,$$

so

$$\int_D f = \int_{x \geqslant 0, y \geqslant 0, x+y \leqslant 1} \frac{1}{2} xy \, (1 - (x + y))^2 \, dxdy.$$

To compute this integral we apply once more the reduction formula. Notice that

$$x \geqslant 0, y \geqslant 0, x + y \leqslant 1, \iff x \geqslant 0, \; 0 \leqslant y \leqslant 1 - x.$$

The last condition needs $1 - x \geqslant 0$, that is $x \leqslant 1$, so

$$\int_{x \geqslant 0, y \geqslant 0, x+y \leqslant 1} xy \, (1 - (x + y))^2 \, dxdy = \int_0^1 \int_0^{1-x} xy \, (1 - (x + y))^2 \, dy \, dx.$$

Starting from the inner integral

$$\int_0^{1-x} xy \, (1 - (x + y))^2 \, dy = x \int_0^{1-x} y(1 - x - y)^2 \, dy = x \frac{(1 - x)^4}{12},$$

therefore

$$\int_{x \geqslant 0, y \geqslant 0, x+y \leqslant 1} xy \, (1 - (x + y))^2 \, dxdy = \frac{1}{12} \int_0^1 x(1 - x)^4 \, dx = \frac{1}{360}$$

from which

$$\int_D f = \frac{1}{720}.$$

#3 Let $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ 0 \leqslant x, y \leqslant 1,\ 0 \leqslant z \leqslant x^2\}$. Clearly D is closed and since, for $(x, y, z) \in D$ we have $0 \leqslant x, y \leqslant 1$ and $0 \leqslant z \leqslant x^2 \leqslant 1$, D is also bounded. Moreover $f(x, y, z) := zy^2\sqrt{x^2 + zy} \in \mathscr{C}(D)$, so f is integrable on D.

To compute the integral we apply the reduction formula:

$$\int_D f = \int_{0\leqslant x,y\leqslant 1} \int_0^{x^2} zy^2\sqrt{x^2 + zy}\ dz\ dxdy = \int_{0\leqslant x,y\leqslant 1} y \int_0^{x^2} zy\sqrt{x^2 + zy}\ dz\ dxdy$$

We notice that

$$y\sqrt{x^2 + zy} = y(x^2 + zy)^{1/2} = \partial_z \frac{2}{3}(x^3 + zy)^{3/2}$$

so, integrating by parts

$$\begin{aligned}
\int_0^{x^2} zy\sqrt{x^2 + zy}\ dz &= \left[\frac{2}{3}z\left(x^2 + zy\right)^{3/2}\right]_{z=0}^{z=x^2} - \int_0^{x^2} \frac{2}{3}\left(x^2 + zy\right)^{3/2}\ dz \\
&= \frac{2}{3}x^5(1 + y)^{3/2} - \frac{2}{3y}\left[\frac{2}{5}\left(x^2 + zy\right)^{5/2}\right]_{z=0}^{z=x^2} \\
&= \frac{2}{3}x^5(1 + y)^{3/2} - \frac{4}{15y}\left[x^5(1 + y)^{5/2} - x^5\right]
\end{aligned}$$

Therefore,

$$\int_D f = \int_{0\leqslant x,y\leqslant 1} y\left(\frac{2}{3}x^5(1 + y)^{3/2} - \frac{4}{15y}\left[x^5(1 + y)^{5/2} - x^5\right]\right)\ dxdy.$$

We have

$$\int_{0\leqslant x,y\leqslant 1} \frac{2}{3}yx^5(1 + y)^{3/2}\ dxdy \overset{RF}{=} \frac{2}{3}\int_0^1 x^5\ dx \int_0^1 y(1 + y)^{5/2}\ dy,$$

and since $\int_0^1 x^5\ dx = \left[\frac{x^6}{6}\right]_{x=0}^{x=1} = \frac{1}{6}$, and

$$\begin{aligned}
\int_0^1 y\ \underbrace{(1 + y)^{5/2}}_{=\partial_y \frac{2}{7}(1+y)^{7/2}}\ dy &= \left[\frac{2}{7}y(1 + y)^{7/2}\right]_{y=0}^{y=1} - \frac{2}{7}\int_0^1 (1 + y)^{7/2}\ dy = \frac{2^{9/2}}{7} - \frac{2}{7}\left[\frac{2}{9}(1 + y)^{9/2}\right]_{y=0}^{y=1} \\
&= \frac{2^{9/2}}{7} - \frac{2^{11/2}}{63} - \frac{4}{63},
\end{aligned}$$

we obtain

$$\int_{0\leqslant x,y\leqslant 1} \frac{2}{3}yx^5(1 + y)^{3/2}\ dxdy = \frac{1}{9}\left(\frac{2^{9/2}}{7} - \frac{2^{11/2}}{63} - \frac{4}{63}\right).$$

