Analytical Methods for Engineering

Answers to LN Exercises

Ex. 1.4.1. i) Done in class.
ii) If X is finite, $\mathscr{F} = \mathscr{P}(X)$, so $\mathscr{F}$ is a $\sigma-$algebra. If X is infinite, then $\mathscr{F}$ is not a $\sigma-$algebra. Indeed, $X \notin \mathscr{F}$ for example.
iii) If X is uncountable, $X \notin F$, so $\mathscr{F}$ cannot be a $\sigma-$algebra. Let's consider the case when X is countable. Then, every subset of X is countable, thus $\mathscr{F} = \mathscr{P}(X)$, so it is a $\sigma-$algebra. □

Ex. 1.4.2. We start noticing that $\mathscr{S}$ is not a $\sigma-$algebra. Now, any $\sigma-$algebra containing $\mathscr{S}$ must also contain $\{a\}^c = \{b, c, d\}$ and $\{a, c\}^c = \{b, d\}$ as well as the (countable) unions of its sets. So, also $\{a, b, d\}$ must be in the $\sigma-$algebra, as well as its complementary $\{a, b, d\}^c = \{c\}$. Therefore, any $\sigma-$algebra containing $\mathscr{S}$ must contain

$$\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}.$$

Since this is a $\sigma-$algebra (easy check), we conclude that it is $\sigma(\mathscr{S})$. □

Ex. 1.4.3. Let $\mathscr{F} := \sigma(\{A, B\})$. We decompose the set X in the following disjoint sets: $A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c$. All these sets must belong to $\sigma(\{A, B\})$, so all possible finite unions of these. Among them, notice that we have

$$A = (A \cap B^c) \cup (A \cap B), \quad B = (B \cap A^c) \cup (B \cap A).$$

Since these 4 sets are disjoint, it is easy to check that the family $\mathscr{F}$ made of all possible finite unions of them is a $\sigma-$algebra that, by construction, must be contained in $\sigma(\mathscr{S})$. On the other hand, since $\{A, B\} \subset \mathscr{F}$, and $\mathscr{F}$ is a $\sigma-$algebra, we have (by definition of $\sigma(\mathscr{S})$), $\sigma(\mathscr{S}) \subset \mathscr{F}$. So,

$$\sigma(\mathscr{S}) = \mathscr{F} = \{\emptyset, A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c, A, B, A^c, B^c,$$
$$A \cup B, A \cup B^c, A^c \cup B, A^c \cup B^c, (A \triangle B), (A \triangle B)^c\}. \quad \square$$

Ex. 1.4.4. We already proved in class that $\mathscr{F}$ is a $\sigma-$algebra. We may notice that, in this case, for every $A \in \mathscr{F}$ only one of A or A^c can be countable. This because X is uncountable, so if for example A is countable, then $A^c = X \backslash A$ is uncountable and vice versa. This remark is important because it says that the function μ is well defined for every $A \in \mathscr{F}$. Indeed: since if $A \in \mathscr{F}$ only one of A, A^c can be countable, the value $\mu(A)$ is well defined.

Now, let's check whether μ is a measure or not. According to the definition we have to check that $\mu(\emptyset) = 0$ and countable additivity. Now, since $\emptyset$ has 0 elements, it is countable, thus $\mu(\emptyset) = 0$ by definition of μ. Let not $(A_n) \subset \mathscr{F}$ be a disjoint family. We have to determine if

$$(\star) \quad \mu\left(\bigsqcup_n A_n\right) = \sum_n \mu(A_n).$$

Since $A_n \in \mathscr{F}$ for every $n \in \mathbb{N}$, either A_n or A_n^c is countable. We have the following alternative:

- either A_n is countable for every $n \in \mathbb{N}$,
- or, at least one of A_n^c is countable, say $\exists N \in \mathbb{N}$ such that A_N^c is countable.

