Analytical Methods for Engineering

Answers to LN Exercises

1.4.1. i) Done in class.
ii) If X is finite, $\mathscr{F} = \mathscr{P}(X)$, so $\mathscr{F}$ is a $\sigma-$algebra. If X is infinite, then $\mathscr{F}$ is not a $\sigma-$algebra. Indeed, $X \notin \mathscr{F}$ for example.
iii) If X is uncountable, $X \notin F$, so $\mathscr{F}$ cannot be a $\sigma-$algebra. Let's consider the case when X is countable. Then, every subset of X is countable, thus $\mathscr{F} = \mathscr{P}(X)$, so it is a $\sigma-$algebra. □

1.4.2. We start noticing that $\mathcal{S}$ is not a $\sigma-$algebra. Now, any $\sigma-$algebra containing $\mathcal{S}$ must also contain $\{a\}^c = \{b, c, d\}$ and $\{a, c\}^c = \{b, d\}$ as well as the (countable) unions of its sets. So, also $\{a, b, d\}$ must be in the $\sigma-$algebra, as well as its complementary $\{a, b, d\}^c = \{c\}$. Therefore, any $\sigma-$algebra containing $\mathcal{S}$ must contain

$$\{\varnothing, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}\,.$$

Since this is a $\sigma-$algebra (easy check), we conclude that it is $\sigma(\mathcal{S})$. □

1.4.3. Let $\mathscr{F} := \sigma(\{A, B\})$. We decompose the set X in the following disjoint sets: $A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c$. All these sets must belong to $\sigma(\{A, B\})$, so all possible finite unions of these. Among them, notice that we have

$$A = (A \cap B^c) \cup (A \cap B), \quad B = (B \cap A^c) \cup (B \cap A).$$

Since these 4 sets are disjoint, it is easy to check that the family $\mathscr{F}$ made of all possible finite unions of them is a $\sigma-$algebra that, by construction, must be contained in $\sigma(\mathcal{S})$. On the other hand, since $\{A, B\} \subset \mathscr{F}$, and $\mathscr{F}$ is a $\sigma-$algebra, we have (by definition of $\sigma(\mathcal{S})$), $\sigma(\mathcal{S}) \subset \mathscr{F}$. So,

$$\sigma(\mathcal{S}) = \mathscr{F} = \{\varnothing, A \cap B^c, A \cap B, B \cap A^c, A^c \cap B^c, A, B, A^c, B^c,$$
$$A \cup B, A \cup B^c, A^c \cup B, A^c \cup B^c, (A \triangle B), (A \triangle B)^c\}. \quad \square$$

1.4.4. We already proved in class that $\mathscr{F}$ is a $\sigma-$algebra. We may notice that, in this case, for every $A \in \mathscr{F}$ only one of A or A^c can be countable. This because X is uncountable, so if for example A is countable, then $A^c = X \backslash A$ is uncountable and vice versa. This remark is important because it says that the function μ is well defined for every $A \in \mathscr{F}$. Indeed: since if $A \in \mathscr{F}$ only one of A, A^c can be countable, the value $\mu(A)$ is well defined.

Now, let's check whether μ is a measure or not. According to the definition we have to check that $\mu(\varnothing) = 0$ and countable additivity. Now, since $\varnothing$ has 0 elements, it is countable, thus $\mu(\varnothing) = 0$ by definition of μ. Let not $(A_n) \subset \mathscr{F}$ be a disjoint family. We have to determine if

$$(\star) \quad \mu\left(\bigsqcup_n A_n\right) = \sum_n \mu(A_n).$$

Since $A_n \in \mathscr{F}$ for every $n \in \mathbb{N}$, either A_n or A_n^c is countable. We have the following alternative:

- either A_n is countable for every $n \in \mathbb{N}$,
- or, at least one of A_n^c is countable, say $\exists N \in \mathbb{N}$ such that A_N^c is countable.

In the first case, $\bigsqcup_n A_n$ is countable (countable union of countable sets), so

$$\mu\left(\bigsqcup_n A_n\right) = 0, \quad \text{and} \quad \sum_n \mu(A_n) = \sum_n 0 = 0,$$

and $(\star)$ holds in this case. In the second case, $\bigsqcup_n A_n \supset A_N$, so $(\bigsqcup_n A_n)^c \subset A_N^c$ is countable, so

$$\mu\left(\bigsqcup_n A_n\right) = 1.$$

In the sum $\sum_n \mu(A_n)$ at least $\mu(A_N) = 1$, so the sum is $\geqslant 1$. If $\mu(A_n) = 0$ for $n \neq N$ we have the conclusion. Assume for a moment that $\mu(A_M) = 1$ for some $M \neq N$. Then, A_M^c would be countable and

$$A_M \cap A_N = \varnothing, \implies X = A_M^c \cup A_N^c,$$

so X would be the union of countable sets, and therefore X itself would be countable, contradicting the assumption. We conclude that $\mu(A_n) = 0$ for all $n \neq N$ and countable additivity follows. □

1.4.7. Let $E, F, G \in \mathscr{F}$. We have

$$\mu(E \cup F \cup G) = \mu(E) + \mu((F \cup G)\backslash E) = \mu(E) + \mu((F\backslash E) \cup (G\backslash E)).$$

We recall that, if $A, B \in \mathscr{F}$ and $\mu(A \cap B) < +\infty$ we have

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$

so

$$\begin{aligned}\mu((F\backslash E) \cup (G\backslash E)) &= \mu(F\backslash E) + \mu(G\backslash E) - \mu((F \cap G)\backslash E)\\ &= (\mu(F) - \mu(E \cap F)) + (\mu(G) - \mu(E \cap G)) - \mu((F \cap G)\backslash E)\\ &= \mu(F) + \mu(G) - (\mu(E \cap F) + \mu(E \cap G)) - \mu((F \cap G)\backslash E).\end{aligned}$$

provided $\mu(E \cap F), \mu(E \cap G), \mu(F \cap G) < +\infty$. Now,

$$\mu((F \cap G)\backslash E) = \mu(F \cap G) - \mu(E \cap F \cap G),$$

because $\mu(E \cap F \cap G) \leqslant \mu(E \cap F) < +\infty$, so

$$\mu(E \cup F \cup G) = \mu(E) + \mu(F) + \mu(G) - (\mu(E \cap F) + \mu(E \cap G) + \mu(F \cap G)) + \mu(E \cap F \cap G). \quad \square$$

1.4.9. i) Let's start from the set S. An element $x \in S$ iff $x \in E_j$ for infinitely many j, that is

$$\exists j_1 < j_2 < \dots : x \in \bigcap_{k=1}^{\infty} E_{j_k}.$$

Of course, indexes j_k depends on the specific point x. So we need to determine a better way to characterize points of S. We may notice that the previous property is equivalent to

$$\forall n,\ \exists j \geqslant n,\ :\ x \in E_j.$$

In this way

$$x \in S, \iff \forall n \in \mathbb{N},\ x \in \bigcup_{j \geqslant n} E_j, \iff x \in \bigcap_n \bigcup_{j \geqslant n} E_j.$$

So,

$$S = \bigcap_n \bigcup_{j \geqslant n} E_j,$$

and since this is a set operation on the $(E_j) \subset \mathscr{F}$ we get $S \in \mathscr{F}$.
ii) To determine the measure of S we have to compute

$$\mu(S) = \mu\left(\bigcap_n \bigcup_{j \geqslant n} E_j\right).$$

Call $F_n := \bigcup_{j \geqslant n} E_j$. It is clear that $F_n \supset F_{n+1}$, so $F_n \searrow$. So, S is a decreasing limit of (F_n) and the idea could be to apply continuity from above to compute $\mu(S)$. This is feasible if $\mu(F_0) < +\infty$. But,

$$\mu(F_0) = \mu\left(\bigcup_{j \geqslant 0} E_j\right) \leqslant \sum_j \mu(E_j) < +\infty,$$

because of the assumption. Therefore, continuity from above applies and

$$\mu(S) = \lim_n \mu(F_n).$$

Finally,

$$\mu(F_n) = \mu\left(\bigcup_{j \geqslant n} E_j\right) \leqslant \sum_{j \geqslant n} \mu(E_j) \longrightarrow 0,$$

being this the tail of a convergent series. □

2.3.1. Suppose, by contradiction, that N^c is not dense in $\mathbb{R}$, that is

$$\exists]a, b[\subset \mathbb{R},\ \ N^c \cap]a, b[= \varnothing.$$

Then $]a, b[\subset N$, so $0 = \lambda(N) \geqslant \lambda(]a, b[) = b - a > 0$, which is impossible. □

2.3.2 We first notice that each C_n is made of a finite union of closed intervals, thus it is a closed set. Therefore, $C_n \in \mathscr{M}_1$ for every n, hence $C := \bigcap_n C_n \in \mathscr{M}_1$. In alternative, we may also

notice that in general, an infinite intersection of closed sets is closed, so C is closed. Since the Lebesgue class $\mathscr{M}_1$ contains both open and closed sets, we deduce $C \in \mathscr{M}_1$.

About $\lambda(C)$ we may notice that $0 \leqslant \lambda(C) \leqslant \lambda(C_n)$ for every n. Now, each C_n is the union of 2^n disjoint intervals each of length $\frac{1}{3^n}$, so $\lambda(C_n) = 2^n \frac{1}{3^n}$, from which $\lambda(C) \leqslant \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n \longrightarrow 0$ when $n \longrightarrow +\infty$. Thus, necessarily, $\lambda(C) \leqslant 0$, from which $\lambda(C) = 0$. □

2.3.5. Let

$$E_{m,n} := \{(x, y) \ : \ mx + ny = 0\},$$

with $(m, n) \in \mathbb{N}\times\mathbb{N}\setminus\{(0, 0)\}$. Since $(m, n) \neq (0, 0)$, $E_{m,n}$ is a plane straight line, so $\lambda_2(E_{m,n}) = 0$, and

$$E = \bigcup_{(m,n)\in\mathbb{N}^2\setminus\{(0,0)\}} E_{m,n},$$

is a countable union. Therefore, by sub-additivity, $\lambda_2(E) \leqslant \sum_{m,n} \lambda_2(E_{m,n}) = 0$. □

2.3.6. By definition,

$$\forall \varepsilon > 0, \ \exists O_\varepsilon, \widetilde{O_\varepsilon} \text{ open } : \ \lambda^*(O_\varepsilon\setminus B) \leqslant \varepsilon, \ \ \lambda^*(\widetilde{O_\varepsilon}\setminus A) \leqslant \varepsilon,$$

2.3.7. The assumption says that $\lambda((A \cap B) \cup (A \cap C) \cup (B \cap C)) = 1$. The thesis is to prove that at least one of $\lambda(A), \lambda(B)\lambda(C)$ must be $\geqslant \frac{2}{3}$. If the conclusion were false, then $\lambda(A), \lambda(B), \lambda(C) < \frac{2}{3}$. Now, we notice that

$$\begin{aligned}
\lambda(E \cup F \cup G) &= \lambda(E \cup F) + \lambda(G) - \lambda((E \cup F) \cap G) \\
&= \lambda(E) + \lambda(F) + \lambda(G) - \lambda(E \cap F) - (\lambda(E \cap G) + \lambda(F \cap G) - \lambda(E \cap F \cap G)) \\
&= \lambda(E) + \lambda(F) + \lambda(G) - (\lambda(E \cap F) + \lambda(E \cap G) + \lambda(F \cap G)) + \lambda(E \cap F \cap G).
\end{aligned}$$

We apply this a first time to $E = A$, $F = B$ and $G = C$ and a second time to $E = A\cap B$, $F = A\cap C$ and $G = B \cap C$. In this last case, by the assumption, we get

$$1 = \lambda(A \cap B) + \lambda(A \cap C) + \lambda(B \cap C) - 3\lambda(A \cap B \cap C) + \lambda(A \cap B \cap C)$$

that is

$$\lambda(A \cap B) + \lambda(A \cap C) + \lambda(B \cap C) = 1 + 2\lambda(A \cap B \cap C),$$

and since, of course, $\lambda(A \cup B \cup C) = 1$, we have

$$1 = \lambda(A) + \lambda(B) + \lambda(C) - (1 + 2\lambda(A \cap B \cap C)) + \lambda(A \cap B \cap C),$$

from which

$$\lambda(A) + \lambda(B) + \lambda(C) = 2 + \lambda(A \cap B \cap C).$$

Now, if $\lambda(A), \lambda(B), \lambda(C) < \frac{2}{3}$ then we would have

$$2 \leqslant 2 + \lambda(A \cap B \cap C) = \lambda(A) + \lambda(B) + \lambda(C) < 3\frac{2}{3} = 2,$$

which is impossible! □

2.3.8. Let $N \subset [0,1]$ with $\lambda(N) = 0$. The goal is to prove that $\lambda(N^2) = 0$ where $N^2 = \{x^2 \ : \ x \in N\}$. Since

$$0 = \lambda(N) = \inf\left\{\sum_n |I_n| \ : \ N \subset \bigcup_n I_n\right\}$$

by the characteristics of inf we have that

$$\forall \varepsilon > 0, \ \exists (I_n^\varepsilon)_n \ : \ N \subset \bigcup_n I_n^\varepsilon, \ \sum_n |I_n^\varepsilon| \leqslant \varepsilon.$$

Since $N \subset [0,1]$, we may assume that $I_n^\varepsilon \subset [0,1]$. Otherwise, we replace I_n^ε with $J_n^\varepsilon = I_n^\varepsilon \cap [0,1]$: J_n^ε is still an interval, being intersection of intervals,

$$N \subset \bigcup_n I_n^\varepsilon, \implies N = N \cap [0,1] \subset \bigcup_n I_n^\varepsilon \cap [0,1] = \bigcup_n J_n^\varepsilon$$

and moreover

$$\sum_n |J_n^\varepsilon| \leqslant \sum_n |I_n^\varepsilon| \leqslant \varepsilon.$$

Now, writing $J_n^\varepsilon = [a_n^\varepsilon, b_n^\varepsilon] \subset [0,1]$, we would have

$$N^2 \subset \bigcup_n (J_n^\varepsilon)^2 = \bigcup_n [(a_n^\varepsilon)^2, (b_n^\varepsilon)^2]$$

and

$$\sum_n |(J_n^\varepsilon)^2| = \sum_n \left((b_n^\varepsilon)^2 - (a_n^\varepsilon)^2\right) = \sum_n \left(b_n^\varepsilon - a_n^\varepsilon\right) \underbrace{\left(b_n^\varepsilon + a_n^\varepsilon\right)}_{0 \leqslant \ldots \leqslant 2} \leqslant 2 \sum_n \left(b_n^\varepsilon - a_n^\varepsilon\right) = 2 \sum_n |J_n^\varepsilon| \leqslant 2\varepsilon.$$

From this and by the definition of λ, we get

$$\lambda(N^2) \leqslant 2\varepsilon,$$

and since ε can be made arbitrarily small, this shows that $\lambda(N^2) = 0$.

In N is bounded, $N \subset [-R, R]$, the previous argument leads to a similar bound $\lambda(N^2) \leqslant 2R\varepsilon$, so we conclude similarly.

Finally, if N is generic, define $N_R := N \cap [-R, R]$. It is clear that $N_R^2 = N^2 \cap [0, R^2] \uparrow N^2$ (when $R \to +\infty$) and since $\lambda(N_R^2) = 0$ for every R, by the continuity from below of λ we obtain also $\lambda(N^2) = 0$. □

3.4.1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be, for example, increasing, so $f(x) \leqslant f(y)$ when $x \leqslant y$. We prove that $\{f \leqslant a\}$ is measurable. Intuitively, $\{f \leqslant a\}$ should be an interval of type $]\infty, \alpha[$ or $]-\infty, \alpha]$ where $\alpha := \sup\{x \ : \ f(x) \leqslant a\}$. Indeed: let α be defined as above. Either $\alpha = +\infty$ or $\alpha < +\infty$. In the first case, $f(x) \leqslant a$ for all $x \in \mathbb{R}$, so $\{f \leqslant a\} = \mathbb{R}$. In the second case, we claim that

$$]-\infty, \alpha[\subset \{f \leqslant a\} \subset]-\infty, \alpha].$$

Indeed: if $x < \alpha$ then, by definition of sup, there exists $\beta > x$ such that $f(\beta) \leqslant a$. But then, being f increasing, $f(x) \leqslant f(\beta) \leqslant a$, so $x \in \{f \leqslant a\}$. This proves that $]-\infty, \alpha[\subset \{f \leqslant a\}$. To prove the second inclusion we prove that, if $x > \alpha$ it cannot be $x \in \{f \leqslant a\}$. Otherwise, $f(x) \leqslant a$, so

$$\alpha = \sup\{x \ : \ f(x) \leqslant a\} \geqslant x > \alpha,$$

obtaining a contradiction. Conclusion: $\{f \leqslant a\}$ can be only $]-\infty, \alpha[$ or $]-\infty, \alpha]$, in both cases it is an interval, so it is a measurable set. □

3.4.2. We have to prove that i) is equivalent to ii) where

i) f is measurable
ii) $\{f > a\} \in \mathscr{F}$ for every $a \in \mathbb{Q}$.

Since i) is equivalent to $\{f > a\} \in \mathscr{F}$ for every $a \in \mathbb{R}$, i) $\Longrightarrow$ ii).