By similar calculations we finally obtain

$$\int_D f = \frac{2}{315}(11 - 4\sqrt{2}). \quad \square$$

5.7.3. The barycenter of D is the point $(\overline{x}, \overline{y})$ defined by

$$\overline{x} = \frac{1}{\lambda_2(D)} \int_D x \, dxdy, \quad \overline{y} = \frac{1}{\lambda_2(D)} \int_D y \, dxdy.$$

Clearly, D is a quarter of disk with radius r, so $\lambda_2(D) = \frac{\pi}{4} r^2$. About the integrals we have

$$\begin{aligned} \int_D x \, dxdy &= \int_{0 \leqslant \rho \leqslant r, \ 0 \leqslant \theta \leqslant \pi/2} (\rho \cos\theta)\rho \, d\rho d\theta \overset{RF}{=} \int_0^r \rho^2 \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta \, d\rho \\ &= \int_0^r \rho^2 \, d\rho \int_0^{\pi/2} \cos\theta \, d\theta = \left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=r} [\sin\theta]_{\theta=0}^{\theta=\pi/2} = \frac{r^3}{3}. \end{aligned}$$

Therefore $\overline{x} = \frac{1}{\frac{\pi}{4} r^2} \frac{r^3}{3} = \frac{4}{3\pi} r$. Swapping x and y we immediately get $\overline{y} = \frac{4}{3\pi} r$.

To calculate the integral, we still use polar coordinates:

$$\begin{aligned} \int_D \frac{x+y}{x^2+y^2} \, dxdy &= \int_{0 \leqslant \rho \leqslant r, 0 \leqslant \theta \leqslant \pi/2} \frac{\rho(\cos\theta + \sin\theta)}{\rho^2} \rho \, d\rho d\theta \\ &= \int_{0 \leqslant \rho \leqslant r, 0 \leqslant \theta \leqslant \pi/2} (\cos\theta + \sin\theta) \, d\rho d\theta \\ &\overset{RF}{=} \int_0^{\pi/2} (\cos\theta + \sin\theta) \int_0^r 1 \, d\rho \, d\theta \\ &= r \left([\sin\theta]_{\theta=0}^{\theta=\pi/2} + [-\cos\theta]_{\theta=0}^{\theta=\pi/2} \right) = 2r. \quad \square \end{aligned}$$

5.7.4. #2 We have

$$\lambda_3(D) = \int_D 1 \, dxdydz.$$

Here it is convenient to introduce *adapted* spherical coordinates:

$$\Phi_{a,b,c}^{-1} \ : \ \begin{cases} x = a\rho\cos\theta\sin\varphi, \\ y = b\rho\sin\theta\sin\varphi, \\ z = c\rho\cos\varphi. \end{cases}$$

In this way $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = \rho^2$. We notice that

$$\left(\Phi_{a,b,c}^{-1}\right)' = \begin{bmatrix} \nabla(\Phi_{a,b,c}^{-1})_1 \\ \nabla(\Phi_{a,b,c}^{-1})_2 \\ \nabla(\Phi_{a,b,c}^{-1})_3 \end{bmatrix}$$

where $(\Phi^{-1}_{a,b,c})_j$ are the components of $\Phi^{-1}_{a,b,c}$. Now, if Φ^{-1} is the standard spherical coordinates map we have

$$\left(\Phi^{-1}_{a,b,c}\right)' = \begin{bmatrix} a\nabla(\Phi^{-1})_1 \\ b\nabla(\Phi^{-1})_2 \\ c\nabla(\Phi^{-1})_3 \end{bmatrix}, \implies |\det(\Phi^{-1}_{a,b,c})'| = |abc\det(\Phi^{-1})'| = abc\rho^2\sin\varphi.$$

Therefore

$$\lambda_3(D) \overset{CV}{=} \int_{0\leqslant\rho\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} abc\rho^2\sin\varphi\ d\rho d\theta d\varphi \overset{RF}{=} abc\int_0^1 \rho^2\ d\rho\int_0^{2\pi} d\theta\int_0^{\pi}\sin\varphi\ d\varphi$$

$$= abc\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=1} 2\pi\,[-\cos\varphi]_{\varphi=0}^{\varphi=\pi} = \frac{4\pi}{3}abc.$$

#4 It is convenient to use cylindrical coordinates: we notice that $(x,y,z) = (\rho\cos\theta, \rho\sin\theta, z) \in D$ iff

$$z \geqslant \rho,\ \ \rho^2 + z^2 \leqslant 1,$$

and since there is no condition on θ, this means $0 \leqslant \theta \leqslant 2\pi$. Therefore

$$\lambda_3(D) = \int_D 1\ dxdydz \overset{CV}{=} \int_{z\geqslant\rho,\ \rho^2+z^2\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi} \rho\ d\rho d\theta dz \overset{RF}{=} 2\pi\int_{z\geqslant\rho,\ \rho^2+z^2\leqslant 1,} \rho\ d\rho dz.$$