In the first case, $\bigsqcup_n A_n$ is countable (countable union of countable sets), so

$$\mu\left(\bigsqcup_n A_n\right) = 0, \text{ and } \sum_n \mu(A_n) = \sum_n 0 = 0,$$

and ($\star$) holds in this case. In the second case, $\bigsqcup_n A_n \supset A_N$, so $(\bigsqcup_n A_n)^c \subset A_N^c$ is countable, so

$$\mu\left(\bigsqcup_n A_n\right) = 1.$$

In the sum $\sum_n \mu(A_n)$ at least $\mu(A_N) = 1$, so the sum is $\geqslant 1$. If $\mu(A_n) = 0$ for $n \neq N$ we have the conclusion. Assume for a moment that $\mu(A_M) = 1$ for some $M \neq N$. Then, A_M^c would be countable and

$$A_M \cap A_N = \varnothing, \implies X = A_M^c \cup A_N^c,$$

so X would be the union of countable sets, and therefore X itself would be countable, contradicting the assumption. We conclude that $\mu(A_n) = 0$ for all $n \neq N$ and countable additivity follows. □

Ex. 1.4.7. Let $E, F, G \in \mathscr{F}$. We have

$$\mu(E \cup F \cup G) = \mu(E) + \mu((F \cup G)\backslash E) = \mu(E) + \mu((F\backslash E) \cup (G\backslash E)).$$

We recall that, if $A, B \in \mathscr{F}$ and $\mu(A \cap B) < +\infty$ we have

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$

so

$$\begin{aligned} \mu((F\backslash E) \cup (G\backslash E)) &= \mu(F\backslash E) + \mu(G\backslash E) - \mu((F \cap G)\backslash E) \\ &= (\mu(F) - \mu(E \cap F)) + (\mu(G) - \mu(E \cap G)) - \mu((F \cap G)\backslash E) \\ &= \mu(F) + \mu(G) - (\mu(E \cap F) + \mu(E \cap G)) - \mu((F \cap G)\backslash E). \end{aligned}$$

provided $\mu(E \cap F), \mu(E \cap G), \mu(F \cap G) < +\infty$. Now,

$$\mu((F \cap G)\backslash E) = \mu(F \cap G) - \mu(E \cap F \cap G),$$

because $\mu(E \cap F \cap G) \leqslant \mu(E \cap F) < +\infty$, so

$$\mu(E \cup F \cup G) = \mu(E) + \mu(F) + \mu(G) - (\mu(E \cap F) + \mu(E \cap G) + \mu(F \cap G)) + \mu(E \cap F \cap G). \quad \square$$

Ex. 1.4.9. i) Let's start from the set S. An element $x \in S$ iff $x \in E_j$ for infinitely many j, that is

$$\exists j_1 < j_2 < \dots \; : \; x \in \bigcap_{k=1}^{\infty} E_{j_k}.$$

Of course, indexes j_k depends on the specific point x. So we need to determine a better way to characterize points of S. We may notice that the previous property is equivalent to

$$\forall n, \; \exists j \geqslant n, \; : \; x \in E_j.$$

In this way

$$x \in S, \quad \Longleftrightarrow \quad \forall n \in \mathbb{N}, \; x \in \bigcup_{j \geqslant n} E_j, \quad \Longleftrightarrow \quad x \in \bigcap_n \bigcup_{j \geqslant n} E_j.$$

So,

$$S = \bigcap_n \bigcup_{j \geqslant n} E_j,$$

and since this is a set operation on the $(E_j) \subset \mathscr{F}$ we get $S \in \mathscr{F}$.
ii) To determine the measure of S we have to compute

$$\mu(S) = \mu\left(\bigcap_n \bigcup_{j \geqslant n} E_j \right).$$

Call $F_n := \bigcup_{j \geqslant n} E_j$. It is clear that $F_n \supset F_{n+1}$, so $F_n \searrow$. So, S is a decreasing limit of (F_n) and the idea could be to apply continuity from above to compute $\mu(S)$. This is feasible if $\mu(F_0) < +\infty$. But,

$$\mu(F_0) = \mu\left(\bigcup_{j \geqslant 0} E_j \right) \leqslant \sum_j \mu(E_j) < +\infty,$$

because of the assumption. Therefore, continuity from above applies and

$$\mu(S) = \lim_n \mu(F_n).$$

Finally,

$$\mu(F_n) = \mu\left(\bigcup_{j \geqslant n} E_j \right) \leqslant \sum_{j \geqslant n} \mu(E_j) \longrightarrow 0,$$

being this the tail of a convergent series. □

2.3.1. Suppone, by contradiction, that N^c is not dense in $\mathbb{R}$, that is

$$\exists]a, b[\subset \mathbb{R}, \;\; N^c \cap]a, b[= \emptyset.$$

Then $]a, b[\subset N$, so $0 = \lambda(N) \geqslant \lambda(]a, b[) = b - a > 0$, which is impossible. □

2.3.2 We first notice that each C_n is made of a finite union of closed intervals, thus it is a closed set. Therefore, $C_n \in \mathscr{M}_1$ for every n, hence $C := \bigcap_n C_n \in \mathscr{M}_1$. In alternative, we may also notice that in general, an infinite intersection of closed sets is closed, so C is closed. Since the Lebesgue class $\mathscr{M}_1$ contains both open and closed sets, we deduce $C \in \mathscr{M}_1$.

