Calculus 2

Answers to LN Exercises

1.8.1. By proving the triangular inequality for the Euclidean norm, we got the formula

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\sum_j x_j y_j.$$

From this,

$$\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|-\vec{y}\|^2 + 2\sum_j x_j(-y_j) = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\sum_j x_j y_j.$$

Therefore,

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\left(\|\vec{x}\|^2 + \|\vec{y}\|^2\right). \quad \square$$

1.8.2. We start with $\|\cdot\|_1$. We have to verify positivity, vanishing, homogeneity and the triangular inequality.

i) Positivity: $\|\vec{x}\|_1 = \sum_j |x_j| \geqslant 0$ because it is a sum of positive numbers.

ii) Vanishing: $\|\vec{x}\|_1 = 0$ iff $\sum_j |x_j| = 0$. Since this is the sum of positive numbers it can be $= 0$ iff $|x_j| = 0$ for every $j = 1, \dots, d$, that is iff $x_j = 0\ \forall j$, so iff $\vec{x} = \vec{0}$.

iii) Homogeneity: we have

$$\|\lambda\vec{x}\|_1 = \sum_j |\lambda x_j| = \sum_j |\lambda||x_j| = |\lambda|\sum_j |x_j| = |\lambda|\|\vec{x}\|_1.$$

(notice we used homogeneity of the modulus, that is $|ab| = |a||b|$)

iv) Triangular inequality: we have

$$\|\vec{x}+\vec{y}\|_1 = \sum_j |x_j + y_j|.$$

Now, because of the triangular inequality for the modulus, $|a+b| \leqslant |a| + |b|$ we have

$$|x_j + y_j| \leqslant |x_j| + |y_j|,\ \forall j.$$

So, summing up on j, we get

$$\|\vec{x}+\vec{y}\|_1 = \sum_j |x_j + y_j| \leqslant \sum_j \left(|x_j| + |y_j|\right) = \sum_j |x_j| + \sum_j |y_j| = \|\vec{x}\|_1 + \|\vec{y}\|_1.$$

Let now discuss the case of $\|\cdot\|_\infty$. We have to verify the same properties as done above.

i) Positivity: $\|\vec{x}\|_\infty = \max_j |x_j \geqslant 0$ (maximum of positive numbers is positive).

ii) Vanishing: $\|\vec{x}\|_\infty = 0$ iff $\max_j |x_j| = 0$. Since the maximum is made on positive numbers, it is clear that it can be $= 0$ iff $|x_j| = 0\ \forall j$, that is $x_j = 0\ \forall j$, so iff $\vec{x} = \vec{0}$.

iii) Homogeneity: we have

$$\|\lambda\vec{x}\|_\infty = \max_j |\lambda x_j| = \max_j |\lambda||x_j|.$$

Now, it is clear that $\max_j c|x_j| = c \max_j |x_j|$ for every $c \geqslant 0$. So $\|\lambda\vec{x}\|_\infty = |\lambda|\|\vec{x}\|_\infty$.

iv) Triangular inequality: we have

$$\|\vec{x}+\vec{y}\|_\infty = \max_j |x_j + y_j|.$$

Now, since

$$|x_j + y_j| \leqslant |x_j| + |y_j| \leqslant \max_k |x_k| + \max_k |y_k| = \|\vec{x}\|_\infty + \|\vec{y}\|_\infty, \ \forall j,$$

we have

$$\|\vec{x}+\vec{y}\|_\infty = \max_j |x_j + y_j| \leqslant \|\vec{x}\|_\infty + \|\vec{y}\|_\infty. \quad \square$$