To apply RF, we notice that

$$z \geqslant \rho,\ \rho^2+z^2 \leqslant 1, \iff z\geqslant\rho,\ z^2 \leqslant 1-\rho^2, \iff \rho \leqslant z \leqslant \sqrt{1-\rho^2},\ \rho \leqslant \sqrt{1-\rho^2}.$$

Now,

$$\rho \leqslant \sqrt{1-\rho^2}, \overset{\rho\geqslant 0}{\iff} \rho^2 \leqslant 1-\rho^2, \iff \rho^2 \leqslant \frac{1}{2}, \iff 0 \leqslant \rho \leqslant \frac{1}{\sqrt{2}}.$$

Therefore,

$$\begin{aligned}
\int_{z\geqslant\rho,\ \rho^2+z^2\leqslant 1,} \rho\ d\rho dz &= \int_{0\leqslant\rho\leqslant 1/\sqrt{2},\ \rho\leqslant z\leqslant\sqrt{1-\rho^2}} \rho\ d\rho dz \overset{RF}{=} \int_0^{1/\sqrt{2}}\int_\rho^{\sqrt{1-\rho^2}} \rho\ d\rho\ dz \\
&= \int_0^{1/\sqrt{2}} \rho\left(\sqrt{1-\rho^2}-\rho\right)\ d\rho \\
&= \int_0^{1/\sqrt{2}} \underbrace{\rho\sqrt{1-\rho^2}}_{=\rho(1-\rho^2)^{1/2}=\partial_x -\frac{1}{3}(1-\rho^2)^{3/2}} d\rho - \int_0^{1/\sqrt{2}} \rho^2\ d\rho \\
&= \left[-\tfrac{1}{3}(1-\rho^2)^{3/2}\right]_{\rho=0}^{\rho=1/\sqrt{2}} - \left[\tfrac{\rho^2}{2}\right]_{\rho=0}^{\rho=1/\sqrt{2}} = \tfrac{1}{3}\left(1-\tfrac{1}{\sqrt{8}}\right) - \tfrac{1}{4},
\end{aligned}$$

from which we deduce that $\lambda_3(D) = \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{8}}\right) - \frac{\pi}{2}$.

#5 By using cylindrical coordinates,

$$\lambda_3(D) = \int_D 1\, dxdydz \overset{CV}{=} \int_{9(1-\rho)^2+4z^2\leqslant 1,\ 0\leqslant\theta\leqslant 2\pi} \rho\, d\rho d\theta dz \overset{RF}{=} 2\pi \int_{9(1-\rho)^2+4z^2\leqslant 1} \rho\, d\rho dz.$$

Now,

$$9(1-\rho)^2 + 4z^2 \leqslant 1, \iff z^2 \leqslant \frac{1-9(1-\rho)^2}{4},$$

which is equivalent to

$$|z| \leqslant \frac{\sqrt{1-9(1-\rho)^2}}{2},\ \text{provided}\ 1-9(1-\rho)^2 \geqslant 0, \iff (1-\rho)^2 \leqslant \frac{1}{9}, \iff |1-\rho| \leqslant \frac{1}{3}$$

that is for $-\frac{1}{3} \leqslant \rho - 1 \leqslant \frac{1}{3}$, or $\frac{2}{3} \leqslant \rho \leqslant \frac{4}{3}$. Therefore,

$$\lambda_3(D) \overset{RF}{=} 2\pi \int_{2/3}^{4/3} \int_{-\sqrt{1-9(1-\rho)^2}/2}^{\sqrt{1-9(1-\rho)^2}/2} \rho\, dz\, d\rho = 4\pi \int_{2/3}^{4/3} \rho\sqrt{1-9(\rho-1)^2}\, d\rho$$

$$\overset{u=3(\rho-1)}{=} \frac{4\pi}{3}\int_{-1}^{1}\left(\frac{u}{3}+1\right)\sqrt{1-u^2}du = \frac{4\pi}{3}\left(\frac{1}{3}\int_{-1}^{1} u\sqrt{1-u^2}\, du + \int_{-1}^{1}\sqrt{1-u^2}\, du\right).$$

Now, $\int_{-1}^{1} u\sqrt{1-u^2}\ du = 0$ (integral of an odd function on a symmetric interval w.r.t. 0) while $\int_{-1}^{1}\sqrt{1-u^2}\, du = \frac{\pi}{2}$ (half area of the unitary circle), so $\lambda_3(D) = \frac{4\pi}{3}\frac{\pi}{2} = \frac{2\pi^2}{3}$.