About $\lambda(C)$ we may notice that $0 \leqslant \lambda(C) \leqslant \lambda(C_n)$ for every n. Now, each C_n is the union of 2^n disjoint intervals each of length $\frac{1}{3^n}$, so $\lambda(C_n) = 2^n \frac{1}{3^n}$, from which $\lambda(C) \leqslant \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \longrightarrow 0$ when $n \longrightarrow +\infty$. Thus, necessarily, $\lambda(C) \leqslant 0$, from which $\lambda(C) = 0$. □

2.3.5. Let

$$E_{m,n} := \{(x,y) \ : \ mx + ny = 0\},$$

with $(m,n) \in \mathbb{N}\times\mathbb{N}\backslash\{(0,0)\}$. Since $(m,n) \neq (0,0)$, $E_{m,n}$ is a plane straight line, so $\lambda_2(E_{m,n}) = 0$, and

$$E = \bigcup_{(m,n)\in\mathbb{N}^2\backslash\{(0,0)\}} E_{m,n},$$

is a countable union. Therefore, by sub-additivity, $\lambda_2(E) \leqslant \sum_{m,n} \lambda_2(E_{m,n}) = 0$. □

2.3.6. By definition,

$$\forall \varepsilon > 0, \ \exists O_\varepsilon, \widetilde{O_\varepsilon} \text{ open } : \ \lambda^*(O_\varepsilon \backslash B) \leqslant \varepsilon, \ \ \lambda^*(\widetilde{O_\varepsilon} \backslash A) \leqslant \varepsilon,$$

2.3.7. The assumption says that $\lambda((A \cap B) \cup (A \cap C) \cup (B \cap C)) = 1$. The thesis is to prove that at least one of $\lambda(A), \lambda(B)\lambda(C)$ must be $\geqslant \frac{2}{3}$. If the conclusion were false, then $\lambda(A), \lambda(B), \lambda(C) < \frac{2}{3}$. Now, we notice that

$$\begin{aligned}
\lambda(E \cup F \cup G) &= \lambda(E \cup F) + \lambda(G) - \lambda((E \cup F) \cap G) \\
&= \lambda(E) + \lambda(F) + \lambda(G) - \lambda(E \cap F) - (\lambda(E \cap G) + \lambda(F \cap G) - \lambda(E \cap F \cap G)) \\
&= \lambda(E) + \lambda(F) + \lambda(G) - (\lambda(E \cap F) + \lambda(E \cap G) + \lambda(F \cap G)) + \lambda(E \cap F \cap G).
\end{aligned}$$

We apply this a first time to $E = A$, $F = B$ and $G = C$ and a second time to $E = A \cap B$, $F = A \cap C$ and $G = B \cap C$. In this last case, by the assumption, we get

$$1 = \lambda(A \cap B) + \lambda(A \cap C) + \lambda(B \cap C) - 3\lambda(A \cap B \cap C) + \lambda(A \cap B \cap C)$$

that is

$$\lambda(A \cap B) + \lambda(A \cap C) + \lambda(B \cap C) = 1 + 2\lambda(A \cap B \cap C),$$

and since, of course, $\lambda(A \cup B \cup C) = 1$, we have

$$1 = \lambda(A) + \lambda(B) + \lambda(C) - (1 + 2\lambda(A \cap B \cap C)) + \lambda(A \cap B \cap C),$$

from which

$$\lambda(A) + \lambda(B) + \lambda(C) = 2 + \lambda(A \cap B \cap C).$$

Now, if $\lambda(A), \lambda(B), \lambda(C) < \frac{2}{3}$ then we would have

$$2 \leqslant 2 + \lambda(A \cap B \cap C) = \lambda(A) + \lambda(B) + \lambda(C) < 3\frac{2}{3} = 2,$$

which is impossible! □

2.3.8. Let $N \subset [0,1]$ with $\lambda(N) = 0$. The goal is to prove that $\lambda(N^2) = 0$ where $N^2 = \{x^2 \ : \ x \in N\}$. Since

$$0 = \lambda(N) = \inf \left\{ \sum_n |I_n| \ : \ N \subset \bigcup_n I_n \right\}$$

by the characteristics of inf we have that

$$\forall \varepsilon > 0, \ \exists (I_n^\varepsilon)_n \ : \ N \subset \bigcup_n I_n^\varepsilon, \ \sum_n |I_n^\varepsilon| \leqslant \varepsilon.$$