1.8.3. i) Let $\vec{x}_n := (e^{-n}, 1)$. Since $e^{-n} \longrightarrow 0$ and $1 \longrightarrow 1$ we conclude that $\vec{x}_n \longrightarrow (0,1)$.
ii) Let $\vec{x}_n := (n, n^2)$. Here we notice that $\|\vec{x}_n\| = \sqrt{n^2+n^4} \longrightarrow +\infty$, so $\vec{x}_n \longrightarrow \infty_2$.
iii) Let $\vec{x}_n := \left(\frac{1}{n}, \frac{1}{n^2}, \sin\frac{1}{n}\right)$. Since $\frac{1}{n} \longrightarrow 0$, $\frac{1}{n^2} \longrightarrow 0$ and $\sin\frac{1}{n} \longrightarrow 0$, we conclude that $\vec{x}_n \longrightarrow (0,0,0)$.
iv) Let $\vec{x}_n := \left(1, 1+\frac{1}{n}, n\right)$. Here, $\|\vec{x}_n\| = \sqrt{1+\left(1+\frac{1}{n}\right)^2+n^4} \geqslant \sqrt{n^4} = n^2 \longrightarrow +\infty$, so $\vec{x}_n \longrightarrow \infty_3$.
v) Let $\vec{x}_n := \left(\tanh n, \frac{\log n}{n}, \frac{\sin n}{n}\right)$. Notice that $\tanh n = \frac{\sinh n}{\cosh n} \sim_{n\to+\infty} \frac{e^n/2}{e^n/2} = 1$, so $\tanh n \longrightarrow 1$. Moreover, since $\log n =_{+\infty} o(n)$, $\frac{\log n}{n} \longrightarrow 0$ and clearly $\frac{\sin n}{n} \longrightarrow 0$. Therefore, $\vec{x}_n \longrightarrow (1,0,0)$.
vi) Let $\vec{x}_n := \left((-1)^n, (-1)^{n+1}\right)$. Here we have that $\vec{x}_{2k} \equiv (1,-1) \longrightarrow (1,-1)$ while $x_{2k+1} \equiv (-1,1) \longrightarrow (-1,1)$ so we conclude that there is no limit for $\vec{x}_n$ when $n \to +\infty$. $\square$

1.8.4. $\Longrightarrow$. Assumption: $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$. Thesis: $x_{n,k} \longrightarrow \ell_k$ for every $k = 1, \dots, d$. By definition

$$\forall \varepsilon > 0, \ \exists N \ : \ \|\vec{x}_n - \vec{\ell}\| \leqslant \varepsilon, \ \forall n \geqslant N.$$

Now,

$$\|\vec{x}_n - \vec{\ell}\| = \sqrt{\sum_{k=1}^d (x_{n,k}-\ell_k)^2} \geqslant \sqrt{(x_{n,k}-\ell_k)^2} = |x_{n,k}-\ell_k|, \ \forall k = 1, \dots, d.$$

Therefore,

$$|x_{n,k}-\ell_k| \leqslant \|\vec{x}_n - \vec{\ell}\| \leqslant \varepsilon, \ \forall n \geqslant N, \ \forall k = 1, \dots, d.$$

and this means that $x_{n,k} \longrightarrow \ell_k$ for every $k = 1, \dots, d$.

$\Longleftarrow$ Now the Assumption is: $x_{n,k} \longrightarrow \ell_k$ for every $k = 1, \dots, d$. The thesis is: $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$. From the assumption we can say that,

$$\forall k = 1, \dots, d, \ \forall \varepsilon > 0, \ \exists N_k \ : \ \|\vec{x}_{n_k} - \vec{\ell}_k\| \leqslant \varepsilon, \ \forall n \geqslant N_k.$$

Notice we wrote N_k because the initial N will depend on the sequence $(x_{n,k})$, so on k. Now, let $N := \max(N_1, \dots, N_d)$. Then if $n \geqslant N \geqslant N_k$ for every $k = 1, \dots, d$, so

$$|x_{n,k} - \ell_k| \leqslant \varepsilon, \ \forall k = 1, \dots, d.$$

Therefore,

$$\|\vec{x}_n - \vec{\ell}\| = \sqrt{\sum_{k=1}^{d} (x_{n,k} - \ell_k)^2} \leqslant \sqrt{\varepsilon^2 + \cdots + \varepsilon^2} = \sqrt{d\varepsilon^2} = \sqrt{d}\varepsilon, \ \forall n \geqslant N.$$

This is exactly the thesis (if you like you can replace ε by $\frac{\varepsilon}{\sqrt{d}}$). □