#6. Using cylindrical coordinates,

$$\lambda_3(D) = \int_D 1\, dxdydz \overset{CV}{=} \int_{\rho^2\leqslant 4, 4\rho^2+z^2\leqslant 64,\ 0\leqslant\theta\leqslant 2\pi} \rho\, d\rho d\theta dz \overset{RF}{=} 2\pi \int_{\rho^2\leqslant 4, 4\rho^2+z^2\leqslant 64} \rho\, d\rho dz.$$

Now, notice that

$$\rho^2 \leqslant 4, 4\rho^2 + z^2 \leqslant 64, \overset{\rho\geqslant 0}{\iff} 0 \leqslant \rho \leqslant 2,\ z^2 \leqslant 64 - 4\rho^2,$$

provided $64 - 4\rho^2 \geqslant 0$, which is true with these conditions becuase $4\rho^2 \leqslant 4\cdot 4 = 16 < 64$. Therefore

$$\lambda_3(D) \overset{RF}{=} 2\pi \int_0^2 \int_{-\sqrt{64-4\rho^2}}^{\sqrt{64-4\rho^2}} \rho\, dz\, d\rho = 4\pi \int_0^2 \underbrace{\rho\sqrt{64-4\rho^2}}_{\partial_\rho - \frac{1}{12}(64-4\rho^2)^{3/2}}\ d\rho$$

$$= -\frac{\pi}{3}\left[(64-4\rho^2)^{3/2}\right]_{\rho=0}^{\rho=2} = -\frac{\pi}{3}\left(48^{3/2} - 64^{3/2}\right) = \frac{\pi}{3}8^{3/2}(8^{3/2} - 6^{3/2}).$$

#7 This is far more complicated, the main difficulty coming with the parametrization of the domain. Moreover, here the use of cartesian is more simple:

$$(x,y,z) \in D, \iff x^2+y^2 \leqslant 1,\ z^2 \leqslant 1-x^2,\ z^2 \leqslant 1-y^2, \iff x^2+y^2 \leqslant 1,\ z^2 \leqslant \min\{1-x^2, 1-y^2\}.$$

Now,

$$1 - x^2 \leqslant 1 - y^2, \iff y^2 \leqslant x^2, \iff |y| \leqslant |x|.$$

Therefore,

$$\begin{aligned}\lambda_3(D) &= \int_D 1\; dxdydz = \int_{x^2+y^2\leqslant 1,\ |y|\leqslant|x|,\ z^2\leqslant 1-x^2} 1\; dxdydz + \int_{x^2+y^2\leqslant 1,\ |x|\leqslant|y|,\ z^2\leqslant 1-y^2} 1\; dxdydz\\ &= 2\int_{x^2+y^2\leqslant 1,\ |y|\leqslant|x|,\ z^2\leqslant 1-x^2} 1\; dxdydz.\end{aligned}$$

Now,

$$\int_{x^2+y^2\leqslant 1,\ |y|\leqslant|x|,\ z^2\leqslant 1-x^2} 1\; dxdydz \overset{RF}{=} \int_{x^2+y^2\leqslant 1,\ |y|\leqslant|x|} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dxdy = 2\int_{x^2+y^2\leqslant 1,\ |y|\leqslant|x|} \sqrt{1-x^2}\, dxdy.$$

For this double integral we have

$$x^2+y^2\leqslant 1,\ |y|\leqslant|x|, \iff y^2\leqslant 1-x^2,\ y^2\leqslant x^2, \iff |x|\leqslant 1,\ |y|\leqslant \min\{|x|,\sqrt{1-x^2}\}.$$

Notice that

$$|x|\leqslant\sqrt{1-x^2}, \iff x^2\leqslant 1-x^2, \iff x^2\leqslant\frac{1}{2}, \iff |x|\leqslant\frac{1}{\sqrt{2}}.$$

Therefore

$$\begin{aligned}\int_{x^2+y^2\leqslant 1,\ |y|\leqslant|x|}\sqrt{1-x^2}\; dxdy &\overset{RF}{=} \int_{|x|\leqslant 1/\sqrt{2}}\sqrt{1-x^2}\int_{-|x|}^{|x|} dy\; dx + \int_{1/\sqrt{2}\leqslant|x|\leqslant 1}\sqrt{1-x^2}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\; dx\\ &= 2\int_0^{1/\sqrt{2}} 2x\sqrt{1-x^2}\; dx + 2\int_{1/\sqrt{2}}^1 \sqrt{1-x^2}2\sqrt{1-x^2}\; dx\\ &= 4\left[(1-x^2)^{3/2}\right]_{x=0}^{x=1/\sqrt{2}} - 4\int_{1/\sqrt{2}}^1 1-x^2\; dx\\ &= 4\left[\tfrac{1}{\sqrt{8}}-1\right] - 4\left(1-\tfrac{1}{\sqrt{2}}\right)+\tfrac{4}{3}.\end{aligned}$$

