Ex. 1.4.4 (**+)

X uncountable set

$\mathcal{F} := \{A \subset X : A$ or A^c is count$\}$

$\mu : \mathcal{F} \to [0, +\infty]$

$$\mu(A) := \begin{cases} 0 & A \text{ count} \\ 1 & A^c \text{ count} \end{cases} \quad A \in \mathcal{F}$$

Check that μ is a well defd measure on $\mathcal{F}$.

Sol: μ is well-defd because, $\forall A \in \mathcal{F}$ only one of A, A^c can be count: indeed if both A, A^c were count

$$\Downarrow$$

$X = A \cup A^c$ would be count

but X is uncount in this pb!

Let' now check the charact. props. of μ :

1) $\mu(\phi) = 0$

2) $\underbrace{\mu\Big(\bigsqcup_n A_n\Big)}_{} \text{\textcircled{=}} \underbrace{\sum_n \mu(A_n)}_{}$

1) $\mu(\phi) = 0$ because ϕ has no elements

2)

$$\mu\Big(\underset{\in \mathcal{F}}{\bigsqcup A_n}\Big)$$

$(A_n) \subset \mathcal{F}$ $\qquad A_n \cap A_m = \phi \quad n \neq m$

if (a) $\bigsqcup A_n$ is count $\Rightarrow \mu(\bigsqcup A_n) = \boxed{0}$

or if (b) $(\bigsqcup A_n)^c$ is count $\Rightarrow \boxed{\mu(\quad) = 1}$

(a) $\bigsqcup A_n$ is count $\Leftrightarrow$ A_n count $\forall n$

$$\Downarrow$$

$$\mu(A_n) = 0$$

$$\Downarrow$$

$$\boxed{\sum \mu(A_n) = 0}$$

(b) $\left(\bigsqcup A_n \right)^c$ is count $\iff$ $\exists N$: A_N uncount, A_N^c cont

$$\sum \mu(A_n) \geq \mu(A_N) = 1$$

Assume, by contradiction, $\exists M \neq N$: A_M, A_M^c count

We know $\boxed{\phi = A_N \cap A_M}$

$$\Downarrow$$

$$X = A_N^c \cup A_M^c \Rightarrow X \text{ count}$$

(!)

$\Rightarrow$ $\forall n \neq N$: A_n is cont $\Rightarrow \mu(A_n) = 0$

$$\Rightarrow \sum_n \mu(A_n) = 1 = \mu\left(\bigsqcup A_n\right) \quad \blacksquare$$

1.4.9 (***) (Borel - Cantelli lemma)

$(X, \mathcal{F}, \mu)$ meas sp

$(E_n) \subset \mathcal{F}$: $\boxed{\sum \mu(E_n) < +\infty}$

$$S := \{ x \in X : x \in E_n \underset{\underline{n}}{\underline{\text{for infinitely many}}} \}$$

Q1. $S \in \mathcal{F}$

Q2. $\mu(S) = ?$

Sol:

Q1. $\bigcap_k E_{2k}$

$$x \in S \iff \exists\, n_1 < n_2 < n_3 < \dots : \quad x \in \bigcap_{j=1}^{\infty} E_{n_j}$$

$$\iff x \in \underbrace{\bigcup_{n_k \,:\, n_1 < n_2 < \dots}}_{\text{pb: it is not a count op.}} \bigcap_{j=1}^{\infty} E_{n_j}$$

(union)

We could also say that

$$x \in S \iff \forall n \in \mathbb{N} \quad \underbrace{\exists k \geq n : x \in E_k}.$$

$$\iff \forall n \in \mathbb{N} \quad x \in \bigcup_{k \geq n} E_k$$

$$\iff x \in \bigcap_{n=0}^{+\infty} \bigcup_{k \geq n} E_k.$$

$$S = \{x \in E : x \in E_n \text{ for infinitely many } n\}$$

$$= \boxed{\bigcap_{n=0}^{\infty} \underbrace{\overbrace{\bigcup_{k \geq n} E_k}^{F_n}}_{\in \mathcal{F}} .} = \bigcap_{n} F_n \in \mathcal{F}.$$