1.8.5. i) True, it follows by that proved in 1.8.4.
ii) False: for example $\vec{x}_n = (0, n) \longrightarrow \infty_2$ but $x_{n,1} \equiv 0 \longrightarrow 0$.
iii) True. Indeed, we may notice that

$$\|\vec{x}_n\| = \sqrt{\sum_{k=1}^{d} x_{n,k}^2} \geqslant \sqrt{x_{n,j}^2} = |x_{n,j}| \longrightarrow +\infty.$$

iv) False: take $\vec{x}_n = (0, (-1)^n)$, $\vec{x}_{2k} \equiv (0, 1) \longrightarrow (0, 1)$, $\vec{x}_{2k+1} \equiv (0, -1) \longrightarrow (0, -1)$, in particular $\lim_n \vec{x}_n$ cannot exists. However $x_{n,1} \equiv 0 \longrightarrow 0$. □

1.8.7. #1,2,4,6 done in class (see slides).
#3. Let $f(x, y) = \frac{y^2 - xy}{x^2 + y^2}$. We have $f(x, 0) \equiv 0 \longrightarrow 0$ when $x \to 0$, $f(0, y) = \frac{y^2}{y^2} \equiv 1 \longrightarrow 1$ when $y \to 0$.

#5. Let $f(x, y) = \frac{xy + \sqrt{y^2+1} - 1}{x^2 + y^2}$. We have $f(x, 0) = \frac{0}{x^2} \equiv 0 \longrightarrow 0$ when $x \to 0$, $f(0, y) = \frac{\sqrt{1+y^2} - 1}{y^2} \longrightarrow \frac{1}{2}$ when $y \to 0$, this because of the fundamental limit $\lim_{t \to 0} \frac{(1+t)^\alpha - 1}{t} = \alpha$ (here $\alpha = 1/2$).

1.8.8. #1,3,5 done in class (see slides).
#2 Let $f(x, y) := \frac{x^2 y^3}{(x^2 + y^2)^2}$. The limit is a $\frac{0}{0}$ indeterminate form. Introducing polar coordinates we have

$$f(x, y) = f(\rho \cos\theta, \rho \sin\theta) = \frac{\rho^5 \cos^2\theta \sin^3\theta}{(\rho^2)^2} = \rho \cos^2\theta \sin^3\theta,$$

so

$$|f(x, y) - 0| = \rho |\cos^2\theta| |\sin^3\theta| \leqslant \rho =: g(\rho) \longrightarrow 0, \ \ \rho \to 0,$$

from which we conclude that $\exists \lim_{(x,y) \to \vec{0}} f = 0$.

#4. Let $f(x,y) := \frac{x\sqrt{|y|}}{\sqrt[3]{x^4+y^4}}$. When $(x,y) \to \vec{0}$, the limit of f yields to an indeterminate form $\frac{0}{0}$. Let's write f in polar coords:

$$\begin{aligned} f(x,y) &= f(\rho\cos\theta, \rho\sin\theta) = \frac{\rho\cos\theta\sqrt{\rho|\sin\theta|}}{\sqrt[3]{\rho^4(\cos^4\theta+\sin^4\theta)}} = \frac{\rho^{3/2}}{\rho^{4/3}}\frac{\cos\theta\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta+\sin^4\theta}} \\ &= \rho^{1/6}\frac{\cos\theta\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta+\sin^4\theta}}. \end{aligned}$$

Since $\rho^{1/6} \longrightarrow 0$ when $\rho \to 0$, we bet on the limit exists and it is equal to 0. To show this we notice that,

$$|f(x,y) - 0| = \rho^{1/6}\frac{|\cos\theta|\sqrt{|\sin\theta|}}{\sqrt[3]{\cos^4\theta+\sin^4\theta}} \leqslant \rho^{1/6}\frac{1}{\sqrt[3]{\cos^4\theta+\sin^4\theta}}.$$

Let $K(\theta) := \sqrt[3]{\cos^4\theta+\sin^4\theta}$. This is a continuous function on $[0, 2\pi]$, so by Weierstrass' theorem there exists a minimum achieved at some $\theta_{min} \in [0, 2\pi]$, that is $K(\theta) \geqslant K(\theta_{min}) =: K_0 \geqslant 0$. Notice that $K_0 > 0$: if $K_0 = 0$ we would have $K(\theta_{min}) = 0$, so $\sqrt[3]{\cos^4\theta_{min} + \sin^4\theta_{min}} = 0$, that is $\cos^4\theta_{min}+\sin^4\theta_{min} = 0$. Since both terms are positive, this is possible iff $\cos^4\theta_{min} = 0$ and (simultaneously) $\sin^4\theta_{min} = 0$, that is $\cos\theta_{min} = \sin\theta_{min} = 0$, which is impossible! Therefore, $K(\theta) \geqslant K_0 > 0$ for every $\theta \in [0, 2\pi]$, so

$$|f(x,y) - 0| \leqslant \rho^{1/6}\frac{1}{K_0} =: g(\rho) \longrightarrow 0,\ \rho \to 0.$$

