Wednesday, 1 October 2025 12:24

Euclidean Norm

Throughout this course we will work with functs of "several vars"

$$f(\underbrace{x_1, x_2, \dots, x_d})$$

$$\mathbb{R}^d := \{ \underbrace{\overbrace{(x_1, \dots, x_d)}^{\text{vector}}}_{\vec{x}} : x_1, \dots, x_d \in \mathbb{R} \}$$

On $\mathbb{R}^d$ we have two main operations:

sum: $\vec{x}, \vec{y} \in \mathbb{R}^d$

$$\vec{x} = (x_1, \dots, x_d)$$

$$\vec{y} = (y_1, \dots, y_d)$$

$$\vec{x} + \vec{y} := (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d)$$

product by scalars

$$\vec{x} \in \mathbb{R}^d, \quad \lambda \in \mathbb{R}$$

$$\lambda\vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_d)$$

Warning: there is not a prod between vectors.

Rmk: You can sum

$$(1,2,-5) + (0,-5,+3) = (1,-3,-2)$$

You can't sum vectors with different number of components:

$(1,2,-5) + (0,1)$ does not make sense

∎

Ex:

$$3(1,0,1) = (3,0,3)$$

$$-1(1,0,1) = (-1,0,-1)$$

$\mathbb{R}^d$, equipped with sum and prod by scalars is a vector space

This means that: the sum is

- (associative) $\vec{x}+(\vec{y}+\vec{z}) = (\vec{x}+\vec{y})+\vec{z}$
- (commutative) $\vec{x}+\vec{y} = \vec{y}+\vec{x}$
- (zero) $\exists\ \vec{0} = (0,\dots\, 0)$ s.t.

$$\vec{x}+\vec{0} = \vec{x} \qquad \forall \vec{x}\in\mathbb{R}^d$$

- (opposite) $\forall \vec{x}\in\mathbb{R}^d \quad \exists\, \vec{y}\in\mathbb{R}^d \quad \vec{x}+\vec{y}=\vec{0}$

(if $\vec{x} = (x_1 — x_d)$, $\vec{y} = (-x_1,\dots,-x_d) = -\vec{x}$)

the product fulfils

- (ass.) $\lambda(\mu\vec{x}) = \underbrace{(\lambda\mu)}_{}\vec{x}$
- (unit) $1\vec{x} = \vec{x}$
- (distrib. prop)

$$\lambda(\vec{x}+\vec{y}) = \lambda\vec{x}+\lambda\vec{y}$$

$$(\lambda+\mu)\vec{x} = \lambda\vec{x}+\mu\vec{x}$$

The goal is to def the op. of limit.

Let's refresh the def of limit for

a seq of real numbers:

Def: (finite limit)

Let $(x_n) \subset \mathbb{R}$ $\quad (x_0, x_1, x_2, \dots)$

We say that $\exists \lim\limits_{n \to +\infty} x_n = \ell \in \mathbb{R}$

if

$$\forall \varepsilon > 0 \quad \exists N : \quad |x_n - \ell| \leq \varepsilon \quad \forall n \geq N$$

The goal is to extend this def

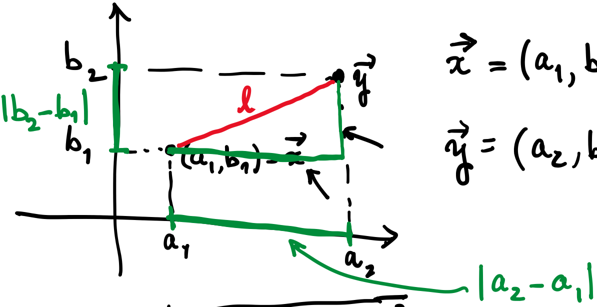
to
$$\lim_{n \to \infty} \vec{x}_n = \vec{\ell}$$

where $(\vec{x}_n) \subset \mathbb{R}^d$, $\vec{\ell} \in \mathbb{R}^d$.

The key point is to understand what should be

$$\underbrace{|\vec{x}_n - \vec{\ell}\,|}_{\text{distance between } \vec{x}_n \text{ and } \vec{\ell}}$$

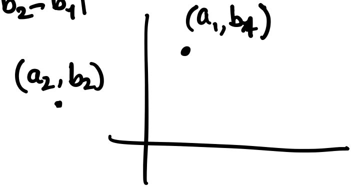
On $\mathbb{R}^2$ $(d=2)$ there's a nat way to do this: $\quad \mathbb{R}^2 = \{(a,b) : a, b \in \mathbb{R}\}$