Let's prove that ii) $\Longrightarrow$ i), that is, let's prove that $\{f > a\} \in \mathscr{F}$ for every $a \in \mathbb{R}$. By ii), this is true if $a \in \mathbb{Q}$. So let $a \in \mathbb{Q}^c$ (irrational). We notice that, if $q \in \mathbb{Q}$ is such that $q > a$, then

$$\{f > q\} \subset \{f > a\}.$$

Since this happens for every $q \in \mathbb{Q}$, $q > a$, we can say that

$$\bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\} \subset \{f > a\}.$$

At left, we have a countable union of measurable sets, so the union is a measurable set. So, if we prove that = holds, we are done! That is, the goal is reduced to prove that

$$\{f > a\} \subset \bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\}.$$

Pick $x \in \{f > a\}$. So, $f(x) > a$. Because of the density of rationals in reals, there exists $r \in \mathbb{Q}$ such that $f(x) > r > a$, so $x \in \{f > r\} \subset \bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\}$. This means that $\{f > a\} \subset \bigcup_{q \in \mathbb{Q},\ q > a} \{f > q\}$ as claimed. □

3.4.4. Notice that

$$\{fg > a\} = \{fg > a,\ g > 0\} \cap \{fg > a,\ g = 0\} \cap \{fg > a,\ g < 0\}.$$

Let's analyze the three sets, starting by the second one (easier), and the first and the third ones being similar. We have

$$\{fg > a,\ g = 0\} = \{0 > a,\ g = 0\} = \begin{cases} \varnothing \in \mathscr{F}, & a \geqslant 0, \\ \{g = 0\} \in \mathscr{F}, & a < 0 \end{cases}$$

For the first set we have

$$\{fg > a,\ g > 0\} = \{f > \frac{a}{g},\ g > 0\} = \bigcup_{q\in\mathbb{Q}} \left\{f > q > \frac{a}{g},\ g > 0\right\} = \bigcup_{q\in\mathbb{Q}} \underbrace{\{f > q\}}_{\in\mathscr{F}} \cap \underbrace{\left\{g > 0, g > \frac{a}{q}\right\}}_{\in\mathscr{F}}$$

from which we see that $\{fg > a,\ g > 0\} \in \mathscr{F}$. Similar argument for the third set. From this the conclusion follows. □

3.4.6. i) Claim: $f_n(x) \longrightarrow 0$ for every $x \in \mathbb{R}$. Take $n \geqslant [x] + 1$. Then $x < [x] + 1 \leqslant n$, from which $f_n(x) = 0$. This means that $(f_n(x))$ is constantly $= 0$ for n large, thus $f_n(x) \longrightarrow 0$.

ii) Claim: $f_n(x) \longrightarrow 1_{]0,+\infty[}(x)$ for every x. Indeed: if $x \leqslant 0$, $f_n(x) \equiv 0 \longrightarrow 0$. If $x > 0$, since $\frac{1}{n} \to 0$ and $n \to +\infty$, for n large enough $\frac{1}{n} < x < n$, so $f_n(x) \equiv 1 \longrightarrow 1$.

iii) We notice that

$$f_{2k}(x) = 1_{[0,1/2]}(x),\ \ f_{2k+1}(x) = 1_{[1/2,1]}(x).$$

For $x < 0$ and $x > 1$ we have $f_n(x) \equiv 0 \longrightarrow 0$. For $x = 1/2$ we have also $f_n(x) \equiv 1 \longrightarrow 1$. If however $0 \leqslant x < 1/2$ we have that $(f_n(x)) = (1, 0, 1, 0, \ldots)$ so there is no limit. Similarly, for $1/2 < x \leqslant 1$, $(f_n(x))$ has no limit. Since the limit of (f_n) does not exist for $x \in [0, 1/2[\cup]1/2, 1]$, which is a positive measure set, we cannot conclude that (f_n) converges pointwise a.e..

3.4.7. We do the proof in dimension $d = 1$ for simplicity, the argument is the same for the general case. Suppose that $g(x) > 0$ for some $x \in \mathbb{R}$. By continuity, there exists a neighborhood U_x of x for which $g(y) > 0, \forall y \in U_x$. We can always assume that $U_x = [x - \varepsilon, x + \varepsilon]$. Therefore

$$\{g \neq 0\} \supset [x - \varepsilon, x + \varepsilon],\ \ 0 = \lambda(\{g \neq 0\}) \geqslant \lambda([x - \varepsilon, x + \varepsilon]) = 2\varepsilon > 0,$$

which is a contradiction. □

4.3.4. The thesis is to prove that $\mu(f < M) = 0$. Suppose $\mu(f < M) > 0$. We claim that

$$\exists \varepsilon > 0,\ :\ \mu(f \leqslant M - \varepsilon) > 0.$$

This can be proved as follows: since $\{f \leqslant M - \frac{1}{n}\} \nearrow \{f < M\}$, by the continuity from below

$$\lim_n \mu\left(f \leqslant M - \frac{1}{n}\right) = \mu(f < M) > 0,$$

so, by the permanence of sign, there exists N such that

$$\mu\left(f \leqslant M - \frac{1}{n}\right) \geqslant \frac{\mu(f < M)}{2},\ \forall n \geqslant N.$$

But then

$$\begin{aligned} M\mu(E) &= \int_E f\, d\mu = \int_{f\leqslant M-\varepsilon} f\, d\mu + \int_{f>M-\varepsilon} f\, d\mu \\ &\leqslant (M-\varepsilon)\mu(f\leqslant M-\varepsilon) + M\mu(f>M-\varepsilon) \\ &= M\mu(E) - \varepsilon\mu(f\leqslant M-\varepsilon) < M\mu(E), \end{aligned}$$

which is a contradiction. □

4.3.5. The assumption says that

$$\int_E f\, d\mu = \pm\int_E |f|\, d\mu.$$

Suppose, for example, that

$$\int_E f\, d\mu = \int_E |f|\, d\mu.$$

We show that $\mu(f<0)=0$ (a similar argument for the other case). Assume, by contradiction, that $\mu(f<0)>0$. As in the previous problem, we can find $\varepsilon>0$ such that $\mu(f<-\varepsilon)>0$. Then

$$\int_E |f|\, d\mu = \int_E f\, d\mu = \int_{f\leqslant-\varepsilon} f\, d\mu + \int_{f>-\varepsilon} f\, d\mu \leqslant -\varepsilon\mu(f\leqslant-\varepsilon) + \int_E |f|\, d\mu < \int_E |f|\, d\mu,$$

which is a contradiction. □

4.3.7. As in the proof of the Chebyshev's inequality, for $\alpha>0$ we have

$$\mu(f\geqslant\alpha) = \int_{f\geqslant\alpha} 1 d\mu$$

Now, ϕ is convex and this means that

$$\phi(\lambda x + (1-\lambda)y) \leqslant \lambda\phi(x) + (1-\lambda)\phi(y),\ \forall x, y\geqslant 0,\ \lambda\in[0,1].$$

Let's consider an x for which $f(x)>\alpha$. Looking at α as convex combination of 0 and $f(x)$, $\alpha = \left(\frac{\alpha}{f(x)}\right) f(x) + \left(1-\frac{\alpha}{f(x)}\right)0$, we have

$$\phi(\alpha) \leqslant \frac{\alpha}{f(x)}\phi(f(x)) + \left(1-\frac{\alpha}{f(x)}\right)\phi(0) = \frac{\alpha}{f(x)}\phi(f(x)),$$

from which

$$\frac{f(x)}{\alpha} \leqslant \frac{\phi(f(x))}{\phi(\alpha)}.$$

Therefore

$$\int_{f\geqslant\alpha} 1\, d\mu \leqslant \int_{f\geqslant\alpha} \frac{f}{\alpha}\, d\mu \leqslant \int_{f\geqslant\alpha} \frac{\phi(f(x))}{\phi(\alpha)}\, d\mu = \frac{1}{\phi(\alpha)}\int_{f\geqslant\alpha}\phi(f)\, d\mu.$$

From this, the conclusion follows. □

4.3.8. Suppose, by contradiction, that $\mu(f < 0) > 0$. As in previous problems, we may always find $\varepsilon > 0$ such that $\mu(f \leqslant -\varepsilon) > 0$. But then

$$0 \leqslant \int_{f\leqslant -\varepsilon} f \, d\mu \leqslant \int_{f\leqslant -\varepsilon} -\varepsilon \, d\mu = -\varepsilon\mu(f \leqslant -\varepsilon) < 0,$$

which is impossible. □

4.3.9. Let's follow the hint and consider the set $E := \{f \geqslant M + \varepsilon\}$ with $\varepsilon > 0$. By Chebyshev inequality

$$\mu(f \geqslant M + \varepsilon) \leqslant \frac{1}{M+\varepsilon} \int_{f\geqslant M+\varepsilon} f \, d\mu \leqslant \frac{1}{M+\varepsilon} \int_X |f| \, d\mu < +\infty.$$

Moreover,

$$\left| \int_E f \, d\mu \right| = \int_E f \, d\mu \geqslant \int_E (M+\varepsilon) \, d\mu = (M+\varepsilon)\mu(E),$$

so, because of the assumption, we would get

$$(M+\varepsilon)\mu(E) \leqslant M\mu(E),$$

and since, as proved above $\mu(E) < +\infty$, either $\mu(E) = 0$ or $M + \varepsilon \leqslant M$, which is impossible. We conclude that $\mu(f \geqslant M + \varepsilon) = 0$ for every $\varepsilon > 0$. In a similar way, $\mu(f \leqslant -M - \varepsilon) = 0$. So, $\mu(|f| \geqslant M + \varepsilon) = 0$ for every $\varepsilon > 0$. When $\varepsilon \downarrow 0$, $\{|f| \geqslant M + \varepsilon\} \downarrow \{|f| > M\}$, and since they have finite measures, continuity from above applies, and

$$\mu(|f| > M) = \lim_n \mu\left(|f| \geqslant M + \frac{1}{n}\right) = 0,$$

from which the conclusion follows. □

6.3.3. Let $f_n(x) := \frac{n}{x(1+x^2)} \sin\frac{x}{n}$. Clearly $f_n \in \mathscr{C}(]0, +\infty[)$, so $f_n \in L([0, +\infty[)$. We apply the dominated convergence theorem. We have

i)
$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \frac{1}{1+x^2} \frac{\sin(x/n)}{x/n} = \frac{1}{1+x^2}, \ \forall x > 0.$$

ii) Let's determine and integrable dominant: since $|\sin t| \leqslant |t|$, we have

$$|f_n(x)| = \left| \frac{1}{1+x^2} \frac{\sin(x/n)}{x/n} \right| \leqslant \frac{1}{1+x^2} =: g(x), \ \forall x > 0, \ \forall n \in \mathbb{N},$$

and clearly $g \in L^1([0, +\infty[)$.

Therefore, the dominated convergence applies and

$$\lim_{n\to+\infty}\int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty}\frac{1}{1+x^2}\,dx = [\arctan x]_{x=0}^{x=+\infty} = \frac{\pi}{2}. \quad \square$$

6.3.4. Let $f_n(x) := \frac{1+nx^2}{(1+x^2)^n}$. Clearly, $f_n \in \mathscr{C}([0,+\infty[) \subset L([0,+\infty[)$. Being f_n continuous, $f_n \in L^1([0,R])$ for every $R > 0$. Thus, the integrability of f_n depends on its behavior at $+\infty$: we have

$$f_n(x) \sim_{+\infty} \frac{nx^2}{x^{2n}} = \frac{n}{x^{2n-2}}, \text{ so } \exists \int^{+\infty} |f_n(x)|\,dx \iff \exists \int^{+\infty} \frac{n}{x^{2n-2}}\,dx \iff 2n-2>1,$$

that is for $n \geqslant 2$. To compute the limit of integrals $\int_0^{+\infty} f_n(x)\,dx$, we apply the dominated convergence theorem. We have

$$\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty}\frac{1+nx^2}{(1+x^2)^n} = \begin{cases} 1, & x=0,\\ 0, & x>0 \end{cases}$$

being, for $x > 0$, $1 + nx^2 \sim x^2 n = o((1+x^2)^n)$. Therefore $\lim_n f_n(x) = 0$ a.e. $x \in [0,+\infty[$.

Let's look for an integrable dominant. We notice that

$$(1+x^2)^n = 1 + nx^2 + \frac{n(n-1)}{2}x^4 + \ldots \geqslant 1 + nx^2 + \frac{n(n-1)}{2}x^4 \geqslant 1 + nx^2 + nx^4,$$

provided $n \geqslant 3$. Therefore, for $n \geqslant 3$,

$$|f_n(x)| \overset{f_n\geqslant 0}{=} \frac{1+nx^2}{(1+x^2)^n} \leqslant \frac{1}{1+\frac{nx^4}{1+nx^2}} \leqslant \frac{1}{1+\frac{nx^4}{n+nx^2}} = \frac{1}{1+\frac{x^4}{1+x^2}} = \frac{x^2+1}{x^4+x^2+1} =: g(x).$$

Since $g \in \mathscr{C}([0,+\infty[) \subset L([0,+\infty[)$, and $g(x) \sim_{+\infty} \frac{1}{x^2}$, we have $g \in L^1([0,+\infty[)$. Conclusion: the dominated convergence theorem applies and

$$\lim_n \int_0^{+\infty} f_n(x)\,dx = \int_0^{+\infty} \lim_n f_n(x)\,dx = \int_0^{+\infty} 0\,dx = 0. \quad \square$$

6.3.6. Let $\mu_f(E) := \int_E f\,d\mu$. We already know that μ_f is a measure provided $f \in L(X)$, $f \geqslant 0$. Let $g = \sum_k c_k 1_{E_k}$ be simple and positive measurable. Then

$$\begin{aligned}\int_X g\,d\mu_f &= \int_X \textstyle\sum_k c_k 1_{E_k}\,d\mu_f = \sum_k c_k \mu_f(E_k) = \sum_k c_k \int_{E_k} f\,d\mu = \sum_k c_k \int_X 1_{E_k} f\,d\mu\\ &= \int_X \underbrace{\sum_k c_k 1_{E_k}}_{g} f\,d\mu = \int_X gf\,d\mu.\end{aligned}$$

We now extend this identity to any $g \in L(X)$ positive ($g \geqslant 0$). We remind that, $\exists(s_n)$ sequence of simple measurable functions such that i) $0 \leqslant s_n \leqslant s_{n+1}$ for every $x \in X$ and $\forall n \in \mathbb{N}$; ii) $s_n(x) \longrightarrow g(x)$ for every $x \in X$. Now, for s_n we have

$$\int_X s_n \, d\mu_f = \int_X s_n f \, d\mu.$$

We now apply to both sides the Beppo-Levi theorem and we conclude that

$$\int_X g \, d\mu_f = \int_X g f \, d\mu.$$

Finally, if $g \in L^1$, the previous identity holds for $g_\pm$. By subracting the corresponding identities the conclusion follows. □

6.3.7. We remind that, if $f \in L(X)$, then

$$f \in L^1(X), \quad \Longleftrightarrow \quad \int_X |f| \, d\mu < +\infty.$$

$\Longrightarrow$. Assume $f \in L^1(X)$. Then

$$\int_X |f| \, d\mu = \sum_n \int_{n \leqslant |f| < n+1} |f| \, d\mu \geqslant \sum_n \int_{n \leqslant |f| < n+1} n \, d\mu = \sum_n n\mu \left(n \leqslant |f| < n+1\right).$$

From this the conclusion follows being $\int_X |f| \, d\mu < +\infty$.

$\Longleftarrow$ The same argument shows that

$$\int_X |f| \, d\mu \leqslant \sum_n (n+1)\mu(n \leqslant |f| < n+1).$$

Now, $(n+1)\mu(n \leqslant |f| < n+1) \sim n\mu(n \leqslant |f| < n+1)$, we have that $\sum_n n\mu(n \leqslant |f| < n+1)$ converges iff $\sum_n (n+1)\mu(n \leqslant |f| < n+1)$, and from this the conclusion follows. □

6.3.8. By assumption,

$$\int_X |f_n| \, d\mu \leqslant \frac{C}{n^\alpha}.$$

Summing these inequalities, nd applying the monotone convergence for series, we have

$$\int_X \sum_n |f_n| \, d\mu = \sum_n \int_X |f_n| \, d\mu \leqslant \sum_n \frac{C}{n^\alpha} < +\infty,$$

being $\alpha > 1$. Therefore $\int_X \sum_n |f_n| \, d\mu < +\infty$ and from this $\sum_n |f_n| < +\infty$ a.e. $x \in X$. As well known, a necessary (but not sufficient) condition for convergence is $|f_n| \longrightarrow 0$, that is $f_n \longrightarrow 0$. □

6.3.9. We notice that

$$\int_E |f|\,d\mu = \int_{E\cap\{|f|\leqslant K\}} |f|\,d\mu + \int_{E\cap\{|f|>K\}} |f|\,d\mu.$$

We have

$$\int_{E\cap\{|f|>K\}} |f|\,d\mu \leqslant \int_{|f|>K} |f|\,d\mu := \nu(|f|>K),$$

where $\nu(F) := \int_F |f|\,d\mu$. As well known, ν is a measure on $(X,\mathscr{F})$ and since $f\in L^1(X)$ we have $\nu(X) = \int_X |f|\,d\nu < +\infty$, so ν is a finite measure. As a consequence of this, continuity from above applies to ν, and since $\{|f|>K\| \downarrow \{|f|=+\infty\}$ we have

$$\nu(|f|>K) \longrightarrow \nu(|f|=+\infty) = \int_{|f|=+\infty} |f|\,d\mu.$$

But since $f\in L^1(X)$, $\mu(|f|=+\infty)=0$, so also $\nu(|f|=+\infty)=0$. We can then choose K in such a way

$$\nu(|f|>K) \leqslant \frac{\varepsilon}{2}.$$

And since

$$\int_{E\cap\{|f|\leqslant K\}} |f|\,d\mu \leqslant K\int_E 1\,d\mu = K\mu(E) \leqslant \frac{\varepsilon}{2}, \iff \mu(E) \leqslant \frac{\varepsilon}{2K} =: \delta.$$

7.3.3. Let

$$F(\xi) := \int_0^1 \frac{x^\xi - 1}{\log x}\,dx = \int_0^1 f(x,\xi)\,dx,$$

We notice that $f(\sharp,\xi)$ is well defined and continuous on $]0,1[$, for every $\xi\in\mathbb{R}$. Let's discuss absolute integrability at $x=0$ and $x=1$. We start noticing that

$$f(x,0)\equiv 0, \implies f(\sharp,0)\in L^1([0,1]).$$

Furthermore,

$$|f(x,\xi)| = \left|\frac{e^{\xi\log x}-1}{\log x}\right| \sim_{x\to 0+} \begin{cases} -\frac{1}{\log x}, & \xi>0,\\ -\frac{e^{\xi\log x}}{\log x}, & \xi<0. \end{cases}$$

so

$$\exists \int_0 |f(x,\xi)|\,dx \iff \begin{cases} \xi>0, & \exists \int_0 -\frac{1}{\log x}\,dx = \int_{-\infty} -\frac{e^y}{y}\,dy, \text{ yes!}\\ \xi<0, & \exists \int_0 -\frac{}{\log x}\,dx = \int_{-\infty} -\frac{e^{\xi y}}{y}e^y\,dy = -\int_{-\infty}\frac{e^{(\xi+1)y}}{y}\,dy,\\ & \iff \xi+1>0, \iff \xi>-1. \end{cases}$$

Similarly, recalling that $e^t - 1 = t + o(t) \sim_{t\to 0} t$, we have

$$|f(x,\xi)| \sim_{x\to 1} \left|\frac{\xi \log x}{\log x}\right| = |\xi|,$$

so we conclude that $f(\sharp, \xi) \in L^1([0,1])$ iff $\xi > -1$.