It is now easy to draw the conlcusion. □

5.7.5. #1,2,3. Let $f(x,y,z)=\sqrt{x^2+y^2}$. Clearly, $f\in\mathscr{C}(D)$ where $D=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\leqslant 16, -5\leqslant z\leqslant 4\}$. The domain is clearly closed and bounded. Therefore, f is integrable on D. Changing variables to cylondrical coordinates,

$$\int_D f \overset{CV}{=} \int_{\rho^2\leqslant 16,\ -5\leqslant z\leqslant 4, 0\leqslant\theta\leqslant 2\pi} \rho\cdot\rho\; d\rho d\theta dz \overset{RF}{=} 2\pi\int_0^4 \rho^2\int_{-5}^4 dz\; d\rho = 384\pi.$$

#5 Since $f \geqslant 0$, we check integrability of $|f| = f$ and, if true, we also get the value of the integral. We change variables by using spherical coordinates

$$\int_D |f| = \int_D f \overset{CV}{=} \int_{\rho\geqslant 0,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} \rho e^{-\rho^2}\, \rho^2 \sin\varphi d\rho d\theta d\varphi \overset{RF}{=} 2\pi \int_0^{+\infty} \rho^3 e^{-\rho^2} \int_0^{\pi} \sin\varphi\, d\varphi\, d\rho$$

$$= 2\pi \int_0^{+\infty} \rho^2 \partial_\rho(-e^{-\rho^2})\, d\rho = 2\pi \left(\underbrace{\left[-\rho^2 e^{-\rho^2}\right]_{\rho=0}^{\rho=+\infty}}_{=0} + \int_0^{+\infty} 2\rho e^{-\rho^2}\, d\rho \right)$$

$$= 2\pi \left[-e^{-\rho^2}\right]_{\rho=0}^{\rho=+\infty} = 2\pi.$$

#6. Since $f \geqslant 0$ on $D = [0, +\infty[^3$, which is closed but unbounded, we check integrability of $|f| = f$ and, if true, we also get the value of the integral. Adapting spherical coordinates in such a way that $x^2 + 2y^2 + 3z^2 = \rho^2$, that is, setting

$$\widetilde{\Phi}^{-1} \ : \ \begin{cases} x = \rho\cos\theta\sin\varphi, \\ \sqrt{2}y = \rho\sin\theta\sin\varphi, \\ \sqrt{3}z = \rho\cos\varphi, \end{cases}$$

we have

$$\det(\widetilde{\Phi}^{-1})' = \det \begin{bmatrix} \nabla(\widetilde{\Phi}^{-1})_1 \\ \nabla(\widetilde{\Phi}^{-1})_2 \\ \nabla(\widetilde{\Phi}^{-1})_3 \end{bmatrix} = \frac{1}{\sqrt{2}\sqrt{3}}\rho^2\sin\varphi.$$

Therefore,

$$\int_D |f| = \int_D f = \int_{\rho\geqslant 0,\ 0\leqslant\theta\leqslant\pi/2,\ 0\leqslant\varphi\leqslant\pi/2} \frac{\rho\cos\theta\sin\varphi}{1+(\rho^2)^2} \frac{1}{\sqrt{6}}\rho^2\sin\varphi\, d\rho d\theta d\varphi$$

$$\overset{RF}{=} \frac{1}{\sqrt{6}} \int_0^{+\infty} \frac{\rho^3}{1+\rho^4} \int_0^{\pi/2} \sin^2\varphi \int_0^{\pi/2} \cos\theta\, d\theta\, d\varphi\, d\rho = \frac{1}{\sqrt{6}} \left(\int_0^{+\infty} \frac{\rho^3}{1+\rho^4}\, d\rho\right)\left(\int_0^{\pi/2} \sin^2\varphi\, d\varphi\right).$$

Since

$$\int_0^{+\infty} \frac{\rho^3}{1+\rho^4}\, d\rho = \frac{1}{4}\left[\log(1+\rho^4)\right]_{\rho=0}^{\rho=+\infty} = +\infty,$$

we conclude that $\int_D |f| = +\infty$, so f is not integrable on D. □

5.7.6. #3 This exercise contains a mistake in the integration domain.

5.7.7. The domain D_a is closed being defined by large inequalities with continuous functions. It is also bounded: from the second constraint, $y \geqslant x^2 \geqslant 0$, so $\frac{1}{x} \geqslant y \geqslant 0$ impleas also $x \geqslant 0$. Therefore $x^2 \leqslant y \leqslant \frac{1}{x}$ implies $x^3 \leqslant 1$ that is $x \leqslant 1$, so $y \leqslant ax^2 \leqslant a$. This proves that $(x, y) \in [0, 1] \times [0, a]$, so D_a is bounded. We may also notice that $x \neq 0$ for $(x, y) \in D_a$, otherwise also $y = 0$ but $(0, 0)$ cannot verify the constraints. In particular, $D_a \subset]0,1]\times]0,a]$, so f is definitely continuous on D_a. Since D_a is closed and bounded we conclude that f is integrable on D_a.