Q2. $\mu(S) = ?$

$$S = \bigcap_{n=0}^{\infty} \bigcup_{k \geq n} E_k = \bigcap_{n=0}^{\infty} F_n.$$

$$\mu(S) = \mu\left(\bigcap_{n=0}^{\infty} F_n\right)$$

$$F_n = \bigcup_{k \geq n} E_k \supset F_{n+1} = \bigcup_{k \geq n+1} F_k$$

$$E_n \cup \boxed{E_{n+1} \cup E_{n+2} \cup \dots} \qquad \boxed{E_{n+1} \cup E_{n+2} \cup \dots}$$

$F_n \searrow$. to apply the cont from above we need to check if

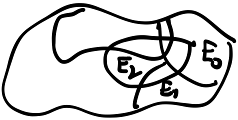
$$\mu(F_0) < +\infty$$

$$\mu(F_0) = \mu\left(\bigcup_k E_k\right)$$

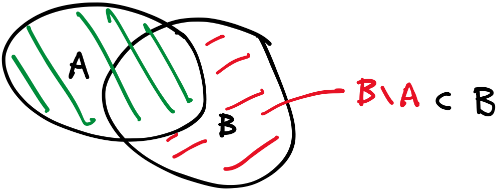
Prop: (countable sub_add)

$(X, \mathcal{F}, \mu)$ meas sp, $(E_n)_n \subset \mathcal{F}$

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$$



Proof: $\mu(A\cup B) = \mu(A) + \underset{\substack{\rotatebox{-90}{$\leq$} \\ \mu(B)}}{\mu(B\setminus A)}$



$$A\cup B = A \sqcup B\setminus A$$

$$\Rightarrow \quad \mu(A\cup B) \leq \mu(A) + \mu(B).$$

$$\Rightarrow \quad \mu\left(\bigcup_{k=0}^{n} E_k\right) \leq \sum_{k=0}^{n} \mu(E_k) \leq \sum_{k=0}^{\infty} \mu(E_k$$

$$\Rightarrow \quad \mu\Big(\underbrace{\bigcup_{k=0}^{n} E_k}_{F_n}\Big) \leq \underset{\text{indep of } n}{\boxed{\sum_{k=0}^{\infty} \mu(E_k)}}$$

$$\Downarrow$$

$$\lim_n \mu\left(\bigcup_{k=0}^{n} E_k\right) \leq \sum_{k=0}^{\infty} \mu(E_k)$$

$$\underset{\text{cont from below.}}{=} \quad \mu\left(\bigcup_{k=0}^{\infty} E_k\right) \leq \sum_k \mu(E_k)$$

∎

Ret to the Q2:

$$\mu(F_0) = \mu\left(\bigcup_{k=0}^{\infty} E_k\right) \underset{\substack{\text{count}\\ \text{sub-add}}}{\leq} \sum_k \mu(E_k) \underset{\text{Pb. Ass.}}{<} +\infty$$

$\Downarrow$ cont from above applies

$$\mu(S) = \mu\left(\bigcap_n F_n\right) = \boxed{\lim_n} \boxed{\mu(F_n)} = 0$$

$$\boxed{\mu(F_n)} = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \boxed{\sum_{k=n}^{+\infty} \mu(E_k)} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow \quad \mu(S) = 0$ ■

We def'd the outer meas on $\mathbb{R}^m$

$$\lambda^*(E) = \inf\left\{ \underbrace{\sum_k |I_k|}_{\geq 0} : E \subset \bigcup I_k,\ \underset{\text{int.}}{(I_k)} \right\}$$

$\lambda^*(E)$ is def'd $\forall E \subset \mathbb{R}^m$

$$\lambda^* : \underbrace{\mathcal{P}(\mathbb{R}^n)}_{\sigma\text{- alg}} \longrightarrow [0, +\infty]$$

Prop:

1) $\lambda^*(\phi) = 0$

($I = [a_1, a_1] \times [a_2, a_2] \times \dots \times [a_n, a_n]$

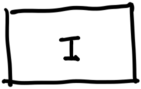
$= \{(a_1, \dots, a_n)\} \supset \phi$

$\sum |I| = |I| = \boxed{0} \in \{ \sum_k |I_k| : \phi \subset \bigcup I_k \}$

$\Downarrow$

$\lambda^*(\phi) = 0 = \inf$ //)

2) $\lambda^*(I) = |I| \quad \forall I$ interval



(I) (covering made by I alone) is a cov. of I

$\boxed{|I|}$ is in

$$\left\{ \underbrace{\sum |I_k|}_{} : I \subset \bigcup_k I_k \right\}$$

$$\boxed{\lambda^*(I) \leq |I|.} < +\infty$$

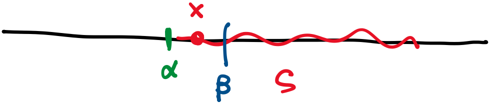
$$\overset{?}{\geq}$$

$$\Downarrow$$

$$0 \leq \lambda^*(I) < +\infty.$$

If $S \subset \mathbb{R}$

$\alpha = \inf S$



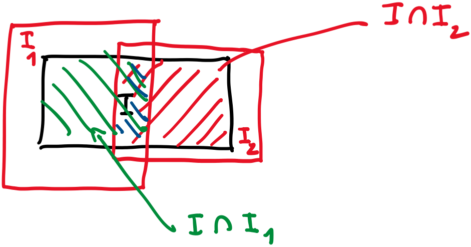
$\Rightarrow$ i) $\alpha \leq x \quad \forall x \in S$

$\rightarrow$ ii) $\forall \beta > \alpha : \quad \exists\, \underline{x \in S}$ s.t $\underline{x \leq \beta}$

So if $$\underbrace{\overset{\alpha}{\lambda^*(I)}}_{\inf\{} < \overset{\beta}{\boxed{|I|}} \leftarrow$$

$$\overset{?}{\Downarrow}$$

$$\exists \; (I_k) : \quad I \subset \bigcup I_k \,, \quad \boxed{\sum |I_k| < |I|}$$



$J_k = \boxed{I \cap I_k}$ is a rect

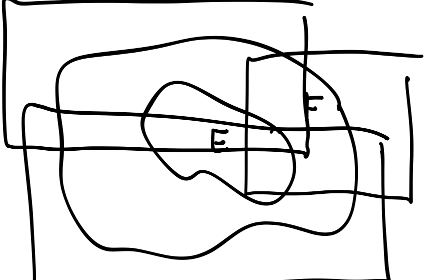
$\bigcup J_k = I$

$|I| \leq \sum |J_k| \leq \sum |I_k| < |I|$!

$\Rightarrow \lambda^*(I) = |I|$ ■

3. λ^* is monot : $E \subset F$

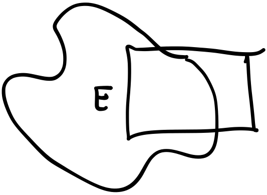
$\lambda^*(E) \leq \lambda^*(F)$.

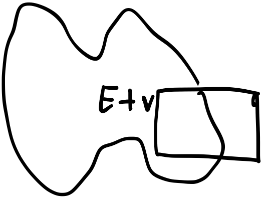


4). λ^* is invariant by translations

$$\lambda^*(E+v) = \lambda^*(E)$$

$$\forall E \subset \mathbb{R}^m, \ \forall v \in \mathbb{R}^m$$



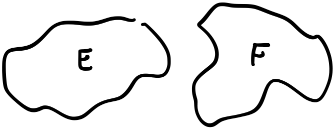


5) λ^* is count sub add

$$\lambda^*\left(\bigcup E_n\right) \leq \sum \lambda^*(E_n).$$

$$\lambda^*\left(\bigsqcup E_n\right) \underset{\text{can be}}{<} \sum \lambda^*(E_n)$$

6) Count add ?



In gen λ^* is not count add.

(Vitali's thm)