To compute $\partial_\xi F$ we apply the differentiation under integral sign theorem:

$$\partial_\xi F(\xi) = \int_0^1 \partial_\xi f(x,\xi)\, dx. \quad (\star)$$

To apply this theorem we notice that:

i)

$$\exists \partial_\xi f(x,\xi) = \partial_\xi \frac{e^{\xi \log x} - 1}{\log x} = \frac{e^{\xi \log x} \log x}{\log x} = e^{\xi \log x} = x^\xi, \ \forall x > 0, \ \forall \xi > -1.$$

In particular, this holds a.e. $x \in [0,1]$, $\forall \xi > -1$.

ii) Integrable dominant:

$$|\partial_x f(x,\xi)| = |x^\xi| \leqslant 1, \ a.e.\ x \in [0,1], \ \forall \xi \geqslant 0.$$

If $a \leqslant \xi < 0$ with a fixed and $a > -1$, we have

$$|\partial_x f(x,\xi)| = |x^\xi| \leqslant x^a = \frac{1}{x^{-a}} \in L^1([0,1]) \iff a > -1.$$

This bound holds for a.e. $x \in [0,1]$ and $\forall \xi \geqslant a$. Since $x^a \geqslant 1$, we conclude that :

$$|\partial_x f(x,\xi)| \leqslant \frac{1}{x^{-a}}, \ a.e.\ x \in [0,1], \ \forall \xi > a.$$

Therefore, we can apply the theorem on $D_a = [a, +\infty[$ and conclude that $(\star)$ holds true for all $\xi \in D_a$. Being $a > -1$ free, we can conclude that the relation holds for $\xi \in]-1, +\infty[$.

In particular,

$$\partial_\xi F(\xi) = \int_0^1 x^\xi \, dx \overset{\xi > -1}{=} \left[\frac{x^{\xi+1}}{\xi+1}\right]_{x=0}^{x=1} = \frac{1}{\xi+1}.$$

Therefore,

$$F(\xi) \log(\xi+1) + k,$$

where k is a constant, and since $F(0) = 0$, we deduce $0 = log1 + k = k$, from which $F(\xi) = \log(\xi+1)$. □

7.3.5. Let

$$F(a,b) := \int_0^{+\infty} \left(e^{-\frac{a}{x^2}} - e^{-\frac{b}{x^2}}\right) dx \overset{y=1/x}{=} \int_0^{+\infty} \frac{e^{-ay^2} - e^{-by^2}}{-y^2}\, dy = \int_0^{+\infty} \frac{e^{-by^2} - e^{-ay^2}}{y^2}\, dy.$$

Let's discuss the existence of the integral $F(a,b)$. We notice that it is not convenient to split the integrals, because $\int_0^{+\infty} \frac{e^{-ay^2}}{y^2}\,dy$ is divergent at $y=0$. However, remembering of $e^t = 1+t+o(t)$, we have

$$f(a,b,y) = \frac{e^{-by^2} - e^{-ay^2}}{y^2} \sim_{y\to 0} \frac{(a-b)y^2}{y^2} = (a-b), \text{ if } a \neq b,$$

while $f(a,a,y) \equiv 0$. Thus, $f(a,b,y)$ is L^1 at $y=0$. Since it is also a continuous function on $]0,+\infty[$, we discuss the integrability at $+\infty$. We notice that if $a=b$ then $f(a,a,y)\equiv 0$ it is integrable at $+\infty$. If $a\neq b$ and one of a,b is negative, for example $a<b$, $a<0$, then

$$f(a,b,y) \sim_{y\to+\infty} \frac{e^{-ay^2}}{y^2} \longrightarrow +\infty,$$

so $f(a,b,\sharp) \notin L^1(+\infty)$. If $a=0$ and $b>0$ (or $a>0$ and $b=0$), then

$$f(0,b,y) = \frac{1-e^{-by^2}}{y^2} \sim \frac{1}{y^2}, \implies f(0,b,y) \in L^1(+\infty).$$

Finally, if both $a,b>0$ and, for example $0<a<b$ (the same for $0<b<a$), then

$$f(a,b,y) \sim_{y\to+\infty} \frac{e^{-ay^2}}{y^2} \leqslant \frac{1}{y^2} \in L^1(+\infty).$$

Conclusion: $f(a,b,\sharp) \in L^1([0,+\infty[)$ if $a=b$ or $a,b \geqslant 0$.

To compute $F(a,b)$ we notice that $F(a,a)\equiv 0$, while

$$\partial_a F(a,b) = \int_0^{+\infty} \partial_a f(a,b,y)\,dy,$$

provided we can exchange derivative and integral. We notice that

i)

$$\exists \partial_a f(a,b,x) = \partial_a \frac{e^{-by^2} - e^{-ay^2}}{y^2} = \frac{1}{y^2}(-e^{-ay^2}(-y^2)) = e^{-ay^2},\ \forall y>0\,(a.e.\ y\in[0,+\infty[),\ \forall a,b.$$

ii) If $\varepsilon>0$, and $a\geqslant\varepsilon$, we have

$$|\partial_a f(a,b,x)| = e^{-ay^2} \leqslant e^{-\varepsilon y^2},\ a.e.\ y\in[0,+\infty[,\ \forall a\geqslant\varepsilon,$$

so,

$$\exists \partial_a F(a,b) = \int_0^{+\infty} e^{-ay^2}\,dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

This holds for every $a\geqslant\varepsilon$, and since $\varepsilon>0$ is arbitrary, we conclude that it holds for every $a>0$.

Similarly,

$$\exists \partial_b F(a,b) = -\frac{1}{2}\sqrt{\frac{\pi}{b}},\ \forall b > 0.$$

To determine $F(a,b)$ for $a, b \geqslant 0$, fix $b > 0$: since

$$\partial_a F(a,b) = \frac{1}{2}\sqrt{\frac{\pi}{a}},\ \implies\ F(a,b) = \sqrt{\pi a} + \varphi(b),$$

where $\varphi(b)$ is a function of b (constant in a). Taking the derivative w.r.t. b, we have

$$\varphi'(b) = \partial_b F(a,b) = -\frac{1}{2}\sqrt{\frac{\pi}{b}},\ \implies\ \varphi(b) = -\sqrt{\pi b} + k,$$

where k is now a constant. Therefore,

$$F(a,b) = \sqrt{\pi}\left(\sqrt{a} - \sqrt{b}\right) + k,$$

and since $F(a,a) = 0$, we deduce $k = 0$. □

7.3.7. Let

$$F(\xi) := \int_0^1 \frac{\log(1+\xi x)}{1+x^2}\, dx.$$

Let's determine the domain of definition of F, that is the set

$$D := \left\{\xi \in \mathbb{R} \ :\ f(\sharp, \xi) \in L^1([0,1])\right\} \text{ where } f(x,\xi) := \frac{\log(1+\xi x)}{1+x^2}.$$

First notice that f is defined provided $1 + \xi x > 0$, and since $x \in [0,1]$, f is well defined for every $\xi \geqslant 0$, while for $\xi < 0$, $1 + \xi x \geqslant 1 + \xi > 0$ iff $\xi > -1$. Notice that, if $\xi = -1$, $f(x,-1) = \frac{\log(1-x)}{1+x^2} \in \mathscr{C}(]0,1])$. Therefore, $D \subset [-1, +\infty[$. If $\xi > -1$, $f(\sharp, \xi) \in \mathscr{C}([0,1]) \subset L^1([0,1])$. For $\xi = -1$, $f(x,-1) = \frac{\log(1-x)}{1+x^2} \in \mathscr{C}(]0,1]) \subset L([0,1])$ and since $\log(1+t) = t + o(t)$,

$$f(x,-1) \sim_{x\to 0} \frac{-x}{1+x^2} = -x,$$

we have that $f(\sharp, -1) \in L^1([0,1])$. We conclude that $D = [-1, +\infty[$.

We compute $\partial_\xi F$ applying the differentiation under integral sign theorem:

$$\partial_\xi F(\xi) = \int_0^1 \partial_\xi f(x,\xi)\, dx.$$

We notice that, for $-1 \leqslant a \leqslant \xi$,

i)

$$\partial_\xi f(x,\xi) = \frac{x}{(1+\xi x)(1+x^2)},\ \forall x \in]0,1],\ \forall \xi \geqslant -1.$$

ii)

$$|\partial_\xi f(x,\xi)| = \frac{1}{|1+\xi x|}\frac{x}{1+x^2} = \begin{cases} \xi \geqslant 0, & \frac{1}{|1+\xi x|}\frac{x}{1+x^2} \leqslant \frac{x}{1+x^2} \leqslant \frac{1}{1+a}\frac{x}{1+x^2}, \\ -1 < a \leqslant \xi < 0, & \leqslant \frac{1}{1+a}\frac{x}{1+x^2} \end{cases}$$

$$\leqslant \frac{1}{1+a}\frac{x}{1+x^2} =: g_a(x) \in L^1([0,1]).$$

Therefore, we can apply the differentiability theorem and conclude that

$$\partial_\xi F(\xi) = \int_0^1 \frac{x}{(1+\xi x)(1+x^2)}\, dx, \ \forall \xi \geqslant a.$$

Since $a > -1$ is arbitrary, we conclude that the previous formula holds for every $\xi > -1$. Now, according to Hermite's decomposition,

$$\frac{x}{(1+\xi x)(1+x^2)} = \frac{A}{1+\xi x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+(Bx+C)(1+\xi x)}{(1+\xi x)(1+x^2)}$$

where, as easily checked, $A = -\frac{\xi}{1+\xi^2}$, $B = \frac{1}{1+\xi^2}$ and $C = \frac{\xi}{1+\xi^2}$. Therefore

$$\begin{aligned} \partial_\xi F(\xi) &= -\frac{\xi}{1+\xi^2}\int_0^1 \frac{1}{1+\xi x}\, dx + \frac{1}{1+\xi^2}\int_0^1 \frac{x+\xi}{1+x^2}\, dx \\ &= -\frac{1}{1+\xi^2}\left[\log(1+\xi x)\right]_{x=0}^{x=1} + \frac{1}{1+\xi^2}\left(\left[\frac{1}{2}\log(1+x^2)\right]_{x=0}^{x=1} + \xi\left[\arctan x\right]_{x=0}^{x=1}\right) \\ &= -\frac{\log(1+\xi)}{1+\xi^2} + \frac{1}{1+\xi^2}\left(\frac{1}{2}\log 2 + \xi\frac{\pi}{4}\right). \end{aligned}$$

Therefore, integrating on $[0,\xi]$

$$F(\xi) - F(0) = -\int_0^\xi \frac{\log(1+x)}{1+x^2}\, dx + \frac{\log 2}{2}\arctan\xi + \frac{\pi}{8}\log(1+\xi^2).$$

Now, $F(0) = \int_0^1 \frac{\log 1}{1+x^2}\, dx = 0$ so we finally obtain

$$F(\xi) = -\int_0^\xi \frac{\log(1+x)}{1+x^2}\, dx + \frac{\log 2}{2}\arctan\xi + \frac{\pi}{8}\log(1+\xi^2).$$

The first integral cannot be further simplified. □

8.3.1. Clearly $\|(x,y)\|_*$ is well defined for every $(x,y) \in \mathbb{R}^2$. It verifies positivity and vanishing. It is also homogeneous as easily checked. However, the triangular inequality is not verified. Take

$(1,1) = (1,0) + (0,1)$. We have that $\|(1,1)\|_* = (\sqrt{1} + \sqrt{1})^2 = 4$, but $\|(1,0)\|_* = \|(0,1)\|_* = 1$, so if the triangular inequality were true, we should have

$$4 = \|(1,1)\|_* = \|(1,0) + (0,1)\|_* \leqslant \|(1,0)\|_* + \|(0,1)\|_* = 1 + 1 = 2,$$

which is manifestly false. □

8.3.2. We check that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^d$ (the other check has been done in class). Clearly, $\|x\|_\infty = \max_j |x_j|$ is well defined (it is the maximum of a finite number of numbers).

Positivity: evident by definition.

Vanishing: we have $\|x\|_\infty = 0$ iff $\max_j |x_j| = 0$. In particular then, $|x_j| \leqslant 0$ for every j, that is $|x_j| = 0$ for every j, so $x_j = 0$ for every j, from which $x = (0, 0, \ldots, 0)$.

Homogeneity: we have

$$\|\alpha x\|_\infty = \max_j |\alpha x_j| = \max_j |\alpha||x_j| = |\alpha| \max_j |x_j| = |\alpha|\|x\|_\infty.$$

Triangular inequality: We have

$$\|x + y\|_\infty = \max_j |x_j + y_j|,$$

and since

$$|x_j + y_j| \leqslant |x_j| + |y_j| \leqslant \max_k |x_k| + \max_k |y_k| = \|x\|_\infty + \|y\|_\infty,$$

for every j, we have

$$\|x + y\|_\infty = \max_j |x_j + y + j| \leqslant \|x\|_\infty + \|y\|_\infty. \quad \square$$

8.3.3. Let $\|f\|_V := \|f\|_\infty + \|f'\|_\infty$. Then

$$\|f\|_V \geqslant \|f\|_\infty,$$

which means that $\|\cdot\|_V$ is stronger than $\|\cdot\|_\infty$. To check that they are not equivalent, we have to find a family $(f_n) \subset V = \mathscr{C}^1([0,1])$ such that $\|f_n\|_\infty \leqslant C$ but $\|f_n\|_V \longrightarrow +\infty$. Since part of $\|f_n\|_V$ is made of $\|f_n\|_\infty$, we look for $f_n \in \mathscr{C}^1$ with "big" derivative. Take

$$f_n(x) := \sin(nx).$$

Clearly $f_n \in \mathscr{C}^1([0,1])$, for every n. Morevoer,

$$\|f_n\|_\infty = \max_{x\in[0,1]} |\sin(nx)| = \max_{y\in[0,n]} |\sin y| = 1,$$

as soon as $n \geqslant 2$. However,

$$\|f_n\|_V = \|f_n\|_\infty + \|f_n'\|_\infty = 1 + \|f_n'\|_\infty,$$

and since

$$\|f_n'\|_\infty = \max_{x\in[0,1]} |n\cos(nx)| = n \max_{y\in[0,n]} |\cos y| = n,$$

we conclude that there cannot be a C such that

$$\|f\|_V \leqslant C\|f\|_\infty,\ \forall f \in V.$$

Indeed, if this is true, then $n = \|f_n\|_V \leqslant C$, for every $n \in \mathbb{N}$. □

8.3.5. i) Let $V := \{f \in \mathscr{C}^1([a,b]) \ : \ f(a) = 0\}$, $\|f\|_V := \int_a^b |f'(x)|\,dx$. Since $f \in \mathscr{C}^1$, $f' \in \mathscr{C}$ so also $|f'| \in \mathscr{C}$, and $\|f\|_V$ is well defined. We check the characteristic properties fulfilled by any norm:

- Positivity: evident.
- Vanishing: we have $\|f\|_V = 0$ iff $\int_a^b |f'(x)|\,dx = 0$. Since $|f'| \in \mathscr{C}$, by a well known result we have $|f'(x)| \equiv 0$, so $f' \equiv 0$ on $[a,b]$. By a well known property of Differential Calculus, this implies $f \equiv C$ (constant), and since $f(a) = 0$, we conclude that $f \equiv 0$.
- Homogeneity: we have

$$\|\alpha f\|_V = \int_a^b |(\alpha f)'(x)|\,dx = \int_a^b |\alpha f'(x)|\,dx = \int_a^b |\alpha||f'(x)|\,dx = |\alpha|\|f\|_V.$$

- Triang. inequality: we have

$$\|f+g\|_V = \int_a^b |f'(x)+g'(x)|\,dx \leqslant \int_a^b |f'(x)| + |g'(x)|\,dx = \|f\|_V + \|g\|_V.$$

ii) We prove that $(V, \|\cdot\|_V) \hookrightarrow (\mathscr{C}([a,b]), \|\cdot\|_\infty)$. Clearly, $V = \mathscr{C}^1([a,b]) \subset \mathscr{C}([a,b])$. We need to prove also that $\|\cdot\|_V$ is stronger than $\|\cdot\|_\infty$. Let $f \in V$ and $x \in [a,b]$. By the Fundamental Thm of Integral Calculus,

$$f(x) = f(a) + \int_a^x f'(y)\,dy = \int_a^x f'(y)\,dy,$$

so

$$|f(x)| = \left|\int_a^x f'(y)\,dy\right| \leqslant \int_a^x |f'(y)|\,dy$$

from which

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)| \leqslant \max_{x\in[a,b]} \int_a^x |f'(y)|\,dy = \int_a^b |f'(y)|\,dy = \|f\|_V. \quad \square$$

8.3.6. i) Let $V := \{f \in \mathscr{C}^2([a,b]) \ : \ f(a) = f(b) = 0\}$ and $\|f\|_V := \int_a^b |f''(x)|\,dx$. Notice that $\|f\|_V$ is well defined for $f \in V$ ($\Longrightarrow f'' \in \mathscr{C}$, $\Longrightarrow |f''| \in \mathscr{C}([a,b]) \Longrightarrow \int_a^b |f''|$ makes sense). Let's now check the characteristic properties fulfilled by any norm.