To compute the integral $\int_{D_a} f$, we apply the change of variable formula. Let $u := xy$, $v := \frac{y}{x^2}$ (**warning: the suggested change of variable is not the more appropriate one!**). Since $x, y > 0$ for $(x,y) \in D_a$, we have that $u, v > 0$. Denoting $\Phi(x,y) := (xy, \frac{y}{x^2})$, Φ is well defined on D_a. Moreover

$$(u,v) = \Phi(x,y) \in \Phi(D_a), \quad \Longleftrightarrow \quad \frac{1}{a} \leqslant xy \leqslant 1, \;\; 1 \leqslant \frac{y}{x^2} \leqslant a, \quad \Longleftrightarrow \quad \frac{1}{a} \leqslant u \leqslant 1, \;\; 1 \leqslant v \leqslant a.$$

that is

$$(u,v) \in \Phi(D_a), \quad \Longleftrightarrow \quad (u,v) \in [1/a, 1] \times [1, a].$$

Therefore

$$I(a) \overset{CV}{=} \int_{1/a \leqslant u \leqslant 1,\; 1 \leqslant v \leqslant a} \frac{1}{v} e^u |\det(\Phi^{-1})'(u,v)| \, dudv.$$

We notice that

$$\det \Phi'(x,y) = \det \begin{bmatrix} \nabla(xy) \\ \\ \nabla \frac{y}{x^2} \end{bmatrix} = \det \begin{bmatrix} y & x \\ \\ -2\frac{y}{x^3} & \frac{1}{x^2} \end{bmatrix} = 3\frac{y}{x^2} = 3v,$$

so

$$|\det(\Phi^{-1})'(u,v)| = \frac{1}{|\det \Phi'(\Phi^{-1}(u,v))|} = \frac{1}{3|v|} = \frac{1}{3v},$$

being $v > 0$ on $\Phi(D_a)$. Returning to the integral,

$$I(a) = \int_{1/a \leqslant u \leqslant 1,\; 1 \leqslant v \leqslant a} \frac{1}{v} e^u \frac{1}{3v} \, dudv \overset{RF}{=} \frac{1}{3} \int_{1/a}^{1} e^u \int_1^a \frac{1}{v^2} \, dv \, du = \frac{1}{3}\left(e - e^{1/a}\right)\left(1 - \frac{1}{a}\right). \quad \square$$

5.7.10. #2 We apply Green's formula to $\vec{F} = (f,g) = (\cos x + 6y^2, 3x - e^{-y^2})$. We have

$$\oint_{\vec{\gamma}} \vec{F} = \int_{B(0,1]} \left(\partial_x g - \partial_y f\right) \, dxdy = \int_{B(0,1]} (3 - 12y) \, dxdy = 3\pi,$$

being $\int_{B(0,1]} y \, dxdy = 0$.

#4 We apply Green's formula to $\vec{F} = (f,g) = (x^3 - y^3, x^3 + y^3)$. We have

$$\begin{aligned} \oint_{\vec{\gamma}} \vec{F} &= \int_{B(0,r]\cap[0,+\infty[^2} \left(\partial_x g - \partial_y f\right) \, dxdy = \int_{B(0,r]\cap\mathbb{R}^2_+} \left(3x^2 - (-3y^2)\right) \, dxdy \\ &= 3\int_{B(0,r]\cap\mathbb{R}^2_+} (x^2 + y^2) \, dxdy \overset{CV}{=} 3\int_{0 \leqslant \rho \leqslant r,\; 0 \leqslant \theta \leqslant \pi/2} \rho^2 \cdot \rho \, d\rho d\theta \\ &\overset{RF}{=} 3\frac{\pi}{2} \int_0^r \rho^3 \, d\rho = \frac{3\pi}{8} r^4. \quad \square \end{aligned}$$

5.7.11. #3. By the area formula

$$\begin{aligned}\lambda_2(\Omega) &= -\oint_{\vec{\gamma}} (y, 0) = -\int_0^{2\pi} y(t)x'(t)\,dt = -\int_0^{2\pi} \sin^2 t 2\cos t(-\sin t)\,dt \\ &= 2\int_0^{2\pi} \underbrace{\sin^3 t\cos t}_{=\partial_t \frac{1}{4}\sin^4 t}\,dt = \frac{1}{2}\left[\sin^4 t\right]_{t=0}^{t=2\pi} = 0.\end{aligned}$$

#4. By the area formula

$$\lambda_2(\Omega) = -\oint_{\vec{\gamma}} (y, 0) = -\int_0^{2\pi} y(t)x'(t)\,dt = -\int_0^{2\pi} \sin^3 t 3\cos^2 t(-\sin t)\,dt = 3\int_0^{2\pi} \sin^4 t\cos^2 t\,dt.$$

We notice that $\cos^2 t\sin t = \partial_t - \frac{1}{3}\cos^3 t$, so

$$\begin{aligned}\int_0^{2\pi} \sin^4 t\cos^2 t\,dt &= \int_0^{2\pi} \sin^3 t\left(\cos^2 t\sin t\right)\,dt = \left[-\frac{1}{3}\cos^3 t\sin^3 t\right]_{t=0}^{t=2\pi} + \int_0^{2\pi} \sin^2 t\cos^4 t\,dt \\ &= \int_0^{2\pi} \sin^2 t\cos^4 t\,dt.\end{aligned}$$