From this we deduce that $\exists \lim_{(x,y)\to\vec{0}} f = 0$. □

1.8.9. #3. Let f be the function of which we aim to compute its limit at $\vec{0}$. The limit is an indeterminate form $\frac{0}{0}$. We notice that

$$f(x,0,0) = \frac{x^4}{\sqrt{x^4}} = \frac{x^4}{x^2} = x^2 \longrightarrow 0,\ x \to 0.$$

So, if a limit exists it must be = 0. Using spherical coordinates

$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi,$$

we get

$$f(x,y,z) = \frac{(\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin\varphi\sin\theta\cos\varphi)^2}{\sqrt{(\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta))^2 + \rho^4\cos^4\varphi}} = \rho^2\frac{(\sin^2\varphi\cos^2\theta + \sin\varphi\sin\theta\cos\varphi)^2}{\sqrt{\sin^4\varphi+\cos^4\varphi}}.$$

From this we can prove that the limit exists and it is equal to 0: indeed,

$$|f(x,y,z)-0| \leqslant \rho^2 \frac{(1+1)^2}{\sqrt{\sin^4\varphi+\cos^4\varphi}}.$$

Let $F(\varphi) := \sqrt{\sin^4\varphi+\cos^4\varphi}$. Clearly $F \in \mathscr{C}([0,2\pi])$, so, by Weierstrass' theorem, there exists $K := \min F_{[0,2\pi]}$. Since the minimum is achieved at some φ_{min} if $K = F(\varphi_{min}) = 0$ we should have $\cos\varphi_{\min} = \sin\varphi_{min} = 0$, which is impossible. We conclude that $K > 0$ and

$$|f(x,y,z)-0| \leqslant \rho^2 \frac{(1+1)^2}{\sqrt{\sin^4\varphi+\cos^4\varphi}} \leqslant \rho^2\frac{4}{K} =: g(\rho) \longrightarrow 0,\ \rho\to 0. \qquad \square$$

#5 We start noticing that $(x,y)\to(0,1)$ iff $(u,v) := (x,y-1)\to(0,0)$, so we are reduced to the limit

$$\lim_{(x,y)\to(0,1)} \frac{x^3\sinh(y-1)}{x^2+y^2-2y+1} = \lim_{(x,y)\to(0,1)} \frac{x^3\sinh(y-1)}{x^2+(y-1)^2} = \lim_{(u,v)\to(0,0)} \frac{u^3\sinh v}{u^2+v^2} =: \lim_{(u,v)\to(0,0)} \widetilde{f}(u,v).$$

The limit is an indeterminate form $\frac{0}{0}$. In polar coordinated for (u,v) we have

$$\widetilde{f}(u,v) = \frac{\rho^3\cos^3\theta\sinh(\rho\sin\theta)}{\rho^2} = \rho\cos^3\theta\sinh(\rho\sin\theta).$$

From this, we guess that the liumit exists and it is equal to 0. Indeed,

$$|\widetilde{f}(u,v)-0| \leqslant \rho|\sinh(\rho\sin\theta)|.$$

Reminding of $\sinh t = t + o(t)$, we have

$$\sinh(\rho\sin\theta) = \rho\sin\theta + o(\rho\sin\theta),$$

so

$$|\sinh(\rho\sin\theta)| = |\rho\sin\theta + o(\rho)| \leqslant \rho + o(\rho),$$

from which

$$|\widetilde{f}(u,v)-0| \leqslant \rho\,(\rho+o(\rho)) = \rho^2 + o(\rho^2) =: g(\rho) \longrightarrow 0,\ \rho\to 0.$$

We can now conclude that $\lim_{(u,v)\to\vec{0}} \widetilde{f}(u,v) = 0$, hence the same holds for the limit of f. $\square$