- Positivity: evident.

- Vanishing: $\|f\|_V = 0$ iff $\int_a^b |f''(x)|\ dx = 0$, from which, by a well known property, $|f''(x)| \equiv 0$ on $[a,b]$, that is $f'' \equiv 0$ on $[a,b]$. Since $f'' = (f')' \equiv 0$, we deduce $f' \equiv C$, so $f(x) = Cx + D$ with C, D constants. Being $f \in V$ we have $0 = f(a) = Ca + D$ and $0 = f(b) = Cb + D$, therefore
$$C(a-b) = 0, \implies C = 0,$$
then $0 = D$. Conclusion: $f \equiv 0$.
- Homogeneity and triangular inequality: straightforward.

ii) Clearly $V \subset \mathscr{C}^1([a,b])$. Let
$$\|f\|_{\mathscr{C}^1} = \|f\|_\infty + \|f'\|_\infty.$$
We aim to bound this by $\|f\|_V$. We notice that, by the Fundamental Thm of Integral calculus,
$$f(x) = f(a) + \int_a^x f'(y)\ dy = \int_a^x f'(y)\ dy.$$
Now, since $f(a) = f(b) = 0$, according to Rolle's theorem there exists $c \in [a,b]$ such that $f'(c) = 0$. Applying the same theorem to f' we have
$$f'(y) = f'(c) + \int_c^y f''(z)\ dz = \int_c^y f''(z)\ dz,$$
so
$$f(x) = \int_a^x \left(\int_c^y f''(z)\ dz \right) dy, \implies |f(x)| \leqslant \int_a^x \left| \int_c^y f''(z)\ dz \right| dy.$$
Now, if $y \geqslant c$,
$$\left| \int_c^y f''(z)\ dz \right| \leqslant \int_c^y |f''(z)|\ dz \leqslant \int_a^b |f''(z)|\ dz = \|f\|_V,$$
while, for $y \leqslant c$,
$$\left| \int_c^y f''(z)\ dz \right| = \left| -\int_y^c f''(z)\ dz \right| = \left| \int_y^c f''(z)\ dz \right| \leqslant \int_y^c |f''(z)|\ dz \leqslant \int_a^b |f''(z)|\ dz = \|f\|_V,$$
In any case
$$\left| \int_c^y f''(z)\ dz \right| \leqslant \|f\|_V,$$
so
$$|f(x)| \leqslant \int_a^x \|f\|_V\ dy = (x-a)\|f\|_V, \implies \|f\|_\infty \leqslant (b-a)\|f\|_V,$$
and
$$|f'(y)| = \left| \int_c^y f''(z)\ dz \right| \leqslant \|f\|_V, \implies \|f'\|_\infty \leqslant \|f\|_V.$$

In conclusion

$$\|f\|_{\mathscr{C}^1} = \|f\|_\infty + \|f'\|_\infty \leqslant (b-a+1)\,\|f\|_V,\ \forall f \in V. \quad \square$$

9.4.2. Let's start showing that

$$\exists C > 0,\ :\ (u+v)^p \leqslant C(u^p + v^p),\ \forall u, v \geqslant 0.$$

Since for $u = 0$ or $v = 0$ it is sufficient $C = 1$, we suppose $u, v > 0$. In this case the inequality is equivalent to

$$\left(\frac{u}{v} + 1\right)^p \leqslant C\left(\left(\frac{u}{v}\right)^p + 1\right),$$

that is, setting $t = \frac{u}{v} > 0$,

$$(t+1)^p \leqslant C(t^p+1), \quad \Longleftrightarrow \quad \frac{(t+1)^p}{t^p+1} \leqslant C,\ \forall t > 0.$$

Now, let $\varphi(t) := \frac{(t+1)^p}{t^p+1} \geqslant 0$. Clearly $\varphi \in \mathscr{C}([0, +\infty[)$ and since $\varphi(+\infty) = 1$, φ must be bounded. We can actually find its maximum:

$$\varphi'(t) = \frac{p(t+1)^{p-1}(t^p+1) - (t+1)^p p t^{p-1}}{(t^p+1)^2} = p(t+1)^{p-1}\frac{t^p + 1 - t^{p-1}(t+1)}{(t^p+1)^2}$$

so

$$\varphi'(t) \geqslant 0, \quad \Longleftrightarrow \quad t^p + 1 - t^p - t^{p-1} \geqslant 0, \quad \Longleftrightarrow \quad t^{p-1} \leqslant 1, \quad \Longleftrightarrow \quad t \leqslant 1.$$

So, φ attains its global maximum at $t = 1$, with $\varphi(1) = \frac{2^p}{2} = 2^{p-1}$. We conclude that the inequality holds with $C = 2^{p-1}$, that is

$$(u+v)^p \leqslant 2^{p-1}(u^p + v^p),\ \forall u, v \geqslant 0.$$

We can now prove that L^p is a vector space: if $f, g \in L^p$ then

$$\int_X |f+g|^p\ d\mu \leqslant \int_X (|f| + |g|)^p\ d\mu \leqslant \int_X 2^{p-1}\left(|f|^p + |g|^p\right)\ d\mu = 2^{p-1}\left(\|f\|_p^p + \|g\|_p^p\right) < +\infty,$$

Easily $\alpha f \in L^p$ if $\alpha \in \mathbb{R}$ and $f \in L^p$. $\square$

9.4.3. i) The concavity of $\log t$ means that

$$\log\left(\lambda t + (1-\lambda)s\right) \geqslant \lambda \log t + (1-\lambda)\log s,\ \forall s, t > 0,\ \forall \lambda \in [0,1].$$

From this

$$\log\left(t^\lambda s^{1-\lambda}\right) \leqslant \log\left(\lambda t + (1-\lambda)s\right),$$

from which

$$t^\lambda s^{1-\lambda} \leqslant \lambda t + (1-\lambda)s.$$

Now, chose $\lambda - \frac{1}{p}$ so $1 - \lambda = 1 - \frac{1}{p} =: \frac{1}{q}$, $t = a^p$ and $s = b^q$ and you have the conclusion.

ii) Noticed that

$$\left|\int_X fg\ d\mu\right| \leqslant \int_X |f||g|\ d\mu,$$

we prove that this last is $\leqslant \|f\|_p\|g\|_q$. We start noticing that, as in CS inequality, the conclusion is trivial if one of $\|f\|_p$ and $\|g\|_q$ is 0: in this case one of f or g would be 0 a.e. and the inequality is trivial. So we assume $\|f\|_p, \|g\|_q > 0$. The conclusion is then equivalent to prove that

$$\int_X \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}\ d\mu \leqslant 1.$$

Applying Young's inequality with $a = \frac{|f|}{\|f\|_p}$ and $b = \frac{|g|}{\|g\|_q}$, we have

$$\int_X \frac{|f|}{\|f\|_p}\frac{|g|}{\|g\|_q}\ d\mu \leqslant \int_X \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q}{\|g\|_q^q}\ d\mu = \frac{1}{p}\frac{\int_X |f|^p\ d\mu}{\|f\|_p^p} + \frac{1}{q}\frac{\int_X |g|^q\ d\mu}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1. \quad \square$$

9.4.4. Following the hint, we have

$$\|f+g\|_p^p \leqslant \int_X |f||f+g|^{p-1}\ d\mu + \int_X |g||f+g|^{p-1}\ d\mu.$$

By Hölder's inequality we have

$$\int_X |f||f+g|^{p-1}\ d\mu \leqslant \|f\|_p\||f+g|^{p-1}\|_q, \quad \int_X |g||f+g|^{p-1}\ d\mu \leqslant \|g\|_p\||f+g|^{p-1}\|_q.$$

Now,

$$\||f+g|^{p-1}\|_q = \left(\int_X \left(|f+g|^{p-1}\right)^q\ d\mu\right)^{1/q} = \left(\int_X |f+g|^{q(p-1)}\ d\mu\right)^{1/q},$$

and since $\frac{1}{q} = 1 - \frac{1}{p} = \frac{p-1}{p}$, $q = \frac{p}{p-1}$ so $q(p-1) = p$, from which

$$\||f+g|^{p-1}\|_q = \left(\int_X |f+g|^p\ d\mu\right)^{1/q} = \left(\left(\int_X |f+g|^p\ d\mu\right)^{1/p}\right)^{p/q} = \|f+g\|_p^{\frac{p}{q}} = \|f+g\|_p^{p-1}.$$

Therefore

$$\|f+g\|_p^p \leqslant \|f\|_p\|f+g\|_p^{p-1} + \|g\|_p\|f+g\|_p^{p-1}.$$

Now, if $\|f+g\|_p = 0$ the triangular inequality is evident. Otherwise, from last inequality and dividing vy $\|f+g\|_p^{p-1}$ we obtain the conclusion. $\square$

9.4.5. i) By CS inequality,

$$\|f\|_1 = \int_X |f|\ d\mu = \int_X |f|\cdot 1\ d\mu \leqslant \|f\|_2\|1\|_2,$$

where $\|1\|_2 = \left(\int_X 1^2 \, d\mu\right)^{1/2} = \sqrt{\mu(X)}$, so

$$\|f\|_1 \leqslant \sqrt{\mu(X)}\|f\|_2.$$

ii) If $\mu(X) = +\infty$, i) is false in general. For example $X = [0, +\infty[$ and $f_n(x) := \frac{1}{1+x}1_{[0,n]}(x)$. Then $f_n \in L^1 \cap L^2$ because $f_n \in \mathscr{C}([0, n])$ and $f \equiv 0$ on $[n, +\infty[$. We have

$$\|f_n\|_1 = \int_0^{+\infty} |f_n(x)| \, dx = \int_0^n \frac{1}{1+x} \, dx = [\log(1+x)]_{x=0}^{x=n} = \log(n+1),$$

and

$$\|f_n\|_2 = \int_0^n \frac{1}{(1+x)^2} \, dx = \left[-\frac{1}{1+x}\right]_{x=0}^{x=n} = 1 - \frac{1}{n+1} \leqslant 1.$$

If $\|\cdot\|_2$ were stronger than $\|\cdot\|_1$ there would be a constant $C > 0$ such that

$$\|f\|_1 \leqslant C\|f\|_2,$$

But then,

$$\log(n+1) = \|f_n\|_1 \leqslant C, \ \forall n \in \mathbb{N},$$

and this is impossible.

iii) As suggested, let's take $X = [0, 1]$ with Lebesgue's measure. We already know that $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$. The vice versa does not hold. Take $f_n(x) = \frac{1}{\sqrt{x}}1_{[1/n,1]}(x)$. Easily, $f_n \in L^1 \cap L^2$. We have

$$\|f_n\|_1 = \int_{1/n}^1 \frac{1}{\sqrt{x}} \, dx = \left[2\sqrt{x}\right]_{x=1/n}^{x=1} = 2\left(1 - \sqrt{\frac{1}{n}}\right) \leqslant 2,$$

while

$$\|f_n\|_2^2 = \int_{1/n}^1 \frac{1}{x} \, dx = [\log x]_{x=1/n}^{x=1} = -\log\frac{1}{n} = \log n.$$

If $\|\cdot\|_1$ were stronger than $\|\cdot\|_1$, there would be a constant C such that

$$\sqrt{\log n} = \|f_n\|_2 \leqslant C\|f_n\|_1 \leqslant 2C, \ \forall n \in \mathbb{N}, \ n \geqslant 1,$$

but this is manifestly impossible.

iv) If X is a finite set, say $X = \{1, \ldots, d\}$, and μ is the counting measure, then

$$\|f\|_1 = \int_X |f| \, d\mu = \sum_{j=1}^d |f(j)|,$$

while

$$\|f\|_2 = \left(\int_X |f| \, d\mu\right)^{1/2} = \left(\sum_{j=1}^d |f(j)|^2\right)^{1/2}.$$

Identifying f with the array $(f(1),\dots,f(d)) \in \mathbb{R}^d$, and recalling that on finite dimensional spaces any two norms are equivalent, there exists $C > 0$ such that

$$\|f\|_2 = \left(\sum_{j=1}^{d} |f(j)|^2\right)^{1/2} \leqslant C \sum_{j=1}^{d} |f(j)| = \|f\|_1. \qquad \square$$

9.4.6. i) The answer is false in general. For example, $f(x) = \frac{1}{1+|x|} \in L^2(\mathbb{R})$ ($f \in \mathscr{C}(\mathbb{R}) \subset L(\mathbb{R})$, $\int_{\mathbb{R}} |f|^2 < +\infty$ being $|f|^2 \sim_{+\infty} \frac{1}{|x|^2}$ which is integrable at $\pm\infty$), but $f \notin L^1(\mathbb{R})$ ($\int_{\mathbb{R}} |f| = +\infty$ being $|f| \sim_{+\infty} \frac{1}{|x|}$ which is not integrable at $\pm\infty$).

ii) We know

$$\int_{\mathbb{R}} |xf(x)|^2 \, dx = \int_{\mathbb{R}} x^2 f(x)^2 \, dx < +\infty.$$

Writing

$$\|f\|_1 = \int_{\mathbb{R}} |f| = \int_{-\infty}^{-1} |f| + \int_{-1}^{1} |f| + \int_{1}^{+\infty} |f|,$$

by the CS inequality we get

$$\int_{-1}^{1} |f| \leqslant \left(\int_{-1}^{1} 1^2 \, dx\right)^{1/2} \left(\int_{-1}^{1} |f|^2 \, dx\right)^{1/2} = \sqrt{2}\|f1_{[-1,1]}\|_2,$$

and

$$\int_{1}^{+\infty} |f| \, dx = \int_{1}^{+\infty} \frac{1}{x}|xf(x)| \, dx \leqslant \left(\int_{1}^{+\infty} \frac{1}{x^2} \, dx\right)^{1/2} \|xf1_{[1,+\infty[}\|_2 = \|xf1_{[1,+\infty[}\|_2$$

and, similarly,

$$\int_{-\infty}^{-1} |f| \, dx \leqslant \|xf1_{]-\infty,1]}\|_2.$$

Therefore

$$\|f\|_1 \leqslant \sqrt{2}\|f1_{[-1,1]}\|_2 + \|xf1_{]-\infty,1]}\|_2 + \|xf1_{[1,+\infty[}\|_2.$$

On the other hand

$$\|xf\|_2^2 = \int_{-\infty}^{-1} |xf|^2 \, dx + \int_{-1}^{1} |xf|^2 + \int_{1}^{+\infty} |xf|^2 \geqslant \|xf1_{]-\infty,1]}\|_2^2 + \|xf1_{[1,+\infty[}\|_2^2,$$

so

$$\|xf\|_2 \geqslant \left(\|xf1_{]-\infty,1]}\|_2^2 + \|xf1_{[1,+\infty[}\|_2^2\right)^{1/2},$$

Since $(a+b)^2 \leqslant 2(a^2+b^2)$, that os $a+b \leqslant \sqrt{2}(a^2+b^2)^{1/2}$, we finally get

$$\begin{aligned}\|f\|_1 &\leqslant \sqrt{2}\|f1_{[-1,1]}\|_2 + \|xf1_{]-\infty,1]}\|_2 + \|xf1_{[1,+\infty[}\|_2 \\ &\leqslant \sqrt{2}\left(\|f1_{[-1,1]}\|_2 + \left(\|xf1_{]-\infty,1]}\|_2^2 + \|xf1_{[1,+\infty[}\|_2^2\right)^{1/2}\right) \\ &\leqslant \sqrt{2}\left(\|f\|_2 + \|xf\|_2\right). \quad \square\end{aligned}$$

9.4.7. i) As in Chebyshev's inequality, for $\alpha > 0$,

$$\mu(|f| \geqslant \alpha) = \int_{|f|\geqslant\alpha} 1 \, d\mu \leqslant \int_{|f|\geqslant\alpha} \left(\frac{|f|}{\alpha}\right)^p \, d\mu = \frac{1}{\alpha^p}\int_{|f|\geqslant\alpha} |f|^p \, d\mu \leqslant \frac{1}{\alpha^p}\|f\|_p^p.$$

ii) We have

$$\alpha^p \mu(|f| \geqslant \alpha) \leqslant \int_{|f|\geqslant\alpha} |f|^p \, d\mu.$$

Defining

$$\nu(E) := \int_E |f|^p \, d\mu,$$

we have a well defined measure on $(X, \mathscr{F})$, with $\nu(X) = \int_X |f|^p \, d\mu = \|f\|_p^p < +\infty$ (because $f \in L^p$). In particular, continuity from above applies, and since $\{|f| \geqslant \alpha\} \downarrow \{|f| = +\infty\}$, we have that

$$\lim_{\alpha\to+\infty} \alpha^p \mu(|f| \geqslant \alpha) \leqslant \lim_{\alpha\to+\infty} \nu(|f| \geqslant \alpha) = \nu(|f| = +\infty) = \int_{|f|=+\infty} |f|^p \, d\mu.$$

Since $\int_X |f|^p \, d\mu < +\infty$, $|f| < +\infty$ μ-a.e., thus $\mu(|f| = +\infty) = 0$, from which $\nu(|f| = +\infty) = 0$. $\square$

9.4.8. i) Applying CS inequality on $[0, x]$, we get

$$\begin{aligned}\left|\int_0^x f(y)\, dy\right| &= \left|\int_0^x y^{-1/4} y^{1/4} f(y)\, dy\right| \overset{CS}{\leqslant} \left(\int_0^x y^{-1/2}\, dy\right)^{1/2} \left(\int_0^x y^{1/2} f(y)^2\, dy\right)^{1/2} \\ &= (2\sqrt{x})^{1/2} \left(\int_0^x y^{1/2} f(y)^2\, dy\right)^{1/2}\end{aligned}$$

from which we have the conclusion.