Denoted $I = \int_0^{2\pi} \sin^4 t\cos^2 t\,dt$, we have

$$2I = \int_0^{2\pi} \sin^4 t\cos^2 t\,dt + \int_0^{2\pi} \sin^2 t\cos^4 t\,dt = \int_0^{2\pi} \sin^2 t\cos^2 t\left(\sin^2 t + \cos^2 t\right)\,dt = \int_0^{2\pi} \sin^2 t\cos^2 t\,dt.$$

Now,

$$\int_0^{2\pi} \sin^2 t\cos^2 t\,dt = \frac{1}{4}\int_0^{2\pi} \sin^2(2t)\,dt = \frac{1}{8}\int_0^{4\pi} \sin^2 u\,du = \frac{1}{4}\int_0^{2\pi} \sin^2 u\,du = \frac{\pi}{2},$$

from which $I = \frac{\pi}{4}$ and $\lambda_2(\Omega) = \frac{3\pi}{4}$. □

7.5.2. #1,5,6 see slides.

#2. We have

$$\cosh(2z) + 1 = 0, \iff \frac{e^{2z} + e^{-2z}}{2} + 1 = 0, \iff e^{2z} + e^{-2z} + 2 = 0, \iff e^{4z} + 1 + 2e^{2z} = 0.$$

Setting $w = e^{2z}$ we have

$$w^2 + 2w + 1 = 0, \iff (w+1)^2 = 0, \iff w = -1,$$

that is

$$e^{2z} = -1, \iff 2z = \log(-1) = \log|-1| + i(\pi + k2\pi) = i(2k+1)\pi,\ k \in \mathbb{Z},$$

from which

$$z = i\left(k + \frac{1}{2}\right),\ k \in \mathbb{Z}.$$

#3. We have

$$e^{iz} = 1, \iff iz = \log 1 + i(0 + k2\pi) = i2k\pi, \iff z = 2k\pi.$$

#4. We have

$$\sinh(iz+1)=0, \quad \Longleftrightarrow \quad \frac{e^{iz+1}-e^{-(iz+1)}}{2}=0, \quad \Longleftrightarrow \quad e^{iz+1}-e^{-(iz+1)}=0, \quad \Longleftrightarrow \quad e^{2(iz+1)}=1,$$

that is,

$$2(iz+1)=\log 1+i(0+k2\pi)=i2k\pi, \quad \Longleftrightarrow \quad iz+1=ik\pi, \quad \Longleftrightarrow \quad z=-i+k\pi,\ k\in\mathbb{Z}.$$

#8. We have

$$e^{iz^2}=i, \quad \Longleftrightarrow \quad iz^2=\log|i|+i\left(\frac{\pi}{2}+k2\pi\right)=i\left(\frac{\pi}{2}+k2\pi\right),\ k\in\mathbb{Z},$$

from which

$$z^2=\frac{\pi}{2}+k2\pi,\ k\in\mathbb{Z}.$$

Here we distinguish between $k\geqslant 0$ and $k<0$.

- if $k\geqslant 0$, then

$$z^2=\frac{\pi}{2}+k2\pi, \quad \Longleftrightarrow \quad z=\pm\sqrt{\frac{\pi}{2}+k2\pi}.$$

- if $k<0$, then

$$z^2=\frac{\pi}{2}+k2\pi, \quad \Longleftrightarrow \quad z=\pm\sqrt{\underbrace{\frac{\pi}{2}+k2\pi}_{<0}}=\pm i\sqrt{-\left(\frac{\pi}{2}+k2\pi\right)}$$

□

7.10.3. Let $z=w=-i$. Then $zw=(-i)^2=-1$, so

$$\log_{\mathbb{C}}(zw)=\log_{\mathbb{C}}(-1)=\log|-1|+i\pi=i\pi.$$

On the other hand,

$$\log_{\mathbb{C}} z=\log_{\mathbb{C}} w=\log_{\mathbb{C}}(-i)=\log|-i|+i\frac{3}{2}\pi,$$

so

$$\log_{\mathbb{C}} z+\log_{\mathbb{C}} w=2i\frac{3}{2}\pi=i3\pi\neq i\pi=\log_{\mathbb{C}}(zw).$$

7.10.4. We check that $f(z):=\operatorname{Im} z$ is not $\mathbb{C}-$differentiable by using the CR equations. We notice that $f(x+iy)=y=u(x,y)+iv(x,y)$, so $u(x,y)=y$ and $v(x,y)=0$. Clearly both u,v are $\mathbb{R}-$differentiable on $\mathbb{R}^2$. So, $f=u+iv$ is $\mathbb{C}-$differentiable at $x+iy$ iff the CR equations hold at point (x,y), that is

$$\begin{cases} \partial_x u=\partial_y v, \\ \\ \partial_y u=-\partial_x v, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} 0=0, \\ \\ 1=0, \end{cases}$$

which is manifestly impossible.