Since $N \subset [0,1]$, we may assume that $I_n^\varepsilon \subset [0,1]$. Otherwise, we replace I_n^ε with $J_n^\varepsilon = I_n^\varepsilon \cap [0,1]$: J_n^ε is still an interval, being intersection of intervals,

$$N \subset \bigcup_n I_n^\varepsilon, \implies N = N \cap [0,1] \subset \bigcup_n I_n^\varepsilon \cap [0,1] = \bigcup_n J_n^\varepsilon$$

and moreover

$$\sum_n |J_n^\varepsilon| \leqslant \sum_n |I_n^\varepsilon| \leqslant \varepsilon.$$

Now, writing $J_n^\varepsilon = [a_n^\varepsilon, b_n^\varepsilon] \subset [0,1]$, we would have

$$N^2 \subset \bigcup_n (J_n^\varepsilon)^2 = \bigcup_n [(a_n^\varepsilon)^2, (b_n^\varepsilon)^2]$$

and

$$\sum_n |(J_n^\varepsilon)^2| = \sum_n \left((b_n^\varepsilon)^2 - (a_n^\varepsilon)^2 \right) = \sum_n \left(b_n^\varepsilon - a_n^\varepsilon \right) \underbrace{\left(b_n^\varepsilon + a_n^\varepsilon \right)}_{0 \leqslant \ldots \leqslant 2} \leqslant 2 \sum_n \left(b_n^\varepsilon - a_n^\varepsilon \right) = 2 \sum_n |J_n^\varepsilon| \leqslant 2\varepsilon.$$

From this and by the definition of λ, we get

$$\lambda(N^2) \leqslant 2\varepsilon,$$

and since ε can be made arbitrarily small, this shows that $\lambda(N^2) = 0$.

In N is bounded, $N \subset [-R, R]$, the previous argument leads to a similar bound $\lambda(N^2) \leqslant 2R\varepsilon$, so we conclude similarly.

Finally, if N is generic, define $N_R := N \cap [-R, R]$. It is clear that $N_R^2 = N^2 \cap [0, R^2] \uparrow N^2$ (when $R \longrightarrow +\infty$) and since $\lambda(N_R^2) = 0$ for every R, by the continuity from below of λ we obtain also $\lambda(N^2) = 0$. □

3.4.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be, for example, increasing, so $f(x) \leqslant f(y)$ when $x \leqslant y$. We prove that $\{f \leqslant a\}$ is measurable. Intuitively, $\{f \leqslant a\}$ should be an interval of type $]\infty, \alpha[$ or $]-\infty, \alpha]$ where $\alpha := \sup\{x \ : \ f(x) \leqslant a\}$. Indeed: let α be defined as above. Either $\alpha = +\infty$ or $\alpha < +\infty$. In the first case, $f(x) \leqslant a$ for all $x \in \mathbb{R}$, so $\{f \leqslant a\} = \mathbb{R}$. In the second case, we claim that

$$]-\infty, \alpha[\subset \{f \leqslant a\} \subset]-\infty, \alpha].$$

Indeed: if $x < \alpha$ then, by definition of sup, there exists $\beta > x$ such that $f(\beta) \leqslant a$. But then, being f increasing, $f(x) \leqslant f(\beta) \leqslant a$, so $x \in \{f \leqslant a\}$. This proves that $]-\infty, \alpha[\subset \{f \leqslant a\}$. To prove the second inclusion we prove that, if $x > \alpha$ it cannot be $x \in \{f \leqslant a\}$. Otherwise, $f(x) \leqslant a$, so

$$\alpha = \sup\{x \,:\, f(x) \leqslant a\} \geqslant x > \alpha,$$

obtaining a contradiction. Conclusion: $\{f \leqslant a\}$ can be only $]-\infty, \alpha[$ or $]-\infty, \alpha]$, in both cases it is an interval, so it is a measurable set. □

3.4.2. We have to prove that i) is equivalent to ii) where

i) f is measurable
ii) $\{f > a\} \in \mathscr{F}$ for every $a \in \mathbb{Q}$.

Since i) is equivalent to $\{f > a\} \in \mathscr{F}$ for every $a \in \mathbb{R}$, i) $\Longrightarrow$ ii).