ii) Let $g(x) := \frac{1}{x}\int_0^x f(y)\,dy$. We have

$$\begin{aligned}\|g\|_2^2 &= \int_0^{+\infty} |g|^2 = \int_0^{+\infty} \frac{1}{x^2}\left(\int_0^x f(y)\,dy\right)^2\,dy\\ &\overset{i)}{\leqslant} \int_0^{+\infty} \frac{1}{x^2} 2x^{1/2}\int_0^x y^{1/2}f(y)^2\,dy\,dx = 2\int_0^{+\infty} x^{-3/2}\int_0^x y^{1/2}f(y)^2\,dy\,dx\\ &\overset{Fubini}{=} 2\int_0^{+\infty} y^{1/2}f(y)^2 \underbrace{\int_y^{+\infty} x^{-3/2}\,dx}_{=\left[-2x^{-1/2}\right]_{x=y}^{x=+\infty}=2y^{-1/2}}\,dy\\ &= 4\int_0^{+\infty} y^{1/2}f(y)^2 y^{-1/2}\,dy = 4\|f\|_2^2. \quad \square\end{aligned}$$

9.4.9. Let $f \in L^p$, $g \in L^q$ and $h \in L^r$ with $1 < p, q, r < +\infty$ such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$. Then

$$\|fgh\|_1 = \int_X |f||g||h|\,d\mu = \int_X |f|\,(|g||h|)\,d\mu \overset{H}{\leqslant} \left(\int_X |f|^p\,d\mu\right)^{1/p}\left(\int_X (|g||h|)^s\,d\mu\right)^{1/s},$$

where $\frac{1}{p}+\frac{1}{s} = 1$, that is $\frac{1}{s} = 1-\frac{1}{p} = \frac{1}{q}+\frac{1}{r}$. Let $\alpha > 0$ be such that $s\alpha = q$, that is $\alpha = \frac{q}{s}$. Notice that, since $\frac{1}{s} > \frac{1}{q}$, $\alpha = \frac{q}{s} > 1$. Similarly, let β be such that $s\beta = r$, that is $\beta = \frac{r}{s}$. As above $\beta > 1$ and moreover

$$\frac{1}{\alpha}+\frac{1}{\beta} = \frac{s}{q}+\frac{s}{r} = s\left(\frac{1}{q}+\frac{1}{r}\right) = s\cdot\frac{1}{s} = 1.$$

Therefore, by Hölder inequality,

$$\int_X |g|^s|h|^s\,d\mu \leqslant \left(\int_X |g|^{\alpha s}\,d\mu\right)^{1/\alpha}\left(\int_X |h|^{\beta s}\,d\mu\right)^{1/\beta} = \|g\|_q^s\|h\|_r^s,$$

from which

$$\left(\int_X (|g||h|)^s\,d\mu\right)^{1/s} \leqslant \left(\|g\|_q^s\|h\|_r^s\right)^{1/s} = \|g\|_q\|h\|_r,$$

and now the conclusion follows. $\square$

10.3.1. #1. Let $f_n(x) := \frac{1}{n}1_{[0,n]}(x)$. For $x \geqslant 0$ fixed, as soon as $n \geqslant [x]+1$ we have $f_n(x) = \frac{1}{n} \longrightarrow 0$. So $f_n \overset{pw}{\longrightarrow} 0$ everywhere on $[0,+\infty[$. Since we hawe pw convergence, the unique possible L^p limit is the pw limit, thus 0. We have

$$\|f_n - 0\|_1 = \|f_n\|_1 = \int_0^{+\infty} \frac{1}{n}1_{[0,n]}\,dx = \frac{1}{n}\int_0^n dx \equiv 1 \not\longrightarrow 0,$$

so $f_n \overset{L^1}{\not\longrightarrow} 0$, hence (f_n) cannot be convergent in L^1 (if $f_n \overset{L^1}{\longrightarrow} f$ then, modulo a subsequence, $f_{n_k} \overset{pw}{\longrightarrow} f$ a.e., but $f_n \overset{pw}{\longrightarrow} 0$, whence $f = 0$ a.e.). Similar argument for L^2 convergence: the unique possible limit candidate is $f = 0$:

$$\|f_n - f\|_2^2 = \|f_n\|_2^2 = \int_0^{+\infty} \left|\frac{1}{n} 1_{[0,n]}\right|^2 = \frac{1}{n^2} \int_0^n 1 \, dx = \frac{1}{n} \longrightarrow 0,$$

so $f_n \overset{L^2}{\longrightarrow} 0$. For L^∞ convergence, since this implies pw convergence, the unique possibility is $f_n \overset{L^\infty}{\longrightarrow} 0$. We have

$$\|f_n - 0\|_\infty = \|f_n\|_\infty = \text{ess. sup}_{x \geqslant 0} |f_n(x)| = \frac{1}{n} \longrightarrow 0.$$

#2. Let $x \geqslant 0$. Here we distinguish $x = 0$ from $x > 0$. In the first case we have $f_n(0) = n \longrightarrow +\infty$. For $x > 0$, as soon as $\frac{1}{n} < x$ (that is, for $n > \frac{1}{x}$, or $n \geqslant [\frac{1}{x}] + 1$) we have $f_n(x) = 0 \longrightarrow 0$. We conclude that $f_n \overset{pw}{\longrightarrow} 0$ a.e. on $[0, +\infty[$. Because of a.e. pw convergence, the unique possibility for $f_n \overset{L^1}{\longrightarrow} f$ is $f = 0$. We have

$$\|f_n - f\|_1 = \|f_n\|_1 = \int_0^{1/n} n \, dx \equiv 1 \not\longrightarrow 0.$$

Thus, (f_n) is not convergent in L^1. Similar for L^2 convergence:

$$\|f_n - f\|_2^2 = \|f_n\|_2^2 = \int_0^{1/n} n^2 \, dx = n \longrightarrow +\infty,$$

so (f_n) is not convergent in L^2. For L^∞ convergence,

$$\|f_n - f\|_\infty = \text{ess. sup}_{x \geqslant 0} |f_n(x)| = n \longrightarrow +\infty,$$

so (f_n) is not convergent in L^∞.

#3. Let $f_n(x) := \sum_{k=1}^n \frac{1}{k} 1_{[k, k+\frac{1}{2^n}]}(x)$. For $x \geqslant 0$ fixed, there exists a unique N such that $x \in [N, N+1[$. In particular, $1_{[k,k+\frac{1}{2^n}]}(x) = 0$ for all k except, at most, for $k = N$. Now, if $x = N$, $1_{[N,N+1/2^n]}(x) = 1$ for every n, so, for $n \geqslant N$,

$$f_n(x) = \frac{1}{N} 1_{[N,N+1/2^n]}(x) = \frac{1}{N} \longrightarrow \frac{1}{N}.$$

If $x \in]N, N+1[$ then, for $n \geqslant N$ and such that $N + \frac{1}{2^n} < x$ (this happens for $n \to +\infty$ being $N + \frac{1}{2^n} \longrightarrow N$), we have

$$f_n(x) = \frac{1}{N} 1_{[N,N+1/2^n]}(x) = 0.$$

Conclusion:

$$f_n(x) \longrightarrow \begin{cases} \frac{1}{N}, & x = N \in \mathbb{N}, \\ 0, & x \notin \mathbb{N}. \end{cases}$$

Thus, $f_n \xrightarrow{pw} 0$ a.e. on $[0, +\infty[$.

About L^1 convergence, the unique possible limit is 0. We have

$$\|f_n - 0\|_1 = \|f_n\|_1 \overset{f_n \geqslant 0}{=} \int_0^{+\infty} \sum_{k=1}^{n} \frac{1}{k} 1_{[k,k+\frac{1}{2^n}]}(x)\, dx = \frac{1}{2^n} \sum_{k=1}^{n} \frac{1}{k}$$

Now, since $\sum_{k=1}^{n} \frac{1}{k} \asymp \log n$ we have $\|f_n - 0\|_1 \asymp \frac{\log n}{2^n} \longrightarrow 0$, so $f_n \xrightarrow{L^1} 0$.

Similarly,

$$\|f_n - 0\|_2^2 = \|f_n\|_2^2 = \int_0^{+\infty} \left(\sum_{k=1}^{n} \frac{1}{k} 1_{[k,k+\frac{1}{2^n}]}(x) \right)^2 dx$$

As said above, for every $x \geqslant 0$ fixed, only at most one of $1_{[k,k+\frac{1}{2^n}]}(x)$ is $= 1$. Therefore

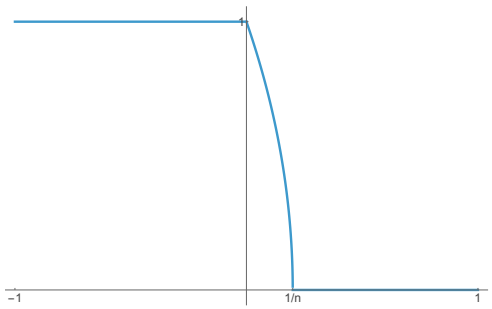
$$\left(\sum_{k=1}^{n} \frac{1}{k} 1_{[k,k+\frac{1}{2^n}]}(x) \right)^2 = \sum_{k=1}^{n} \frac{1}{k^2} 1_{[k,k+\frac{1}{2^n}]}(x)$$

whence

$$\|f_n - 0\|_2^2 = \int_0^{+\infty} \sum_{k=1}^{n} \frac{1}{k^2} 1_{[k,k+\frac{1}{2^n}]}(x)\, dx = \frac{1}{2^n} \sum_{k=1}^{n} \frac{1}{k^2} \longrightarrow 0,$$

being $\sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$. □

10.3.3. A quick plot of f_n:



We see that, for $x \leqslant 0$, $f_n(x) \equiv 1 \longrightarrow 1$. For $x > 0$, taking n such that $\frac{1}{n} < x$, that is $n > \frac{1}{x}$ or $n \geqslant \left[\frac{1}{x}\right] + 1$, we have $f_n(x) = 0 \longrightarrow 0$. Thus

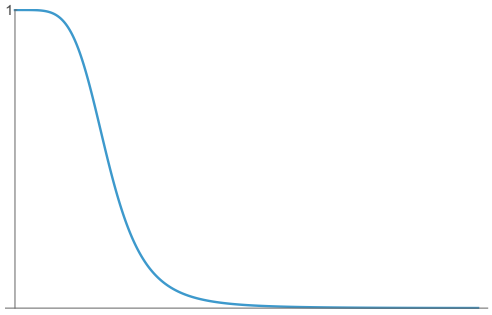
$$f_n \xrightarrow{pw} 1_{[-1,0]}.$$

In this case $1_{[-1,0]}$ can be the unique possible L^2 limit. We notice that $1_{[-1,0]} \in L^2$ and

$$\|f_n - 1_{[-1,0]}\|_2^2 = \int_0^{1/n} \left|\sqrt{1-nx}\right|^2 \, dx = \int_0^{1/n} 1 - nx \, dx = \frac{1}{n} - n\left[\frac{x^2}{2}\right]_{x=0}^{x=1/n} = \frac{1}{2n} \longrightarrow 0,$$

so $f_n \xrightarrow{L^2} 1_{[-1,0]}$. □

10.3.5. The graph of f_n:



Clearly, $f_n \in \mathscr{C}([0,+\infty[)$ and since $f_n(x) \sim_{+\infty} \frac{1}{x^n}$ is integrable at $+\infty$ for every $n \geqslant 2$, we have $f_n \in L^1([0,+\infty[)$ for every $n \geqslant 2$. We notice that

$$x^n \longrightarrow \begin{cases} 0, & 0 \leqslant x < 1, \\ 1, & x = 1, \\ +\infty, & x > 1. \end{cases} \quad \Longrightarrow \quad f_n(x) = \frac{1}{1+x^n} \longrightarrow \begin{cases} 1, & 0 \leqslant x < 1, \\ \frac{1}{2}, & x = 1, \\ 0, & x > 1. \end{cases}$$

So, $f_n \xrightarrow{pw\ a.e.} 1_{[0,1]}$. Since this is the unique possible L^1 limit, let's check this. We have

$$\|f_n - f\|_1 = \int_0^{+\infty} |f_n - f| = \int_0^1 \left|\frac{1}{1+x^n} - 1\right| dx + \int_1^{+\infty} \frac{1}{1+x^n} \, dx = 1 - \int_0^1 \frac{1}{1+x^n} \, dx + \int_1^{+\infty} \frac{1}{1+x^n} \, dx.$$

On $[0,1]$ we have

i) $\frac{1}{1+x^n} \longrightarrow 1$ a.e. $x \in [0,1]$;

ii) $0 \leqslant \frac{1}{1+x^n} \leqslant 1 \in L^1([0,1]), \forall x \in [0,1], \forall n \in \mathbb{N}$.

Therefore, dominated convergence applies and

$$\int_0^1 \frac{1}{1+x^n} \, dx \longrightarrow \int_0^1 1 \, dx = 1.$$

On $[1,+\infty[$, noticed that $x^n \uparrow$ in n, $\frac{1}{1+x^n} \downarrow$ in n, for every $x \geqslant 1$, and since $\frac{1}{1+x^2} \in L^1([1,+\infty[)$, decreasing monotone convergence (or dominated convergence) applies and

$$\int_1^{+\infty} \frac{1}{1+x^n} \, dx \longrightarrow \int_1^{+\infty} 0 \, dx = 0.$$

Gathering these facts together, we get $\|f_n - f\|_1 \longrightarrow 0$, thus $f_n \xrightarrow{L^1} f$. □

10.3.6. All the proposed norms are true norm. This is evident for $\|\cdot\|_*$ being $\|\cdot\|_\infty$ a norm. For $\|\cdot\|_{**}$ we have that it is well defined ($f \in \mathscr{C}^1$ means $f' \in \mathscr{C}$, thus $\|f'\|_\infty$ makes sense). Let's check the characteristic properties:

i) Positivity: evident.

ii) Vanishing: we have

$$\|f\|_{**} = 0, \iff \begin{cases} |f(0)| = 0, \\ \\ \|f'\|_\infty = 0, \end{cases} \iff \begin{cases} f(0) = 0, \\ \\ f'(x) \equiv 0. \end{cases}$$

From the second, $f \equiv C$ (constant), and since $f(0) = 0$, we get $f \equiv 0$.

iii) Homogeneity and triangular inequality: straightforward.

For $\|\cdot\|_{***}$ we have $\|f\|_{***} = |f(1)| + \|f'\|_1$. Again, this is well defined because $f' \in \mathscr{C}([0,1])$. Let's check the characteristic properties:

i) Positivity: evident.

ii) Vanishing: we have

$$\|f\|_{***} = 0, \iff \begin{cases} |f(1)| = 0, \\ \\ \|f'\|_1 = 0, \end{cases} \iff \begin{cases} f(1) = 0, \\ \\ f'(x) \equiv 0 \text{ (being } f' \in \mathscr{C}) \end{cases}$$

From the second, $f \equiv C$ (constant), and since $f(1) = 0$, we get $f \equiv 0$.

iii) Homogeneity and triangular inequality: straightforward.

Let now $f_n(x) := \frac{1}{n}\sin(n^2 x)$. We notice that $f_n \xrightarrow{pw} 0$. We have

$$\|f_n\|_* = \|f_n\|_\infty + \|f_n'\|_\infty,$$

where

$$\|f_n\|_\infty = \max_{x\in[0,1]} \tfrac{1}{n}|\sin(n^2 x)| = \tfrac{1}{n}\max_{y\in[0,n]^2} |\sin y| = \tfrac{1}{n},$$

$$\|f_n'\|_\infty = \max_{x\in[0,1]} \tfrac{1}{n}|n^2\cos(n^2 x)| = n\max_{y\in[0,n]} |\cos y| = n,$$

so $\|f_n\|_* = \frac{1}{n} + n \longrightarrow +\infty$, thus (f_n) is unbounded, whence it cannot have a limit in $\|\cdot\|_*$. Similarly, being $f_n(0) = 0$,

$$\|f_n\|_{**} = n \longrightarrow +\infty,$$

so (f_n) cannot be convergent for $\|\cdot\|_{**}$ too.

Finally, $|f_n(1)| = \frac{1}{n}\sin n^2$ and

$$\|f_n'\|_1 = \int_0^1 |n\cos(n^2 x)|\, dx = \frac{1}{n}\int_0^{n^2} |\cos y|\, dy.$$

Since $\int_0^{\pi/2} |\cos y|\ dy = 1$, if $K \in \mathbb{N}$ is such that $K\frac{\pi}{2} \leqslant n^2 < (K+1)\frac{\pi}{2}$, that is $K = \left[\frac{2}{\pi}n^2\right]$, we have

$$K = \int_0^{K\frac{\pi}{2}} |\cos y|\ dy \leqslant \int_0^{n^2} |\cos y|\ dy \leqslant \int_0^{(K+1)\frac{\pi}{2}} |\cos y|\ dy = K+1$$

so

$$\left[\frac{2}{\pi}n^2\right] \leqslant \int_0^{n^2} |\cos y|\ dy \leqslant \left[\frac{2}{\pi}n^2\right] + 1$$

from which

$$\frac{\left[\frac{2}{\pi}n^2\right]}{n} \leqslant \frac{1}{n}\int_0^{n^2} |\cos y|\ dy \leqslant \frac{\left[\frac{2}{\pi}n^2\right]+1}{n},$$

and since $[\frac{2}{\pi}n^2] \sim \frac{2}{\pi}n^2$ we deduce that $\frac{1}{n}\int_0^{n^2} |\cos y|\ dy \longrightarrow +\infty$. Conclusion: (f_n) is not convergent also for $\|\cdot\|_{***}$.

