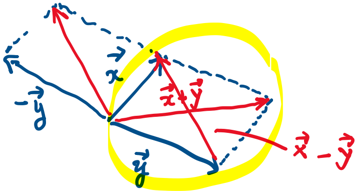
Friday, 3 October 2025 10:35

Ex 1.8.1

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\left(\|\vec{x}\|^2 + \|\vec{y}\|^2\right)$$



Sol: $\rightarrow \|\vec{x}+\vec{y}\|^2 = \underbrace{\|\vec{x}\|^2} + \underbrace{\|\vec{y}\|^2} + \underbrace{\sum x_j \cdot y_j}$

$(=: \vec{x} \bullet \vec{y}$

scalar prod of $\vec{x}$ and $\vec{y})$

$+$

$$\rightarrow \|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|-\vec{y}\|^2 + \sum x_j(-y_j)$$

$$= \underbrace{\|\vec{x}\|^2} + \underbrace{\|\vec{y}\|^2} - \underbrace{\sum x_j y_j}$$

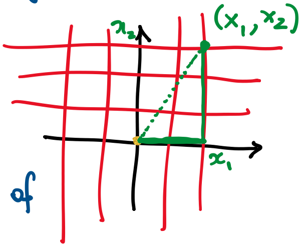
$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 \qquad \blacksquare$$

1.8.2

$$\|\vec{x}\|_1 := \sum_{j=1}^{d} |x_j| \qquad \vec{x} \in \mathbb{R}^d$$

(Manhattan norm)

$$\|\vec{x}\|_\infty := \max_{j=1\sim d} |x_j|$$



Check that they verify the same props of the Eucl. norm.

Let's do for $\|\cdot\|_1$ norm:

- positivity : $\|\vec{x}\|_1 \geqslant 0 \quad \forall \vec{x} \in \mathbb{R}^d$. Since $\|\vec{x}\|_1 = \sum |x_j| \geqslant 0$
- vanishing
- homog.
- triang ineq

Vanishing $\|\vec{x}\|_1 = 0 \overset{?}{\Longleftrightarrow} \vec{x} = \vec{0}$

$$\|\vec{x}\|_1 = 0 \Leftrightarrow \underbrace{\sum_{j=1}^{d} |x_j|}_{\geqslant 0} = 0$$

$$\Leftrightarrow |x_j| = 0 \quad \forall j$$

$$\Leftrightarrow x_j = 0 \quad \forall j$$

$$\Leftrightarrow \vec{x} = \vec{0}.$$

Homog: $\|\lambda \vec{x}\|_1 = |\lambda| \, \|\vec{x}\|_1 \quad \forall \lambda \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}^d$

$$\|\lambda\vec{x}\|_1 = \sum_j \underset{\overset{\shortparallel}{\boxed{|\lambda|}\,|x_j|}}{|\lambda x_j|} = |\lambda| \underbrace{\sum_j |x_j|}_{\|\vec{x}\|_1}$$

Triang ineq: $\boxed{\|\vec{x}+\vec{y}\|_1} \leq \|\vec{x}\|_1 + \|\vec{y}\|_1$ (goal)

$$\|\vec{x}+\vec{y}\|_1 \boxed{=} \sum_j \underbrace{|x_j+y_j|}_{}$$

$$|a+b| \leq |a| + |b|$$

$$|x_j+y_j| \leq |x_j| + |y_j|$$

$$\Downarrow$$

$$\sum |x_j+y_j| \leq \sum \left(|x_j| + |y_j|\right)$$

$$\shortparallel$$

$$\|\vec{x}+\vec{y}\|_1 \boxed{\leq} \underbrace{\sum |x_j|}_{\|\vec{x}\|_1} + \underbrace{\sum |y_j|}_{\|\vec{y}\|_1}$$

☑

We introduced the def of lim. of

a seq. of vectors:

$(\vec{x}_n) \subset \mathbb{R}^d$, $\exists \lim\limits_{n\to+\infty} \vec{x}_n = \vec{\ell} \in \mathbb{R}^d$

$$\Leftrightarrow \boxed{\forall \varepsilon>0 \ \exists N : \underbrace{\|\vec{x}_n - \vec{\ell}\|}_{\delta_n} \le \varepsilon \ \forall n \ge N.}$$

Rmk: $\delta_n := \|\vec{x}_n - \vec{\ell}\| \ge 0$

(δ_n) is a seq of positive numbers:

$$\lim_n \vec{x}_n = \vec{\ell} \Leftrightarrow \forall \varepsilon>0 \ \exists N : 0 \le \delta_n \le \varepsilon \ \forall n \ge N$$

$$\Downarrow$$

$$\boxed{\forall \varepsilon>0 \ \exists N : |\delta_n| \le \varepsilon \ \forall n \ge N}$$