#6 Since $(x,y)\to(1,1)$, $(u,v) := (x-1,y-1)\to(0,0)$, so

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(u,v)\to(0,0)} \frac{u^2v^7}{(u^2+v^2)^{5/2}} =: \lim_{(u,v)\to(0,0)} \widetilde{f}(u,v).$$

Clearly, the limit is an indeterminate form $\frac{0}{0}$. Let's pass to polar coordinates $u = \rho\cos\theta$, $v = \rho\sin\theta$

$$\widetilde{f}(u,v) = \frac{\rho^9\cos^2\theta\sin^7\theta}{(\rho^2)^{5/2}} = \rho^4\cos^2\theta\sin^7\theta,$$

so

$$|\widetilde{f}(u,v) - 0| \leqslant \rho^4 \longrightarrow 0, \ \rho \to 0,$$

and from this it follows that

$$\lim_{(x,y)\to(1,1)} f(x,y) = \lim_{(u,v)\to(0,0)} \widetilde{f}(u,v) = 0. \quad \square$$

1.8.10 #2 The limit does not exist: indeed, $f(x,0) = x^4 - x^2 \longrightarrow +\infty$ is $|x| \to +\infty$. $f(x,x) \equiv 0 \longrightarrow 0$ when $|x| \to +\infty$.

#3 We have $f(x,0) = x^2 \longrightarrow +\infty$ when $|x| \to +\infty$. Let's look at f under polar coordinates: we get

$$f(x,y) = \rho^4\cos^2\theta\sin^2\theta + \rho^2 - \rho^2\cos\theta\sin\theta.$$

Apparently, the first term is the strongest one. However, if one of the two coords vanishes, the first term is constantly $= 0$. This suggests that this term is not particularly determinant. Being also positive, we may notice that

$$f(x,y) \geqslant \rho^2 - \rho^2\cos\theta\sin\theta = \rho^2\left(1 - \frac{1}{2}\sin(2\theta)\right) \geqslant \frac{1}{2}\rho^2 \longrightarrow +\infty, \ \rho \to +\infty.$$

This is sufficient to establish that $\exists \lim_{(x,y)\to\infty_2} f(x,y) = +\infty$.

#4. We have $f(x,0,0) = x^4 \longrightarrow +\infty$ when $|x| \to +\infty$, so if a limit exists it must be $= +\infty$. To compute the limit, an idea could be to use spherical coordinates. This, however, does not simplify the term $x^4 + y^4 + z^4$. We might expect that this term is somehow correlated to $(x^2+y^2+z^2)^2$ and indeed

$$(x^2+y^2+z^2)^2 = x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 \geqslant x^4 + y^4 + z^4.$$

This is an upper bound for $x^4+y^4+z^4$. To get a more useful lower bound we remind of the elementary inequality $2ab \leqslant a^2 + b^2$, so

$$(x^2+y^2+z^2)^2 = x^4 + y^4 + z^4 + \underbrace{2x^2y^2}_{\leqslant (x^2)^2+(y^2)^2 = x^4+y^4} + \underbrace{2x^2z^2}_{\leqslant x^4+z^4} + \underbrace{2y^2z^2}_{\leqslant y^4+z^4} \leqslant 3(x^4+y^4+z^4),$$

so

$$x^4+y^4+z^4 \geqslant \frac{1}{3}\left(x^2+y^2+z^2\right)^2.$$

Now, with this we can say that, in spherical coordinates

$$f(x,y,z) \geqslant \frac{1}{3}\left(x^2+y^2+z^2\right)^2 - xyz = \frac{1}{3}\rho^4 - \rho^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta \geqslant \frac{1}{3}\rho^4 - \rho^3 \longrightarrow +\infty$$

when $\rho \to +\infty$. This shows that $\exists \lim_{(x,y,z)\to\infty_3} f = +\infty$. $\square$

#5 Notice that $f(x,0,0) = x^2 \longrightarrow +\infty$ when $|x| \to +\infty$. So, if a limit exists, it must be equal to $+\infty$. Before applying the spherical coordinates, we write

$$f(x,y,z) = x^2 + y^2 + z^2 - xz + z^4 - z^2.$$

Let's focus on the first part $x^2 + y^2 + z^2 - xz$. Using spherical coords, we have

$$x^2 + y^2 + z^2 - xz = \rho^2 - \rho^2 \sin\varphi\cos\theta\cos\varphi = \rho^2\left(1 - \frac{1}{2}\sin(2\varphi)\cos\theta\right) \geqslant \rho^2\frac{1}{2}.$$