About relations: since $|f(0)| \leqslant \|f\|_\infty$, it is clear that $\|f\|_{**} \leqslant \|f\|_*$, so $\|\cdot\|_*$ is stronger than $\|\cdot\|_{**}$. The viceversa also holds: from the fundamental formula of integral calculus,

$$f(x) = f(0) + \int_0^x f'(y)\ dy,$$

so,

$$|f(x)| \leqslant |f(0)| + \left|\int_0^x f'(y)\ dy\right| \leqslant |f(0)| + \int_0^x |f'(y)|\ dy \leqslant |f(0)| + \|f'\|_\infty.$$

Therefore, $\|f\|_\infty \leqslant |f(0)| + \|f'\|_\infty$, so

$$\|f\|_* = \|f\|_\infty + \|f'\|_\infty \leqslant |f(0)| + 2\|f'\|_\infty \leqslant 2\|f\|_{**}.$$

We can see also that $\|\cdot\|_*$ is stronger than $\|\cdot\|_{***}$ (so also $\|\cdot\|_{**}$ it is). Indeed,

$$\|f\|_{***} = |f(1)| + \|f'\|_1 \leqslant \|f\|_\infty + \|f'\|_\infty = \|f\|_*.$$

We claim that $\|\cdot\|_{***}$ is not equivalent to $\|\cdot\|_*$. Take $f_n(x) = x^n$. We have

$$\|f_n\|_\infty = \max_{x\in[0,1]} |x^n| = 1, \quad \|f_n'\|_\infty = \max_{x\in[0,1]} |nx^{n-1}| = n.$$

Therefore, $\|f_n\|_* = n+1$. However

$$\|f_n\|_{***} = |f_n(1)| + \|f_n'\|_1 = 1 + \int_0^1 nx^{n-1}\ dx = 1 + [x^n]_{x=0}^{x=1} = 2.$$

In particular, there cannot be a constant $C > 0$ such that $\|f\|_* \leqslant C\|f\|_{***}$ for all $f \in V$. □

10.3.7. We notice that $f_n(0) = n \longrightarrow +\infty$ while, for $0 < x \leqslant 1$,

$$f_n(x) \sim_{n\to+\infty} \frac{n}{n^9x^3} = \frac{1}{x^3n^8} \longrightarrow 0.$$

So $f_n \overset{pw\ a.e.}{\longrightarrow} 0$ on $[0,1]$. The unique possible L^p limit is $f = 0$. We have

$$\|f_n\|_\infty = \text{ess. sup}_{x\in[0,1]} |f_n(x)| = \text{ess. sup}_{x\in[0,1]} \frac{n}{1+n^9x^3} = n \longrightarrow +\infty,$$

so (f_n) is not convergent in L^∞ norm. Let's check L^2 norm:

$$\|f_n\|_2^2 = \int_0^1 \frac{n^2}{(1+n^9x^3)^2}\, dx \overset{y=n^3x}{=} \frac{1}{n}\int_0^{n^3} \frac{1}{(1+y^3)^2}\, dy,$$

and since $\int_0^{n^3} \frac{1}{(1+y^3)^2}\, dy \longrightarrow \int_0^{+\infty} \frac{1}{(1+y^3)^2}\, dy < +\infty$ we have $\|f_n\|_2^2 = o\left(\frac{1}{n}\right) \longrightarrow 0$, so $f_n \xrightarrow{L^2} 0$. Now, recalling that (see Ex. 9.4.5), when $\mu(X) < +\infty$, $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$ norm, and precisely

$$\|f\|_1 \leqslant \sqrt{\mu(X)}\|f\|_2,$$

we have $\|f_n\|_1 \longrightarrow 0$, so $f_n \xrightarrow{L^1} 0$. □

10.3.9. Let $1_{E_n} \xrightarrow{L^1} f$ for some $f \in L^1$. Modulo a sub-sequence, we can say that (1_{E_n}) converges a.e.. Since $1_{E_n}(x) \in \{0,1\}$ this means that $f(x) \in \{0,1\}$ for a.e. $x \in X$. Let $E := \{x \in X \ : \ f(x) = 1\}$ so $f(x) = 1_E(x)$. We need to check that E is a measurable set. We may notice that in order $\lim_n 1_{E_n}(x) = 1$ we must have $1_{E_n}(x) = 1$ for all $n \geqslant N$ for some suitable N. Thus,

$$x \in E_n,\ \forall n \geqslant N,\ \implies\ x \in \bigcap_{n\geqslant N} E_n,\ \text{for some } N,\ \implies\ x \in \bigcup_N \bigcap_{n\geqslant N} E_n.$$

The vice versa also holds, thus,

$$x \in E \iff x \in \bigcup_N \bigcap_{n\geqslant N} E_n, \iff E = \bigcup_N \bigcap_{n\geqslant N} E_n.$$

Since the E_n are measurable, also E it is. □

11.4.3. We have

$$\begin{aligned} f * g(-x) &= \int_{\mathbb{R}} f(y)g(-x-y)\, dy = \int_{\mathbb{R}} f(y)g(x+y)\, dy \\ &\overset{y=-z}{=} \int_{\mathbb{R}} f(-z)g(x-z)\, dz = \int_{\mathbb{R}} f(z)g(x-z)\, dz \\ &= f * g(x). \quad \square \end{aligned}$$

11.4.5. We have

$$\begin{aligned}|f*g(x)| &= \left|\int_{\mathbb{R}} f(y)g(x-y)\,dy\right| \leqslant \int_{\mathbb{R}} |f(y)||g(x-y)|\,dy \\ &\overset{CS}{\leqslant} \left(\int_{\mathbb{R}} |f(y)|^2\,dy\right)^{1/2}\left(\int_{\mathbb{R}} |g(x-y)|^2\,dy\right)^{1/2} \overset{z=x-y}{=} \|f\|_2\left(\int_{\mathbb{R}} |g(z)|^2\,dz\right)^{1/2} \\ &= \|f\|_2\|g\|_2.\end{aligned}$$

This holds a.e. $x \in \mathbb{R}$. Therefore

$$\|f*g\|_\infty \leqslant \|f\|_2\|g\|_2.$$

For the extension the unique difference is using Hölder inequality:

$$\begin{aligned}|f*g(x)| &= \left|\int_{\mathbb{R}} f(y)g(x-y)\,dy\right| \leqslant \int_{\mathbb{R}} |f(y)||g(x-y)|\,dy \\ &\overset{H}{\leqslant} \left(\int_{\mathbb{R}} |f(y)|^p\,dy\right)^{1/p}\left(\int_{\mathbb{R}} |g(x-y)|^q\,dy\right)^{1/q} \overset{z=x-y}{=} \|f\|_p\left(\int_{\mathbb{R}} |g(z)|^q\,dz\right)^{1/q} \\ &= \|f\|_p\|g\|_q,\end{aligned}$$

a.e., from which $\|f*g\|_\infty \leqslant \|f\|_p\|g\|_q$. □

11.4.6. i) Let $f \in L^1$ and $g \in L^\infty$. From

$$|f*g(x)| \leqslant \int_{\mathbb{R}} |f(y)||g(x-y)|\,dy =\leqslant \|g\|_\infty \int_{\mathbb{R}} |f(y)|\,dy = \|g\|_\infty\|f\|_1,$$

from which

$$\|f*g\|_\infty \leqslant \|f\|_1\|g\|_\infty.$$

ii) Let now $f \in L^1$ and $g \in L^2$. We have integrating w.r.t. x we get

$$\|f*g\|_2^2 = \int_{\mathbb{R}} |f*g(x)|^2\,dx \leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)||g(x-y)|\,dy\right)^2\,dx.$$

As suggested we write $|f(y)||g(x-y)| = |f(y)|^{1/2}|f(y)|^{1/2}|g(x-y)|$ and we apply the CS inequality:

$$\begin{aligned}\int_{\mathbb{R}} |f(y)||g(x-y)|\,dy &= \int_{\mathbb{R}} \left(|f(y)|^{1/2}\right)\left(|f(y)|^{1/2}|g(x-y)|\right)\,dy \\ &\leqslant \left(\int_{\mathbb{R}} |f(y)|\,dy\right)^{1/2}\left(\int_{\mathbb{R}} |f(y)||g(x-y)|^2\,dy\right)^{1/2}\end{aligned}$$

so

$$\begin{aligned}\int_{\mathbb{R}} |f*g(x)|^2\,dx &\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)|\,dy\right)\left(\int_{\mathbb{R}} |f(y)||g(x-y)|^2\,dy\right)\,dx\\ &= \|f\|_1 \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)||g(x-y)|^2\,dy\,dx \overset{Fub.}{=} \|f\|_1 \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)||g(x-y)|^2\,dx\,dy\\ &= \|f\|_1\|f\|_1\|g\|_2^2 = \|f\|_1^2\|g\|_2^2;\end{aligned}$$

from which $\|f*g\|_2 \leqslant \|f\|_1\|g\|_2$.

iii) For the $f \in L^1$ and $g \in L^p$ $1 < p < +\infty$ case the trick is similar. We start noticing that

$$\|f*g\|_p^p = \int_{\mathbb{R}} |f*g(x)|^p\,dx \leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)||g(x-y)|\,dy\right)^p\,dx.$$

Following the idea of the case $p = 2$, we write $|f(y)||g(x-y)| = |f(y)|^{1/q}|f(y)|^{1/p}|g(x-y)|$, where $\frac{1}{p} + \frac{1}{q} = 1$, and we apply the Hölder's inequality:

$$\begin{aligned}\int_{\mathbb{R}} |f(y)||g(x-y)|\,dy &= \int_{\mathbb{R}} \left(|f(y)|^{1/q}\right)\left(|f(y)|^{1/p}|g(x-y)|\right)\,dy\\ &\leqslant \left(\int_{\mathbb{R}} |f(y)|\,dy\right)^{1/q}\left(\int_{\mathbb{R}} |f(y)||g(x-y)|^p\,dy\right)^{1/p}\end{aligned}$$

so

$$\begin{aligned}\int_{\mathbb{R}} |f*g(x)|^p\,dx &\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(y)|\,dy\right)^{p/q}\left(\int_{\mathbb{R}} |f(y)||g(x-y)|^p\,dy\right)\,dx\\ &= \|f\|_1^{p/q} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)||g(x-y)|^p\,dy\,dx \overset{Fub.}{=} \|f\|_1^{p/q} \int_{\mathbb{R}}\int_{\mathbb{R}} |f(y)||g(x-y)|^p\,dx\,dy\\ &= \|f\|_1^{p/q}\|f\|_1\|g\|_p^p = \|f\|_1^p\|g\|_p^p,\end{aligned}$$

from which $\|f*g\|_p \leqslant \|f\|_1\|g\|_p$. □

11.4.8. We have

$$f*g(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)\,dy = \int_{-R}^{R} f(y)g(x-y)\,dy.$$

Since $y \in [-R,R]$, $-R \leqslant x-y \leqslant R$ iff $-2R \leqslant -R+y \leqslant x \leqslant R+y \leqslant 2R$, so if $|x| > 2R$, $x-y \notin [-R,R]$, thus in particular $g(x-y) = 0$ for every $y \in [-R,R]$. So

$$f*g(x) = 0,\ \forall |x| > 2R. \quad \square$$

11.4.9. The exercise misses an hypothesis: g' must be bounded, that is $g' \in L^\infty$. Indeed, we apply the differentiation under integral sign:

$$\partial_x (f * g)(x) = \int_{-\infty}^{+\infty} \partial_x \big(f(y)g(x-y)\big)\, dy = \int_{-\infty}^{+\infty} f(y)g'(x-y)\, dy.$$

This provided the theorem applies. To this aim we have

- $\exists \partial_x \big(f(y)g(x-y)\big) = f(y)g'(x-y)$, a.e. $y \in \mathbb{R}$, $\forall x \in \mathbb{R}$ because $g \in \mathscr{C}^1$.
- $|\partial_x \big(f(y)g(x-y)\big)| = |f(y)||g'(x-y)| \leqslant \|g'\|_\infty |f(y)| \in L^1(\mathbb{R})$, a.e. y and for every $x \in \mathbb{R}$.

Therefore, th edifferentiation theorem applies and $\partial_x (f * g)(x) = f * g'(x)$. □

12.4.1. i) Notice that $\|\cdot\|$ is, by definition, well defined for every $f \in V_\alpha$. If $f, g \in V_\alpha$ then $f, g \in \mathscr{C}([0, +\infty[)$, so $f + g \in \mathscr{C}([0, +\infty[)$ and

$$\|f + g\| = \sup_{x \geqslant 0} (e^{ax}|f(x) + g(x)|) \leqslant \sup_{x \geqslant 0} (e^{ax}|f(x)| + e^{ax}|g(x)|) \leqslant \|f\| + \|g\| < +\infty.$$

At once, this proves that $f + g \in V_\alpha$ and also the triangular inequality for the $\|\cdot\|$. Similarly, $af \in V_\alpha$ for every $a \in \mathbb{R}$ and $\|af\| = |a|\|f\|$, thus homogeneity holds. From these it follows that V_α is a vector space. It remains to check positivity and vanishing. Positivity is evident from the definition of $\|\cdot\|$. Assuming that $0 = \|f\| = \|e^{\alpha \sharp} f\|_\infty$, we have $e^{\alpha x} f(x) \equiv 0$, that is $f \equiv 0$. Therefore, vanishing holds and $\|\cdot\|$ is a well defined norm on V_α.

ii) Let $(f_n) \subset V_\alpha$ be a Cauchy sequence, that is

$$\forall \varepsilon > 0,\ \exists N\ :\ \|f_n - f_m\| \leqslant \varepsilon,\ \forall n, m \geqslant N.$$

We already noticed that $\|f\| = \|e^{\alpha \sharp} f\|_\infty$. So, setting $g_n(x) := e^{\alpha x} f_n(x)$ we have that

$$\forall \varepsilon > 0,\ \exists N\ :\ \|g_n - g_m\|_\infty \leqslant \varepsilon,\ \forall n, m \geqslant N.$$

Thus, (g_n) is a Cauchy sequence w.r.t. $\|\cdot\|_\infty$, and since $(g_n) \subset B([0, +\infty[)$ ($\|g_n\|_\infty = \|e^{\alpha \sharp} f_n\| < +\infty$ being $f_n \in V_\alpha$), $g_n \xrightarrow{\|\cdot\|_\infty} g \in B(X)$. Moreover, since uniform limit of continuous functions is a continuous function, $g \in \mathscr{C}([0, +\infty[)$. Setting $f(x) := e^{-\alpha x} g(x) \in \mathscr{C}([0, +\infty[)$ we have $\|f\| = \|e^{\alpha \sharp} f\|_\infty = \|g\|_\infty < +\infty$, so $f \in V_\alpha$, and

$$\|f_n - f\| = \|e^{\alpha \sharp} f_n - e^{\alpha \sharp} f\|_\infty = \|g_n - g\|_\infty \longrightarrow 0,$$

so $f_n \xrightarrow{\|\cdot\|} f$, and this shows that V_α is a Banach space. □

12.4.2.

12.4.4.

12.4.5. Let $(f_n) \subset \mathscr{C}^1([a, b])$ be a Cauchy sequence, that is

$$\forall \varepsilon > 0,\ \exists N,\ :\ \|f_n - f_m\| = \|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty \leqslant \varepsilon,\ \forall n, m \geqslant N.\ (\star)$$

The goal is to show that $\exists f \in \mathscr{C}^1([a,b])$ such that $f_n \xrightarrow{\|\cdot\|} f$. We start noticing that, from the Cauchy property, (f_n) is a Cauchy sequence in uniform norm. As we know, this norm is complete, so $f_n \xrightarrow{\|\cdot\|_\infty} f$ for some $f \in \mathscr{C}([a,b])$. We now have a limit, the next step is to show that this $f \in \mathscr{C}^1([a,b])$. As suggested, since $f_n \in \mathscr{C}^1([a,b])$ the fundamental formula of integral calculus applies:

$$f_n(x) = f_n(a) + \int_a^x f_n'(y)\, dy.$$

We show now that we can pass this to the limit. Indeed, since $f_n \xrightarrow{\|\cdot\|_\infty} f$, in particular $f_n \xrightarrow{pw} f$, so $f_n(x) \longrightarrow f(x)$ and $f_n(a) \longrightarrow f(a)$. About (f_n') we also may notice that, from $(\star)$, also $(f_n') \subset \mathscr{C}([a,b])$ is a Cauchy sequence in uniform norm: therefore $f_n' \xrightarrow{\|\cdot\|_\infty} g$, for some $g \in \mathscr{C}([a,b])$. Then

$$\left|\int_a^x f_n'\, dy - \int_a^x g\, dy\right| = \left|\int_a^x (f_n' - g)\, dy\right| \leqslant \int_a^x |f_n' - g|\, dy \leqslant (x-a)\|f_n' - g\|_\infty \leqslant (b-a)\|f_n' - g\|_\infty$$

so

$$\int_a^x f_n'\, dy \longrightarrow \int_a^x g\, dy,\ \forall x \in [a,b].$$

So we can say that

$$f(x) = f(a) + \int_a^x g(y)\, dy,\ \forall x \in [a,b],\ \implies\ \exists f'(x) = g(x).$$

This shows that $f \in \mathscr{C}^1([a,b])$, so $f \in V$. To finish, we prove that $f_n \xrightarrow{\|\cdot\|} f$. Letting $m \to +\infty$ into $(\star)$ we get

$$\|f_n - f\| = \|f_n - f\|_\infty + \|f_n' - f'\|_\infty \leqslant \varepsilon,\ \forall n \geqslant N,$$

which is the conclusion. □

12.4.6. We have only to prove the $\implies$. Let $(f_n) \subset V$ be a generic Cauchy sequence,

$$\forall \varepsilon > 0,\ \exists N,\ :\ \|f_n - f_m\| \leqslant \varepsilon,\ \forall n, m \geqslant N.\ (\star)$$

We first show that $(\|f_n\|)$ is a Cauchy sequence in $\mathbb{R}$: this follows by the triangular inequality because

$$\|f_n\| = \|f_n - f_m + f_m\| \leqslant \|f_n - f_m\| + \|f_m\|,\ \implies\ \|f_n\| - \|f_m\| \leqslant \|f_n - f_m\|,$$

and switching f_n with f_m we also have

$$\|f_m\| - \|f_n\| \leqslant \|f_n - f_m\|$$

from which

$$-\|f_n - f_m\| \leqslant \|f_n\| - \|f_m\| \leqslant \|f_n - f_m\|,\ \iff\ |\|f_n\| - \|f_m\|| \leqslant \|f_n - f_m\|.$$