Let's prove that ii) $\Longrightarrow$ i), that is, let's prove that $\{f > a\} \in \mathscr{F}$ for every $a \in \mathbb{R}$. By ii), this is true if $a \in \mathbb{Q}$. So let $a \in \mathbb{Q}^c$ (irrational). We notice that, if $q \in \mathbb{Q}$ is such that $q > a$, then

$$\{f > q\} \subset \{f > a\}.$$

Since this happens for every $q \in \mathbb{Q}$, $q > a$, we can say that

$$\bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\} \subset \{f > a\}.$$

At left, we have a countable union of measurable sets, so the union is a measurable set. So, if we prove that = holds, we are done! That is, the goal is reduced to prove that

$$\{f > a\} \subset \bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\}.$$

Pick $x \in \{f > a\}$. So, $f(x) > a$. Because of the density of rationals in reals, there exists $r \in \mathbb{Q}$ such that $f(x) > r > a$, so $x \in \{f > r\} \subset \bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\}$. This means that $\{f > a\} \subset \bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\}$ as claimed. □

3.4.4. Notice that

$$\{fg > a\} = \{fg > a,\ g > 0\} \cap \{fg > a,\ g = 0\} \cap \{fg > a,\ g < 0\}.$$

Let's analyze the three sets, starting by the second one (easier), and the first and the third ones being similar. We have

$$\{fg > a,\ g = 0\} = \{0 > a,\ g = 0\} = \begin{cases} \emptyset \in \mathscr{F}, & a \geqslant 0, \\ \\ \{g = 0\} \in \mathscr{F}, & a < 0 \end{cases}$$

For the first set we have

$$\{fg > a,\ g > 0\} = \{f > \frac{a}{g},\ g > 0\} = \bigcup_{q\in\mathbb{Q}} \left\{f > q > \frac{a}{g},\ g > 0\right\} = \bigcup_{q\in\mathbb{Q}} \underbrace{\{f > q\}}_{\in\mathscr{F}} \cap \underbrace{\left\{g > 0, g > \frac{a}{q}\right\}}_{\in\mathscr{F}}$$

from which we see that $\{fg > a,\ g > 0\} \in \mathscr{F}$. Similar argument for the third set. From this the conclusion follows. □

3.4.6. i) Claim: $f_n(x) \longrightarrow 0$ for every $x \in \mathbb{R}$. Take $n \geqslant [x] + 1$. Then $x < [x] + 1 \leqslant n$, from which $f_n(x) = 0$. This means that $(f_n(x))$ is constantly $= 0$ for n large, thus $f_n(x) \longrightarrow 0$.

ii) Claim: $f_n(x) \longrightarrow 1_{]0,+\infty[}(x)$ for every x. Indeed: if $x \leqslant 0$, $f_n(x) \equiv 0 \longrightarrow 0$. If $x > 0$, since $\frac{1}{n} \to 0$ and $n \to +\infty$, for n large enough $\frac{1}{n} < x < n$, so $f_n(x) \equiv 1 \longrightarrow 1$.

iii) We notice that

$$f_{2k}(x) = 1_{[0,1/2]}(x), \quad f_{2k+1}(x) = 1_{[1/2,1]}(x).$$

For $x < 0$ and $x > 1$ we have $f_n(x) \equiv 0 \longrightarrow 0$. For $x = 1/2$ we have also $f_n(x) \equiv 1 \longrightarrow 1$. If however $0 \leqslant x < 1/2$ we have that $(f_n(x)) = (1, 0, 1, 0, \dots)$ so there is no limit. Similarly, for $1/2 < x \leqslant 1$, $(f_n(x))$ has no limit. Since the limit of (f_n) does not exist for $x \in [0, 1/2[\cup]1/2, 1]$, which is a positive measure set, we cannot conclude that (f_n) converges pointwise a.e..

3.4.7. We do the proof in dimension $d = 1$ for simplicity, the argument is the same for the general case. Suppose that $g(x) > 0$ for some $x \in \mathbb{R}$. By continuity, there exists a neighborhood U_x of x for which $g(y) > 0, \forall y \in U_x$. We can always assume that $U_x = [x - \varepsilon, x + \varepsilon]$. Therefore

$$\{g \neq 0\} \supset [x - \varepsilon, x + \varepsilon], \quad 0 = \lambda(\{g \neq 0\}) \geqslant \lambda([x - \varepsilon, x + \varepsilon]) = 2\varepsilon > 0,$$

which is a contradiction. □