$|\delta_n - 0| \le \varepsilon$

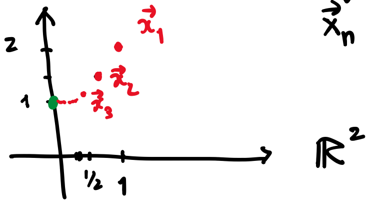
$$\Leftrightarrow \lim_n \delta_n = 0$$

$$\Leftrightarrow \lim_n \|\vec{x}_n - \vec{\ell}\| = 0$$

$$\lim_n \vec{x}_n = \vec{\ell} \in \mathbb{R}^d \Longleftrightarrow \lim_n \|\vec{x}_n - \vec{\ell}\| = 0$$

Ex (*)

Q. Show that $\lim_n \underbrace{\left(\frac{1}{n}, \frac{n+1}{n}\right)}_{\vec{x}_n} = \overset{\vec{\ell}}{(0,1)}$ $\quad n \geq 1$



$$\vec{x}_1 = (1, 2)$$

$$\vec{x}_2 = (1/2, 3/2)$$

$$\vec{x}_3 = \left(\frac{1}{3}, 4/3\right)$$

Sol: $\vec{x}_n - \vec{\ell} = \left(\frac{1}{n}, \frac{n+1}{n}\right) - (0,1)$

$$= \left(\frac{1}{n} - 0, \frac{n+1}{n} - \cancel{1}\right)$$

$$\cancel{1} + 1/n$$

$$= \left(\frac{1}{n}, \frac{1}{n}\right)$$

$$\|\vec{x}_n - \vec{\ell}\| = \left\|\left(\frac{1}{n}, \frac{1}{n}\right)\right\|$$

$$= \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \sqrt{\frac{2}{n^2}}$$

$$= \frac{\sqrt{2}}{n}$$

$$\|\vec{x}_n - \vec{\ell}\| = \frac{\sqrt{2}}{n}$$

$$\lim_n \|\vec{x}_n - \vec{\ell}\| = \lim_n \frac{\sqrt{2}}{n} = 0$$

$$\Rightarrow \quad \exists \lim_{n\to+\infty} \left(\frac{1}{n}, \frac{n+1}{n}\right) = (0,1).$$

$\boxtimes$

Rmk: $\vec{x}_n = (\underbrace{x_{n,1}, \dots, x_{n,d}}_{\text{components of } \vec{x}_n})$

$$\vec{\ell} = (\ell_1, \dots, \ell_d)$$

Can we say that

$\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d \iff$

$\left(\lim_n \vec{x}_n = \vec{\ell}\right)$

$x_{n,1} \longrightarrow \ell_1 \in \mathbb{R}$

$x_{n,2} \longrightarrow \ell_2 \in \mathbb{R}$

$\vdots$

$x_{n,d} \longrightarrow \ell_d \in \mathbb{R}$

Ex 1.8.3 (*)

i) $\vec{x}_n = (\boxed{e^{-n}}, 1) \in \mathbb{R}^2$

$\exists \lim \vec{x}_n = ?$

Here

$$x_{n,1} = e^{-n} \longrightarrow 0$$

$$= \frac{1}{\underbrace{e^n}_{\to +\infty}} \longrightarrow 0$$

$$x_{n,2} = 1 \longrightarrow 1$$

$$\Rightarrow \quad \vec{x}_n \longrightarrow (0, 1). \quad ■$$

iii) $\vec{x}_n = \left(\frac{1}{n}, \frac{1}{n^2}, \sin\frac{1}{n}\right) \in \mathbb{R}^3$

Here

$$x_{n,1} = \frac{1}{n} \longrightarrow 0$$

$$x_{n,2} = \frac{1}{n^2} \longrightarrow 0 \qquad \Rightarrow \vec{x}_n \longrightarrow \vec{0}$$

$$x_{n,3} = \sin\frac{1}{n} \to 0$$

■

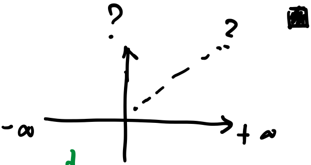
Ex : $\vec{x}_n = \left(\frac{1}{n}, (-1)^n\right)$

$x_{n,1} = \frac{1}{n} \longrightarrow 0$

$x_{n,2} = (-1)^n \not\longrightarrow$

$\Rightarrow$ $\vec{x}_n$ has no limit in $\mathbb{R}^2$.

$-\infty$
$+\infty$



Def: (infinite limit)

We say that $(\vec{x}_n) \subset \mathbb{R}^d$? has infinite limit

$$\left(\lim_{n \to \infty} \vec{x}_n = \infty_d \right)$$

if
$$\boxed{\lim_n \|\vec{x}_n\| = +\infty}$$

Ex : $\left(\frac{1}{n}, n\right) =: \vec{x}_n$ Since

$\frac{1}{n} \downarrow 0$, $n \downarrow +\infty$

$x_{n,1} = \frac{1}{n} \longrightarrow 0$ but

$x_{n,2} = n \longrightarrow +\infty$

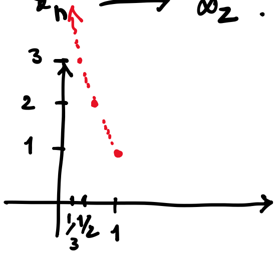
we can say that

$$\nexists \lim \vec{x}_n \in \mathbb{R}^2$$

However since

$$\|\vec{x}_n\| = \sqrt{\underbrace{\underset{\downarrow\, 0}{\frac{1}{n^2}} + \underset{\downarrow\, +\infty}{n^2}}_{\to +\infty}} \longrightarrow +\infty$$