If $f(x+iy) = |x+iy| = \sqrt{x^2+y^2} = u(x,y) + iv(x,y)$, we see that both u, v are $\mathbb{R}$−differentiable on $\mathbb{R}^2 \backslash \{(0,0)\}$. The CR equations are

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v, \end{cases} \iff \begin{cases} \frac{x}{\sqrt{x^2+y^2}} = 0, \\ \\ \frac{y}{\sqrt{x^2+y^2}} = 0, \end{cases}$$

which are never verifies at $(x,y) \neq (0,0)$. We conclude that f is not $\mathbb{C}$−differentiably at each $z \in \mathbb{C}$. □

7.10.5. iii) Let $u(x,y) := x^2$. Clearly u is $\mathbb{R}$−differentiable on $\mathbb{R}^2$. In order $f = u + iv$ be $\mathbb{C}$−differentiable at (x,y), v must be differentiable and such that the CR equations hold, that is

$$\begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v, \end{cases} \iff \begin{cases} \partial_x v = -\partial_y u = 0, \\ \\ \partial_y v = \partial_x u = 2x, \end{cases}$$

From the first equation,

$$v(x,y) = c(y),$$

and plugging this into the second equation we get $c'(y) = 2x$, which is impossible. We deduce that it is impossible to find v such that $f = u + iv$ be $\mathbb{C}$−differentiable on $\mathbb{C}$.

iv) Let $u(x,y) = x^2 - y^2$. Clearly, u is $\mathbb{R}$−differentiable on $\mathbb{R}^2$. So, $f = u + iv$ is $\mathbb{C}$−differentiable on $\mathbb{R}^2$ iff v is $\mathbb{R}$−differentiable and the CR equations hold true,

$$\begin{cases} \partial_x v = -\partial_y u = 2y, \\ \\ \partial_y v = \partial_x u = 2x, \end{cases}$$

From the first,

$$v(x,y) = \int 2y \, dx = 2xy + c(y),$$

and plugging thisa into the second equation we get

$$2x + c'(y) = 2x, \iff c'(y) = 0, \iff c \equiv \text{constant}.$$

Therefore $v(x,y) = 2xy + c$. Clearly, such v is $\mathbb{R}$−differentiable on $\mathbb{R}^2$. Hence, $f(x+iy) = x^2 - y^2 + i2xy + c = (x+iy)^2 + c$, so $f(z) = z^2 + c$. □

7.10.7. Let $f(x+iy) = u(x,y) + iv(x,y)$. Since f is $\mathbb{C}$-differentiable on $\mathbb{C}$, both u and v are $\mathbb{R}^2$−differentiable on $\mathbb{R}^2$ and th CR equations hold:

$$\begin{cases} \partial_x u(x,y) \equiv \partial_y v(x,y), \\ \\ \partial_y u(x,y) = -\partial_x v(x,y). \end{cases}$$

Now, let $g(z) := \overline{f(\bar{z})}$. We have

$$g(x+iy) = \overline{f(\overline{x+iy})} = \overline{f(x-iy)} = \overline{u(x,-y) + iv(x,-y)} = u(x,-y) - iv(x,-y) = U(x,y) + iV(x,y),$$

where $U(x,y) = u(x,-y)$ and $V(x,y) = -v(x,-y)$. To show that also g is $\mathbb{C}$−differentiable we have to verify that i) both U, V are $\mathbb{R}^2$−differentiable; ii) U, V verify the Cauchy-Riemann equations.

i) Since u, v are $\mathbb{R}^2$-differentiable, also $U(x,y) = u(x,-y)$ and $V(x,y) = -v(x,-y)$ are $\mathbb{R}^2$-differentiable.

ii) We have

$$\partial_x U(x,y) = \partial_x \left(u(x,-y)\right) = \partial_x u(x,-y) \overset{CR}{=} \partial_y v(x,-y),$$

and since

$$\partial_y V(x,y) = \partial_y \left(-v(x,-y)\right) = -\partial_y v(x,-y)(-1) = \partial_y v(x,-y) = \partial_x U(x,y),$$

we have that the first CR equation is fulfilled by U, V. A similar check shows that also the second CR equation is fulfilled:

$$\partial_y U(x,y) = \partial_y \left(u(x,-y)\right) = -\partial_y u(x,-y) \overset{CR}{=} -\left(-\partial_x v(x,-y)\right) = \partial_x v(x,-y),$$

and since

$$-\partial_x V(x,y) = -\partial_x \left(v(x,-y)\right) = -\partial_x v(x,-y) = -\partial_y U(x,y),$$

the conclusion follows.

With this we show that $g = U + iV$ is $\mathbb{C}-$differentiable. □