Therefore,

$$|\|f_n\| - \|f_m\|| \leqslant \varepsilon,\ \forall n, m \geqslant N.$$

Since $(\mathbb{R}, |\cdot|)$ is a complete space, $\exists \lim_n \|f_n\| = \ell \geqslant 0$. If $\ell = 0$ we have $f_n \xrightarrow{\|\cdot\|} 0$, so (f_n) converges. Otherwise $\ell > 0$. By the permanence of sign, $2\ell \geqslant \|f_n\| \geqslant \frac{\ell}{2} > 0$ for $n \geqslant M$, so, without any restriction, we may assume $2\ell \geqslant \|f_n\| \geqslant \frac{\ell}{2}$ for every n. Define now

$$u_n := \frac{f_n}{\|f_n\|} \in S.$$

We claim that u_n is a Cauchy sequence. We have

$$\begin{aligned} u_n - u_m &= \frac{f_n}{\|f_n\|} - \frac{f_m}{\|f_m\|} = \frac{1}{\|f_n\|}(f_n - f_m) + \left(\frac{1}{\|f_n\|} - \frac{1}{\|f_m\|}\right) f_m \\ &= \frac{1}{\|f_n\|}(f_n - f_m) + \frac{\|f_m\| - \|f_n\|}{\|f_n\|\|f_m\|} f_m \end{aligned}$$

so

$$\|u_n - u_m\| \leqslant \frac{1}{\|f_n\|}\|f_n - f_m\| + \frac{|\|f_m\| - \|f_n\||}{\|f_n\|\|f_m\|}\|f_m\| \leqslant \frac{2}{\ell}\varepsilon + \frac{4}{\ell^2}\varepsilon 2\ell = \frac{6}{\ell}\varepsilon,\ \forall n, m \geqslant N.$$

By the hypothesis, $u_n \xrightarrow{\|\cdot\|} u \in S$. Let $f := \ell u$. We claim that $f_n \xrightarrow{\|\cdot\|} f$. Indeed,

$$\begin{aligned} \|f_n - f\| &= \big\|\|f_n\|u_n - \ell u\big\| = \big\|\|f_n\|(u_n - u) + (\|f_n\| - \ell)u\big\| \\ &\leqslant \|f_n\|\|u_n - u\| + \big|\|f_n\| - \ell\big|\|u\| \longrightarrow \ell \cdot 0 + 0 \cdot 1 = 0. \qquad \square \end{aligned}$$

12.4.7. Let $s_n := \sum_{k=1}^n f_k$. We check that (s_n) is a Cauchy sequence: for $n > m$ we have

$$\|s_n - s_m\| = \left\|\sum_{k=m+1}^n f_k\right\| \leqslant \sum_{k=m+1}^n \|f_k\| = \sigma_n - \sigma_m,$$

where

$$\sigma_n := \sum_{k=0}^n \|f_k\|.$$

By hypothesis, $(\sigma_n) \subset \mathbb{R}$ is convergent, so it is a Cauchy sequence. This means that

$$\forall \varepsilon > 0,\ \exists N,\ :\ |\sigma_n - \sigma_m| \leqslant \varepsilon,\ \forall n, m \geqslant N.$$

So, if $n > m \geqslant N$, noticed that $\sigma_n \geqslant \sigma_m$ we have

$$\|s_n - s_m\| \leqslant \sigma_n - \sigma_m = |\sigma_n - \sigma_m| \leqslant \varepsilon,$$

that is (s_n) is a Cauchy sequence. Since $(V, \|\cdot\|)$ is a Banach space, $\exists \lim_n s_n \in V$. $\square$

13.3.2. i) If p, q are polynomials, $p \cdot q$ is a polynomials so $|p(x)q(x)| \leqslant a|x|^n + b$, from which

$$\int_{\mathbb{R}} |p(x)q(x)e^{-x^2}|\ dx \leqslant \int_{\mathbb{R}} (a|x|^n + b)e^{-x^2}\ dx < +\infty,$$

and the scalar product is well defined. The characteristic properties of the scalar product are straightforward (vanishing in strong form since functions are continuous).

ii) We have

$$\langle x^m, x^n\rangle_V = \int_{\mathbb{R}} x^{n+m}e^{-x^2}\ dx.$$

If $n+m$ is odd, x^{n+m} is odd, thus the integral vanishes and $\langle x^n, x^m\rangle_V = 0$. Now, $n+m$ is odd iff n is odd (even) and m is even (odd). In all other cases, setting $k = n+m$,

$$\begin{aligned} I_k &:= \int_{\mathbb{R}} x^k e^{-x^2}\ dx = -\frac{1}{2}\int_{\mathbb{R}} x^{k-1}\underbrace{\left(-2xe^{-x^2}\right)}_{=\partial_x e^{-x^2}}\ dx \\ &= -\frac{1}{2}\left[\left[x^{k-1}e^{-x^2}\right]_{x=-\infty}^{x=+\infty} - (k-1)\int_{\mathbb{R}} x^{k-2}e^{-x^2}\ dx\right] \\ &= \frac{k-1}{2}I_{k-2}. \end{aligned}$$

Therefore, if $k = 2h$,

$$I_{2h} = \frac{(2h-1)}{2}I_{2(h-1)} = \frac{(2h-1)(2h-3)}{2^2}I_{2(h-2)} = \ldots = \frac{(2h-1)(2h-3)\cdots 3\cdot 1}{2^h}I_0$$

Since $I_0 = \int_{\mathbb{R}} e^{-x^2}\ dx = \sqrt{\pi}$, we have

$$I_{2h} = \frac{\pi^{1/2}}{2^{2h}}\frac{(2h)!}{h!}.$$

Therefore, if $n+m$ is even,

$$\langle x^n, x^m\rangle_V = \frac{\pi^{1/2}}{2^{n+m}}\frac{(n+m)!}{\left(\frac{n+m}{2}\right)!}.$$

iii) We have

$$\begin{aligned} \|x^2 - (ax+b)\|_V^2 &= \langle x^2 - (ax+b), x^2 - (ax+b)\rangle_V \\ &= \langle x^2, x^2\rangle_V - 2b\langle x^2, 1\rangle_V + a^2\langle x, x\rangle_V + +b^2\langle 1, 1\rangle_V \\ &= \pi^{1/2}\left(\tfrac{3}{4} - b + \tfrac{1}{2}a^2 + b^2\right) =: f(a,b). \end{aligned}$$

We have $\lim_{(a,b)\to\infty_2} f(a,b) = +\infty$, and being also $f \in \mathsf{C}(\mathbb{R}^2)$ we conclude that a min point exists and must fulfill $\nabla f(a,b) = \vec{0}$, that is $(a, 2b-1) = (0,0)$ from which $a = 0$ and $b = 1/2$. □

13.3.3. We have

$$\begin{aligned}\langle f_n, f_m\rangle &= \int_{[0,1]^d} e^{i2\pi n\cdot x} e^{-i2\pi m\cdot x}\, dx = \int_{[0,1]^d} e^{i2\pi(n-m)\cdot x}\, dx \\ &= \int_{[0,1]^d} \prod_{j=1}^d e^{i2\pi(n_j-m_j)x_j}\, dx_1\cdots dx_d \\ &= \prod_{j=1}^d \int_0^1 e^{i2\pi(n_j-m_j)x_j}\, dx_j\end{aligned}$$

Now, being

$$\int_0^1 e^{i2\pi(n_j-m_j)x_j}\, dx_j = \begin{cases} (n_j \neq m_j), & = \left[\frac{e^{i2\pi(n_j-m_j)x}}{i2\pi(n_j-m_j)}\right]_{x_j=0}^{x_j=1} = 0, \\ (n_j = m_j), & = \int_0^1 1\, dx_j = 1, \end{cases}$$

we have that $\langle f_n, f_m\rangle = 0$ when $n \neq m$ and $\langle f_n, f_m\rangle = 1$ iff $n = m$. □

13.3.4. From the Cauchy-Schwarz inequality,

$$|\langle f, g\rangle| \leqslant \|f\|\|g\| = \|f\|, \text{ if } \|g\| = 1.$$

Therefore

$$\sup_{g\in V\,:\,\|g\|=1} |\langle f, g\rangle| \leqslant \|f\|.$$

If $f = 0$ clearly equality holds. So assume $f \neq 0$. Taking $g = \frac{f}{\|f\|}$ we have $\|g\| = 1$ and $\langle f, g\rangle = \frac{1}{\|f\|}\langle f, f\rangle = \frac{\|f\|^2}{\|f\|} = \|f\|$. So, $\|f\| \leqslant \sup|\langle f, g\rangle|$ (where $\|g\| = 1$). □

13.3.5. There is no issue with the definition. Let's prove the characteristic properties.

- positivity: $\langle A, A\rangle = \mathrm{Tr}(A^*A)$. Let $A^*A = [m_{ij}]$. Then, if $e_i = (\delta_{i1}, \ldots, \delta_{id})$ are the vectors of the canonical basis,
$$m_{ii} = (A^*A)e_i \cdot e_i = (Ae_i)\cdot(Ae_i) = \|Ae_i\|^2 \geqslant 0,$$
where $\|\cdot\|$ is the euclidean norm of $\mathbb{R}^d$. So $m_{ii} \geqslant 0$ for every i, therefore $\langle A, A\rangle = \sum_i m_{ii} \geqslant 0$.
- vanishing: suppose $\langle A, A\rangle = 0$, that is $\mathrm{Tr}(A^*A) = 0$. Using notations of previous point, $0 = \sum_i m_{ii} = \sum_i \|Ae_i\|^2$, so $Ae_i = 0$ for every i. But then, for every vector $v \in \mathbb{R}^d$, writing $v = \sum_i v_i e_i$ we have
$$Av = \sum_i v_i Ae_i = 0,$$
that is $A = 0$ (matrix with all entries = 0).

- linearity: we have

$$\begin{aligned}\langle \lambda A + \mu B, C\rangle &= \operatorname{Tr}[(\lambda A + \mu B)^* C] = \operatorname{Tr}[(\lambda A^* + \mu B^*)C] \\ &= \lambda \operatorname{Tr}[A^* C] + \mu \operatorname{Tr}[B^* C] = \lambda\langle A, C\rangle + \mu\langle B, C\rangle.\end{aligned}$$

- symmetry: since $\operatorname{Tr} M = \operatorname{Tr} M^*$ we have

$$\langle A, B\rangle = \operatorname{Tr}[A^* B] = \operatorname{Tr}\left[(B^* A)^*\right] = \operatorname{Tr}[B^* A] = \langle B, A\rangle. \qquad \square$$

13.3.7. We prove that (f_n) is a Cauchy sequence, the convergence will follow from the completeness of V (Hilbert space). To this aim we recall that, by the parallelogram identity,

$$\|f_n + f_m\|^2 + \|f_n - f_m\|^2 = 2\left(\|f_n\|^2 + \|f_m\|^2\right) = 2,$$

being $\|f_n\| = 1$ for every n. So

$$\|f_n - f_m\|^2 = 2 - \|f_n + f_m\|^2.$$

If $\lim_{n,m\to+\infty} \|f_n + f_m\|^2 = 2$ (there is a typo in the text), for $\varepsilon > 0$ fixed there exists N such that

$$\|f_n - f_m\|^2 = 2 - \|f_n + f_m\|^2 \leqslant \varepsilon, \ \forall n, m \geqslant N,$$

so (f_n) is a Cauchy sequence. $\square$

13.3.9. i) $\iff$ ii) Let $s_n := \sum_{k=0}^n f_k$ and $\sigma_n := \sum_{k=0}^n \|f_n\|^2$. We notice that if $n > m$,

$$\|s_n - s_m\|^2 = \left\| \sum_{k=m+1}^n f_k \right\|^2 \overset{Pyth.}{=} \sum_{k=m+1}^n \|f_k\|^2 = \sigma_n - \sigma_m = |\sigma_n - \sigma_m|.$$

So, (s_n) is a Cauchy sequence in $(V, \langle\cdot,\cdot\rangle)$ iff (σ_n) is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. Since both spaces are complete (V by Hypothesis), Cauchy sequence means convergent so we have the conclusion.

i) $\iff$ iii). If $\sum_n f_n$ converges to $s \in V$, then

$$\langle s, g\rangle = \left\langle \lim_{n\to+\infty} s_n, g \right\rangle = \lim_n \langle s_n, g\rangle = \lim_n \left\langle \sum_{k=0}^n f_k, g \right\rangle = \lim_n \sum_{k=0}^n \langle f_k, g\rangle.$$

This means that $\sum_n \langle f_n, g\rangle$ is convergent for every $g \in V$.

Vice versa, assume $\sum_n \langle f_n, g\rangle$ is convergent for every $g \in V$. Let's prove that $\sum_n \|f_n\|^2 < +\infty$. Assume, by contradiction, that $\sum_n \|f_n\|^2 = +\infty$. We can assume $f_n \neq 0$ for every n (otherwise we eliminate the vector). Let $g = \sum_n c_n f_n$. By i), $g \in V$ iff $\sum_n \|c_n f_n\|^2 = \sum_n c_n^2 \|f_n\| < +\infty$. On the other hand

$$\sum_n \langle f_n, g\rangle = \sum_n c_n \|f_n\|^2.$$

So, the goal is to build (c_n) in such a way that

$$\sum_n c_n^2 \|f_n\|^2 < +\infty, \quad \sum_n c_n \|f_n\|^2 = +\infty.$$

Let $a_n := \|f_n\|^2$, so $\sum_n a_n = +\infty$. We define N_k as the first integer for which $\sum_{n=0}^{N_k} a_n \geqslant 2^k$ and we set $c_n = \frac{1}{2^k}$ for $N_k \leqslant n < N_{k+1}$. Then

$$\sum_n c_n^2 a_n = \sum_k \sum_{n=N_k}^{N_{k+1}} c_n^2 a_n = \sum_k \frac{1}{2^{2k}} \underbrace{\sum_{n=N_k}^{N_{k+1}} a_n}_{\leqslant 2^{k+1}} \leqslant \sum_k \frac{1}{2^{2k}} 2^{k+1} = 2 \sum_k \frac{1}{2^k} < +\infty$$

while

$$\sum_n c_n a_n = \sum_k \sum_{n=N_k}^{N_{k+1}} c_n a_n = \sum_k \frac{1}{2^k} \underbrace{\sum_{n=N_k}^{N_{k+1}} a_n}_{\geqslant 2^k} \leqslant \sum_k \frac{1}{2^k} 2^k = \sum_k 1 = +\infty$$

(rmk: $\sum_{n=N_k}^{N_{k+1}} a_n \geqslant 2^k$ because, if $< 2^k$ we would have $\sum_{n=0}^{N_{k+1}} a_n = \sum_{n=0}^{N_k - 1} a_n + \sum_{n=N_k}^{N_{k+1}} a_n$; since, by definition of N_k, $\sum_{n=0}^{N_k - 1} a_n < 2^k$, if $\sum_{n=N_k}^{N_{k+1}} a_n < 2^k$ then $\sum_{n=0}^{N_{k+1}} a_n < 2^k + 2^k = 2^{k+1}$). □

14.3.1. We have to solve for

$$\min_{a,b,c \in \mathbb{R}} \|\cos x - (ax^2 + bx + c)\|_2. \ (\star)$$

Let $H = L^2([0, 2\pi])$ be the Hilbert space equipped with standard svalar product $\langle f, g \rangle_2 = \int_0^{2\pi} f(x) g(x)\, dx$, and $U := \mathrm{Span}(1, x, x^2)$. Since U is finite dimensional it is closed. According to the orthogonal projection thm, the solution of problem $(\star)$ is $\Pi_U \cos$ verifying

$$\langle \cos - \Pi_U \cos, u \rangle_2 = 0, \ \forall u \in U.$$

Equivalently, since $\Pi_U \cos = ax^2 + bx + c$, we must verify

$$\langle \cos - (ax^2 + bx + c), x^k \rangle_2 = 0, \ k = 0, 1, 2. \ (\star\star)$$

We notice that,

$$\langle \cos, 1 \rangle_2 = \int_0^{2\pi} \cos x \, dx = [\sin x]_{x=0}^{x=2\pi} = 0,$$

while, for $k \geqslant 1$,

$$\langle \cos, x^k \rangle_2 = \int_0^{2\pi} x^k \cos x \, dx = [x^k \sin x]_{x=0}^{x=2\pi} - \int_0^{2\pi} k x^{k-1} \sin x \, dx = -k \int_0^{2\pi} x^{k-1} \sin x \, dx.$$

Therefore

$$\langle \cos, x \rangle_2 = - \int_0^{2\pi} \sin x \, dx = [\cos x]_{x=0}^{x=2\pi} = 0,$$

and

$$\langle \cos, x^2 \rangle_2 = -2 \int_0^{2\pi} x \sin x \, dx = 2 \left[[x \cos x]_{x=0}^{x=2\pi} - \int_0^{2\pi} \cos x \, dx \right] = 4\pi.$$

Moreover,

$$\langle x^j, x^k \rangle_2 = \int_0^{2\pi} x^{j+x} \, dx = \left[\frac{x^{j+k+1}}{j+k+1} \right]_{x=0}^{x=2\pi} = \frac{(2\pi)^{j+k+1}}{j+k+1}.$$

From this, the orthogonality condition ($\star\star$) yields the system

$$\begin{cases} -a\frac{(2\pi)^3}{3} - b\frac{(2\pi)^2}{2} - c2\pi = 0, \\ \\ -a\frac{(2\pi)^4}{4} - b\frac{(2\pi)^3}{3} - c\frac{(2\pi)^2}{2} = 0, \\ \\ 4\pi - a\frac{(2\pi)^5}{5} - b\frac{(2\pi)^4}{4} - c\frac{(2\pi)^3}{3} = 0. \end{cases}$$

STraightforward calculations lead to $a = \frac{45}{2\pi^4}$, $b = -\frac{45}{\pi^3}$ and $c = \frac{15}{\pi^2}$. □