$\Rightarrow \quad \vec{x}_n \longrightarrow \infty_2 .$



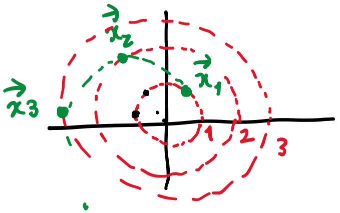
$\vec{x}_1 = (1,1)$

$\vec{x}_2 = \left(\frac{1}{2}, 2\right)$

$\vec{x}_3 = \left(\frac{1}{3}, 3\right)$

⊠

Ex: $\vec{x}_n = (n\cos n, n\sin n) = n(\cos n, \sin n)$



$\vec{x}_1 = (\cos 1, \sin 1)$

$\vec{x}_2 = 2(\cos 2, \sin 2)$

$\vec{x}_3 = 3(\cos 3, \sin 3)$

Now notice that

$$\|\vec{x}_n\| = \sqrt{(n\cos n)^2 + (n\sin n)^2}$$

$$= \sqrt{n^2 \underbrace{(\cos^2 n + \sin^2 n)}_{=1}}$$

$$= n \longrightarrow +\infty . \qquad \Rightarrow \vec{x}_n \longrightarrow \infty_2$$

Notice that

$$x_{n,1} = n\cos n \not\longrightarrow$$

$$x_{n,2} = n\sin n \not\longrightarrow$$

Warning: it is clear that if one of the components of $\vec{x}_n$ goes to $\pm\infty \Rightarrow \vec{x}_n \longrightarrow \infty_d$

$(\vec{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,d})$

$$\overbrace{x_{n,2}} \longrightarrow +\infty$$

$$\|\vec{x}_n\| = \sqrt{\sum_j x_{n,j}^2} \geq \sqrt{x_{n,2}^2} = |x_{n,2}|$$

$$\sum_j x_{n,j}^2 \geq x_{n,2}^2$$

$$|x_{n,2}| \longrightarrow +\infty$$

$$\Rightarrow \quad \|\vec{x}_n\| \longrightarrow +\infty$$

BUT this is not an IFF $\Longleftrightarrow$

The example $\vec{x}_n = (n\cos n, n\sin n)$ shows that $\vec{x}_n \longrightarrow \infty_2$ but none of the components have a lim!

Do 1.8.3/4/5

Limit of a function

In this course we consider several types of functs.

$$\underbrace{f(x,y)}_{\in \mathbb{R}^2} = \underbrace{xy^2}_{\in \mathbb{R}} \qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$f(\underbrace{1,-5}) = 1\cdot(-5)^2 = \boxed{25}$$

$$f\underbrace{(x,y,z)}_{\in\mathbb{R}^3} = \underbrace{x + y\sin z}_{\in\mathbb{R}} \qquad f:\mathbb{R}^3 \longrightarrow \mathbb{R}$$

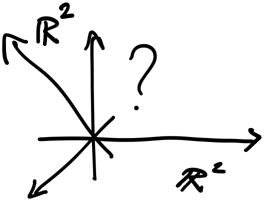
In general :

$$f(x_1,\dots,x_d) = f(\vec{x})$$

$$f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$$

f is a numerical funct of vect var.

$$\vec{F}\underbrace{(x,y)}_{\in\mathbb{R}^2} = \underbrace{(x+y,\ xy)}_{\in\mathbb{R}^2} \qquad \vec{F}:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$



$$\vec{F}\underset{\mathbb{R}^2}{(x,y)} = \underset{\mathbb{R}^3}{(x+y,\ xy,\ x/y)}$$

In general, we consider functs

$$\vec{F}(\underset{\mathbb{R}^d}{\underset{\in}{\vec{x}}}) \in \mathbb{R}^m$$

$$\boxed{\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m}$$

We want to def

$$\lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}) = \vec{\ell}$$

Let's refresh a poss. def of lim for $f = f(x)$ $x \in \mathbb{R}$:

Def: (Calc 1)

$f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, $p \in \mathrm{Acc}(D)$. We say

$$\lim_{x \to p} f(x) = \ell \in \mathbb{R}$$

if

$$\forall (x_n) \subset D,\ x_n \neq p,\ \underbrace{x_n \to p}_{} \Rightarrow f(x_n) \longrightarrow \ell.$$

Def (Calc 2)

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$,

$\vec{p} \in \mathrm{Acc}\, D$. We say

$$\exists \lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}) = \vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}$$

if

$$\forall (\vec{x}_n) \subset D,\ \vec{x}_n + \underline{\vec{p},\ \vec{x}_n \longrightarrow \vec{p}} \Rightarrow \vec{F}(\vec{x}_n) \longrightarrow \vec{\ell}$$