On the other side, we notice that there exists a constant C such that $z^4 - z^2 \geqslant C$ for every $z \in \mathbb{R}$. Indeed, if $h(z) = z^4 - z^2$, for $z > 0$ we have

$$h'(z) = 4z^3 - 2z = 2z(2z^2 - 1) \geqslant 0, \quad \Longleftrightarrow \quad 2z^2 - 1 \geqslant 0, \quad \Longleftrightarrow \quad z \geqslant \frac{1}{\sqrt{2}}.$$

This means that $h \searrow$ on $[0, \frac{1}{\sqrt{2}}]$ and $h \nearrow$ on $[\frac{1}{\sqrt{2}}, +\infty[$, so $z = \frac{1}{\sqrt{2}}$ is a minimum for h on $[0,+\infty[$. Since $h(-z) = h(z)$ we have that it is also a minimum for all $z \in \mathbb{R}$. Thus,

$$h(z) \geqslant h(\frac{1}{\sqrt{2}}) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}, \quad \Longleftrightarrow \quad z^4 - z^2 \geqslant -\frac{1}{4}.$$

We can now put the pieces together. Combining the two bounds we have

$$f(x,y,z) = \underbrace{x^2 + y^2 + z^2 - xz}_{\geqslant \frac{\rho^2}{2}} + \underbrace{z^4 - z^2}_{\geqslant -\frac{1}{4}} \geqslant \frac{\rho^2}{2} - \frac{1}{4} =: g(\rho) \longrightarrow +\infty, \ \rho \to +\infty.$$

From this $\exists \lim_{(x,y,z)\to\infty_3} f = +\infty$. $\square$

#6. Also here we have $f(0,0,z) = z^2 - z \longrightarrow +\infty$ when $|z| \to +\infty$. So, the possible candidate limit is $+\infty$. Before applying spherical coords, we notice that

$$f(x,y,z) = \sqrt{x^2 + y^2} + |z| + \left(z^2 - z - |z|\right).$$

Now, let $h(z) := z^2 - z - |z|$. As in the previous exercise, h is bounded from below. Indeed: for $z < 0$, $h(z) = z^2$, so $h \geqslant 0$; for $z \geqslant 0$, $h(z) = z^2 - 2z$, $h'(z) = 2z - 2 = 2(z-1)$, so $z = 1$ is a global minimum with $h(1) = -1$, so $h(z) \geqslant -1$ for every $z \in \mathbb{R}$. On the other hand, by using spherical coords,

$$\sqrt{x^2 + y^2} + |z| = \sqrt{\rho^2 \sin^2\varphi} + \rho|\cos\varphi| = \rho\left(|\sin\varphi| + |\cos\varphi|\right).$$

The coefficient $C(\varphi) := |\sin\varphi| + |\cos\varphi|$ is continuous for $[0,\pi]$, so by Weierstrass' theorem there exists the minimum of C. Since $C(\varphi) > 0$ for every φ, we deduce that $C(\varphi) \geqslant C(\varphi_{min}) =: K > 0$, Therefore, merging the two arguments,

$$f(x,y,z)=\underbrace{\sqrt{x^2+y^2}+|z|}_{\leqslant K\rho}+\underbrace{\left(z^2-z-|z|\right)}_{\geqslant -1}\geqslant K\rho-1=:g(\rho)\longrightarrow+\infty,\ \rho\to+\infty.$$

We conclude that $\exists \lim_{(x,y,z)\to\infty_3} f=+\infty$. $\square$

#7. Here we might have some suspect about the existence of the limit. The root term is positive and of size ρ^2, while the xyz is or order ρ^3 with variable sign. Take

$$f(x,x,x)=\sqrt{4x^4+x^4}-x^3=\sqrt{5}x^2-x^3\longrightarrow\begin{cases}-\infty, & x\to+\infty,\\ \\ +\infty, & x\to-\infty.\end{cases}$$

We conclude that $\nexists \lim_{(x,y,z)\to\infty_3} f$. $\square$

Next: do ex. 1.8.12/13/14/15