14.3.2. We can recast the problem as follows: let $H = L^2([-1,1])$ equipped with usual product, then

$$\min_{a,b,c} \int_{-1}^{1} |x^3 + ax^2 + bx + c|^2 \, dx = \min_{a,b,c} \int_{-1}^{1} |x^3 - (ax^2 + bx + c)|^2 \, dx = \min_{u \in U} \|\sharp^3 - u\|_2^2$$

where $U = \mathrm{Span}(1, x, x^2)$. Since U is finite dimensional, U is closed, and the solution to the previous problem is $\Pi_U \sharp^3$. This is characterized by solving

$$\langle \sharp^3 - (a\sharp^2 + b\sharp + c), \sharp^k \rangle_2 = 0, \ k = 0, 1, 2.$$

We notice that

$$\langle \sharp^j, \sharp^k \rangle = \int_{-1}^{1} x^j x^k \, dx = \int_{-1}^{1} x^{j+k} \, dx = \left[\frac{x^{j+k+1}}{j+k+1} \right]_{x=-1}^{x=1} = \begin{cases} 0, & \text{if } j+k \text{ is odd}, \\ \\ \frac{2}{j+k+1}, & \text{if } j+k \text{ is even}. \end{cases}$$

Therefore, the orthogonality condition yields the system

$$\begin{cases} -a\frac{2}{3} - 2c = 0, \\ \\ \frac{2}{5} - b\frac{2}{3} = 0, \\ \\ -a\frac{2}{5} - c\frac{2}{3} = 0, \end{cases} \iff a = c = 0, \ b = \frac{3}{5}. \quad \square$$

14.3.3. Let $H := L^2([0, +\infty[)$ equipped with standard scalar product $\langle f, g \rangle = \int_0^{+\infty} fg \, dx$. The problem consists in folving for

$$\min_{u \in U} \|e^{-\sharp} - u\|_2,$$

where $U = \mathrm{Span}(e^{-2x}, e^{-3x})$. Since H is a Hilbert space, U is finite dimensional hence closed, the solution is $\Pi_U e^{-\sharp}$ characterized by the orthogonality condition

$$\langle e^{-\sharp} - (ae^{-2\sharp} + be^{-3\sharp}), e^{-k\sharp}\rangle_2 = 0,\ k = 2, 3.$$

We notice that

$$\langle e^{-j\sharp}, e^{-k\sharp}\rangle_2 = \int_0^{+\infty} e^{-(j+k)x}\, dx = \left[\frac{e^{-(j+k)x}}{-(j+k)}\right]_{x=0}^{x=+\infty} = \frac{1}{j+k}.$$

Therefore, the orthogonslity condition yields the system

$$\begin{cases} \frac{1}{3} - a\frac{1}{4} - b\frac{1}{5} = 0, \\ \\ \frac{1}{4} - a\frac{1}{5} - b\frac{1}{6} = 0. \end{cases} \iff a = \frac{10}{3},\ b = -\frac{5}{2}. \quad \square$$

14.3.4. We equip $L^2([0,1])$ with standard scalar product. We may recast the problem as

$$\max_{f\in L^2,\ \|f\|=1} \langle f, e^{\sharp}\rangle_2.$$

By CS inequality, $|\langle f, e^{\sharp}\rangle_2| \leqslant \|f\|_2\|e^{\sharp}\|_2 = \|e^{\sharp}\|_2$, where

$$\|e^{\sharp}\|_2^2 = \int_0^1 (e^x)^2\, dx = \int_0^1 e^{2x}\, dx = \left[\frac{e^{2x}}{2}\right]_{x=0}^{x=1} = \frac{e^2-1}{2}.$$

Therefore, $\langle f, e^{\sharp}\rangle_2 \leqslant |\langle f, e^{\sharp}\rangle_2| \leqslant \sqrt{\frac{e^2-1}{2}}$ for every $f \in L^2$ such that $\|f\|_2 = 1$. Now, taking $f^* = \sqrt{\frac{2}{e^2-1}}e^{\sharp}$ we have $f^* \in L^2([0,1])$, $\|f^*\|_2 = 1$ and

$$\langle f^*, e^{\sharp}\rangle_2 = \sqrt{\frac{2}{e^2-1}}\|e^{\sharp}\|_2^2 = \sqrt{\frac{e^2-1}{2}}.$$

From this,

$$\max_{f\in L^2,\ \|f\|=1} \langle f, e^{\sharp}\rangle_2 = \sqrt{\frac{e^2-1}{2}},$$

and optimal f is f^*. $\square$

14.3.5. See slides. $\square$

14.3.6. See slides. $\square$

17.2.1. #1. Clearly $x\mathrm{rect}_a(x) \in L^1(\mathbb{R})$ and

$$\widehat{\sharp\mathrm{rect}_a}(\xi) = \int_{\mathbb{R}} x1_{[-a,a]}(x)e^{-i\xi x}\, dx = \int_{-a}^{a} xe^{-i\xi x}\, dx.$$

Here we may compute the integral by parts or, in an alternative way, noticed that $xe^{-i\xi x} = \frac{1}{-i}\partial_\xi e^{-i\xi x} = i\partial_\xi e^{-i\xi x}$, so,

$$\widehat{\sharp \mathrm{rect}_a}(\xi) = \int_{-a}^{a} i\partial_\xi e^{-i\xi x}\, dx.$$

Now, since differentiability under integral sign applies (we already check that $\partial_x e^{-i\xi x} = -ixe^{-i\xi x}$ exists $\forall x \in [-a,a]$, $\forall \xi \in \mathbb{R}$ and $|\partial_\xi e^{-i\xi x}| = |-ixe^{-i\xi x}| = |x| \in L^1([-a,a])$), we have

$$\widehat{\sharp \mathrm{rect}_a}(\xi) = i\partial_\xi \int_{-a}^{a} e^{-i\xi x}\, dx = i\partial_\xi \widehat{\mathrm{rect}_a}(\xi) = i\partial_\xi 2a\frac{\sin(a\xi)}{a\xi} = 2i\frac{a\xi\cos(a\xi) - \sin(a\xi)}{\xi^2}.$$

This for $\xi \neq 0$. For $\xi = 0$ we have directly,

$$\widehat{\sharp \mathrm{rect}_a}(0) = \int_{-a}^{a} x\, dx = 0.$$

#2. $\widehat{f}(\xi) = 2\frac{1-\cos(a\xi)}{\xi^2}$.

#3. Clearly $f(x) = \cos x \mathrm{rect}_{\pi/2}(x) \in L^1(\mathbb{R})$. By Euler's formulas, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ so

$$\widehat{f}(\xi) = \frac{1}{2}\left(\widehat{e^{i\sharp}\mathrm{rect}_{\pi/2}}(\xi) + \widehat{e^{-i\sharp}\mathrm{rect}_{\pi/2}}(\xi)\right).$$

Now, recalling that, for $g \in L^1$, $\widehat{e^{im\sharp}g}(\xi) = \widehat{g}(\xi - m)$, we have

$$\widehat{f}(\xi) = \frac{1}{2}\left(\widehat{\mathrm{rect}_{\pi/2}}(\xi-1) + \widehat{\mathrm{rect}_{\pi/2}}(\xi+1)\right) = \frac{\sin(\frac{\pi}{2}(\xi-1))}{\xi-1} + \frac{\sin(\frac{\pi}{2}(\xi+1))}{\xi+1}. \quad \square$$

#4. Clearly, $f(x) = e^{-|x|}\operatorname{sgn}(x) \in L^1(\mathbb{R})$. We have

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-|x|}\operatorname{sgn}(x)e^{-i\xi x}\, dx = \int_{-\infty}^{0} -e^{x}e^{-i\xi x}\, dx + \int_{0}^{+\infty} e^{-x}e^{-i\xi x}\, dx \\ &= \left[-\frac{e^{(1-i\xi)x}}{1-i\xi}\right]_{x=-\infty}^{x=0} + \left[\frac{e^{-(1+i\xi)x}}{-(1+i\xi)}\right]_{x=0}^{x=+\infty} = -\frac{1}{1-i\xi} + \frac{1}{1+i\xi} = -\frac{2i\xi}{1+\xi^2}. \quad \square\end{aligned}$$

#5. Clearly $f(x) = e^{-x}1_{[0,+\infty[}(x) \in L^1$. Now,

$$\widehat{f}(\xi) = \int_{0}^{+\infty} e^{-x}e^{-i\xi x}\, dx = \left[\frac{e^{-(1+i\xi)x}}{-(1+i\xi)}\right]_{x=0}^{x=+\infty} = \frac{1}{1+i\xi}. \quad \square$$

17.2.4. If f is even and real valued, then

$$\overline{\widehat{f}(\xi)} = \overline{\int_{\mathbb{R}} f(x)e^{-i\xi x}\,dx} = \int_{\mathbb{R}} \overline{f(x)e^{-i\xi x}}\,dx = \int_{\mathbb{R}} f(x)e^{i\xi x}\,dx$$

$$\overset{y=-x}{=} \int_{\mathbb{R}} f(-y)e^{-i\xi y}\,dy = \int_{\mathbb{R}} f(y)e^{-i\xi y}\,dy = \widehat{f}(\xi).$$

Therefore $\widehat{f}(\xi) \in \mathbb{R}$, $\forall \xi \in \mathbb{R}$. □

17.2.8. Warning! The exercise contains a typo: the correct statement is

$$|\widehat{f}(\xi)| < \widehat{f}(0),\ \forall \xi \neq 0.$$

We start noticing that

$$|\widehat{f}(\xi)| \leqslant \int_{\mathbb{R}} |f(x)|\,dx \overset{f\geq 0}{=} \int_{\mathbb{R}} f(x)\,dx = \widehat{f}(0),\ \forall \xi \in \mathbb{R}.$$

The problem is showing that the inequality must be strict. We notice that

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\,(\cos(-\xi x) + i\sin(-\xi x))\,\,dx = \int_{\mathbb{R}} f(x)\cos(\xi x)\,dx - i\int_{\mathbb{R}} f(x)\sin(\xi x)\,dx.$$

So,

$$|\widehat{f}(\xi)|^2 = \left(\int_{\mathbb{R}} f(x)\cos(\xi x)\,dx\right)^2 + \left(\int_{\mathbb{R}} f(x)\sin(\xi x)\,dx\right)^2.$$

By CS inequality,

$$\left(\int_{\mathbb{R}} f\cos\right)^2 = \left(\int_{\mathbb{R}} f^{1/2} f^{1/2}\cos\right)^2 \leqslant \int_{\mathbb{R}} f\,dx \int_{\mathbb{R}} f\cos^2$$

and = holds iff $f^{1/2} = \lambda f^{1/2}\cos$ a.e. for some λ. Since $f > 0$ always, $\lambda\cos(\xi x) = 1$ a.e. x which is impossible. Thus, = never holds, and < holds. Similarly,

$$\left(\int_{\mathbb{R}} f\sin\right)^2 < \int_{\mathbb{R}} f\,dx \int_{\mathbb{R}} f\sin^2.$$

Therefore

$$|\widehat{f}(\xi)|^2 < \int_{\mathbb{R}} f\left(\int_{\mathbb{R}} f(\cos^2 + \sin^2)\right) = \left(\int_{\mathbb{R}} f\right)^2 = \widehat{f}(0)^2. \quad \square$$

17.2.9. Since $f \equiv 0$ for $|x| \geqslant R$, we have

$$\widehat{f}(\xi) = \int_{-R}^{R} f(x)e^{-i\xi x}\,dx.$$

Now, recall that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},\ \forall z \in \mathbb{C}.$$

Therefore

$$e^{-i\xi x} = \sum_{n=0}^{\infty} \frac{(-i\xi x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-i\xi)^n}{n!} x^n,$$

so,

$$\widehat{f}(\xi) = \int_{-R}^{R} f(x) \sum_{n=0}^{\infty} \frac{(-i\xi)^n}{n!} x^n \, dx \overset{?}{=} \sum_{n=0}^{\infty} \int_{-R}^{R} f(x) \frac{(-i\xi)^n}{n!} x^n \, dx.$$

To justify $\overset{?}{=}$ we apply the dominated convergence for series. We remind that if $(f_n) \in L^1(E)$ are such that

$$\sum_n \int_E |f_n| \, d\mu < +\infty, \implies \int_E \sum_n f_n \, d\mu = \sum_n \int_E f_n \, d\mu.$$

In our case,

$$\sum_n \int_{-R}^{R} \left| f(x) \frac{(-i\xi)^n}{n!} x^n \right| \, dx \leqslant \sum_n \int_{-R}^{R} |f(x)| \frac{(|R\xi|)^n}{n!} \, dx = \sum_n \frac{(|R\xi|)^n}{n!} \|f\|_1 = e^{R|\xi|} \|f\|_1 < +\infty,$$

so

$$\widehat{f}(\xi) = \sum_{n=0}^{\infty} \frac{(-i)^n \int_{-R}^{R} x^n f(x) \, dx}{n!} \xi^n \equiv \sum_n c_n \xi^n,\ \forall \xi \in \mathbb{R},$$

that is $\widehat{f}$ is a power series. □

18.5.1. We notice that $1_{[-1,1]}, x1_{[-1,1]}, x^2 1_{[-1,1]} \in L^1(\mathbb{R})$. By the well known rule $\partial_\xi \widehat{f} = \widehat{-i\sharp f}(\xi)$, we have

$$\begin{aligned}(\widehat{-i\sharp)^2 1_{[-1,1]}}(\xi) &= \partial_\xi \widehat{-i\sharp 1_{[-1,1]}}(\xi) = \partial_{\xi\xi} \widehat{1_{[-1,1]}}(\xi) = \partial_{\xi\xi} \left(2\frac{\sin \xi}{\xi} \right) \\ &= 2\partial_\xi \frac{\xi \cos \xi - \sin \xi}{\xi^2} = 2\frac{-\xi^3 \sin \xi - (\xi \cos \xi - \sin \xi)2\xi}{\xi^4}. \quad \square\end{aligned}$$

18.5.3. Clearly, $f_a \in L^1(\mathbb{R})$ for every $a > 0$, so $f_a * f_b$ is well defined and L^1 if $a, b > 0$. By the rule of the FT of a convolution product, we have

$$\widehat{f_a * f_b}(\xi) = \widehat{f_a}(\xi) \widehat{f_b}(\xi).$$

Let's compute $\widehat{f_a}$. By the definition

$$\widehat{f_a}(\xi) = \int_0^{+\infty} e^{-ax} e^{-i\xi x}\, dx = \int_0^{+\infty} e^{-(a+i\xi)x}\, dx = \left[\frac{e^{-(a+i\xi)x}}{-(a+i\xi)}\right]_{x=0}^{x=+\infty}.$$

Now, since for $a > 0$ we have

$$|e^{-(a+i\xi)x}| = e^{-ax} \overset{x\to+\infty}{\longrightarrow} 0,$$

we conclude that

$$\widehat{f_a}(\xi) = \frac{1}{a+i\xi}.$$

Therefore,

$$\widehat{f_a * f_b}(\xi) = \frac{1}{(a+i\xi)(b+i\xi)}. \quad \square$$

18.5.5. Let $f(x) = e^{-x^2/2}$. As noticed, $f'(x) = -xf(x)$, so

$$\widehat{f'}(\xi) = \widehat{-\sharp f}(\xi), \iff i\xi\widehat{f}(\xi) = -i\partial_\xi\widehat{f}(\xi), \iff \partial_\xi\widehat{f}(\xi) = -\xi\widehat{f}(\xi),$$

from which

$$\widehat{f}(\xi) = ke^{-\frac{\xi^2}{2}}.$$

To determine k we notice that

$$k = \widehat{f}(0) = \int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}. \quad \square$$

18.5.9. (Warning! this exercise contains a typo and it demands the une of the inversion formula) i) We verify that

$$f_\varepsilon(x) := \frac{1}{2\pi}\widehat{g_\varepsilon}(-x), \;\; g_\varepsilon(\xi) = \frac{1}{2\pi}e^{-\frac{1}{2}\varepsilon^2\xi^2}\widehat{f}(\xi),$$

is well defined. Since $f \in L^1$, $\widehat{f}$ is continuous and bounded, so if $g_\varepsilon \in \mathscr{C}(\mathbb{R}) \subset L(\mathbb{R})$ and, because of the bound $|\widehat{f}(\xi)| \leqslant \|f\|_1$, we have

$$\int_{\mathbb{R}} |g_\varepsilon(\xi)|\, d\xi \leqslant \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{1}{2}\varepsilon^2\xi^2}\|f\|_1\, d\xi < +\infty,$$

so $g_\varepsilon \in L^1$. Therefore, $f_\varepsilon = \widehat{g_\varepsilon}$ is well defined.

ii) Recall that

$$\widehat{e^{-\frac{\sharp^2}{2\sigma^2}}}(\xi) = \sqrt{2\pi\sigma^2}e^{-\frac{1}{2}\sigma^2\xi^2}, \implies e^{-\frac{1}{2}\varepsilon^2\xi^2} = \widehat{\frac{1}{\sqrt{2\pi\varepsilon^2}}e^{-\frac{\sharp^2}{2\varepsilon^2}}}(\xi) =: \widehat{\delta_\varepsilon}(\xi),$$

from which

$$g_\varepsilon(\xi) = \frac{1}{2\pi}e^{-\frac{1}{2}\varepsilon^2\xi^2}\widehat{f}(\xi) = \frac{1}{2\pi}\widehat{\frac{1}{\sqrt{2\pi\varepsilon^2}}e^{-\frac{\sharp^2}{2\varepsilon^2}}}(\xi)\widehat{f}(\xi) = \frac{1}{2\pi}\widehat{\delta_\varepsilon * f}(\xi).$$

Therefore, by the inversion formula

$$f_\varepsilon(x) = \frac{1}{2\pi}\widehat{\widehat{\delta_\varepsilon * f}}(-x) = (\delta_\varepsilon * f)(x).$$

Now, let $\delta_\varepsilon(x)$ is an approximate unit. It is a well known fact that $f_\varepsilon = f * \delta_\varepsilon \xrightarrow{L^1} f$ when $\varepsilon \to 0$, from which the conclusion follows. □