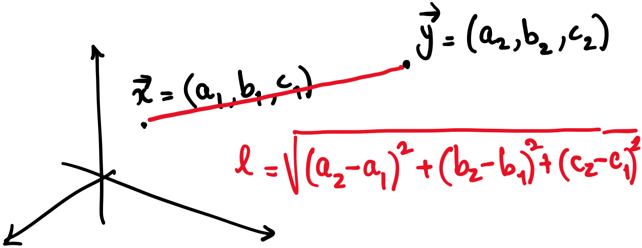
$\vec{x} = (a_1, b_1)$

$\vec{y} = (a_2, b_2)$

$$\ell = \sqrt{|a_2 - a_1|^2 + |b_2 - b_1|^2}$$



$$|\vec{x} - \vec{y}| = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$$



$$\ell = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2 + (c_2 - c_1)^2}$$

$\mathbb{R}^3$

In gen

$$\vec{x} = (x_1, \dots, x_d)$$

$$\vec{y} = (y_1, \dots, y_d)$$

$$\text{dist}(\vec{x}, \vec{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \dots + (y_d - x_d)^2}$$

Def: Given $\vec{x} \in \mathbb{R}^d$, we call **euclidean norm** of $\vec{x} = (x_1 — x_d)$ the quantity

$$\|\vec{x}\| := \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$$

$$\underset{\uparrow}{\|\vec{x}\|} := \sqrt{\sum_{j=1}^{d} \underset{\uparrow}{x_j^2}}$$

Warnings: $\|\vec{x}\|$ is a positive (≥ 0) number

Ex:

On $\mathbb{R}^2$ $$\|\underbrace{\overset{\text{numbers/scalars}}{(x, y)}}_{\text{vect}}\| = \sqrt{x^2 + y^2}$$

On $\mathbb{R}^3$ $\quad \|\underbrace{(x,y,z)}_{\text{vect}}\| = \sqrt{x^2+y^2+z^2}$.

$$\|(1,1)\| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\|(1,-1)\| = \sqrt{1^2+(-1)^2} = \sqrt{2}$$ ← correct

$$= \sqrt{\underbrace{1^2-1^2}_{\overset{\shortparallel}{0}}} = 0$$ ← wrong

$$\|(1,-2)\| = \sqrt{1^2-2^2} = \sqrt{1-4} = \sqrt{-3} \;!!!$$

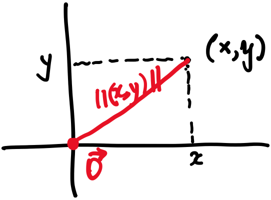
$$= i\sqrt{3}$$

$$\|(1,0,3)\| = \sqrt{1^2+0^2+3^2} = \sqrt{10}.$$

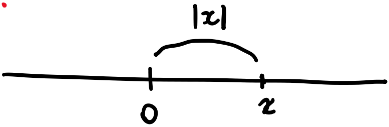
∎

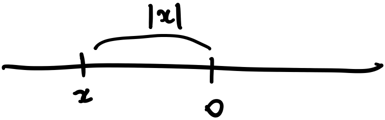
Rmk: What $\|\vec{x}\|$ represents?

$\mathbb{R}^2$: $\|(x,y)\| = \sqrt{x^2+y^2}$



$\|\vec{x}\|$ is the Euclidean dist between $\vec{x}$ and $\vec{0}$. ∎





Prop: The Eucl. norm verifies the following properties:

1. (positivity) $\|\vec{x}\| \geqslant 0 \quad \forall \vec{x} \in \mathbb{R}^d$
2. (vanishing) $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$
3. (homogeneity) $\underline{\|\lambda \vec{x}\|} = |\lambda|\,\|\vec{x}\| \quad \lambda \in \mathbb{R}, \ \vec{x} \in \mathbb{R}^d$
4. (triang ineq.) $\underbrace{\|\vec{x} + \vec{y}\|} \leqslant \|\vec{x}\| + \|\vec{y}\|$

Rmk: (on triang ineq)

We can have $\|\vec{x} + \vec{y}\| < \|\vec{x}\| + \|\vec{y}\|$

$\vec{x}, \quad \vec{y} = \underbrace{-\vec{x}}$

$$\|\vec{x}+\vec{y}\| = \|\vec{0}\| = 0$$

$$\neq$$

$$\|\vec{x}\| + \|\vec{y}\| = 2\|\vec{x}\|$$

$$\|-\vec{x}\| = |-1|\,\|\vec{x}\| = \|\vec{x}\|$$

$\blacksquare$

Proof:

1. true by def : $\|\vec{x}\| = \sqrt{\text{pos. numb}} \geq 0$

2. $\|\vec{x}\| = 0 \iff \sqrt{x_1^2+\dots+x_d^2} = 0$

$$\iff \underset{\overset{\vee\!|}{0}}{x_1^2} + \text{——} + \underset{\overset{\vee\!|}{0}}{x_d^2} = 0$$

$$\iff x_1^2 = x_2^2 = \dots = x_d^2 = 0$$

$$\iff x_1 = x_2 = \dots = x_d = 0$$

$$\iff \vec{x} = (0, 0, \dots, 0) = \vec{0}$$

3. $\|\lambda\vec{x}\| = \sqrt{(\lambda x_1)^2 + (\lambda x_2)^2 + \dots + (\lambda x_d)^2}$

$$\underbrace{\lambda^2}(x_1^2 + x_2^2 + \dots + x_d^2)$$

$\vec{x} = (x_1 \text{——} x_d)$

$\lambda\vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_d)$

$$= \underbrace{\sqrt{\lambda^2}}_{|\lambda|} \underbrace{\sqrt{x_1^2 + \text{——} + x_d^2}}_{\|\vec{x}\|}$$

$$= |\lambda| \, \|\vec{x}\|.$$

4. $\vec{x} = (x_1 \text{———} x_d)$

$\vec{y} = (y_1 \text{———} y_d) \Rightarrow \vec{x} + \vec{y} = (x_1 + y_1, \text{——}, x_d + y_d)$

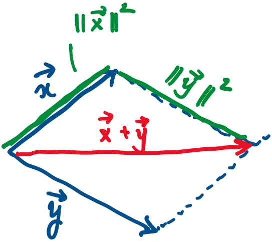
$$\|\vec{x} + \vec{y}\| = \sqrt{\sum_{j=1}^{d} (x_j + y_j)^2}$$

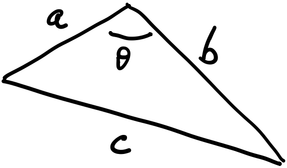
$$\Downarrow$$

$$\|\vec{x} + \vec{y}\|^2 = \sum_{j=1}^{d} \underset{\substack{\parallel \\ (x_j^2 + y_j^2 + 2x_j y_j)}}{(x_j + y_j)^2}$$

$$= \underbrace{\sum_{j=1}^{d} x_j^2}_{\|\vec{x}\|^2} + \underbrace{\sum_{j=1}^{d} y_j^2}_{\|\vec{y}\|^2} + 2 \sum_{j=1}^{d} x_j y_j$$

$$\|\vec{x}+\vec{y}\|^2 = \underbrace{\|\vec{x}\|^2} + \|\vec{y}\|^2 + 2\sum_{j=1} x_j y_j$$





$$c^2 = a^2 + b^2 + 2\underbrace{ab}\cos\theta$$

Cauchy - Schwarz Inequality:

$$\text{(CS)} \quad \left| \sum_{j=1}^{d} x_j y_j \right| \leq \|\vec{x}\|\,\|\vec{y}\| \quad \forall \vec{x},\vec{y} \in \mathbb{R}^d$$

(and = holds $\Leftrightarrow$ $\vec{x} = \alpha \vec{y}$ or $\vec{y} = \alpha \vec{x}$)

With the (CS) ineq we have:

$$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\sum x_j y_j$$

$$\underbrace{-\|\vec{x}\|\,\|\vec{y}\| \leq}_{\times} \boxed{\sum x_j y_j \leq \|\vec{x}\|\,\|\vec{y}\|}$$

$$\left(|x| \leq \alpha \iff -\alpha \leq x \leq \alpha \right)$$

$$\leq \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\|\vec{y}\|$$

$$= \left(\|\vec{x}\| + \|\vec{y}\|\right)^2$$

$$\Rightarrow \quad \|\vec{x}+\vec{y}\|^2 \leq \left(\|\vec{x}\| + \|\vec{y}\|\right)^2$$

$$\Rightarrow \quad \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|. \qquad \blacksquare$$

Proof of CS ineq:

$$\underbrace{\left|\sum x_j y_j\right|} \leq \boxed{\|\vec{x}\|\|\vec{y}\|}$$

$$\rotatebox{90}{$\leq$}$$

$$\leq \sum |x_j y_j| = \sum |x_j||y_j|$$

↑ triang ineq for abs val.

So we prove

$$(*) \quad \sum |x_j||y_j| \leq \underbrace{\boxed{\|\vec{x}\|\|\vec{y}\|}}_{\neq 0} \quad \forall \vec{x}, \vec{y}.$$

If $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0} \Rightarrow$ ineq true. $(0 \leq 0)$

So we can assume $\vec{x}, \vec{y} \neq \vec{0} \Rightarrow \|\vec{x}\|, \|\vec{y}\| \neq 0$

$$(*) \iff \sum \underbrace{\boxed{\frac{|x_j|}{\|\vec{x}\|}}}_{a}\underbrace{\boxed{\frac{|y_j|}{\|\vec{y}\|}}}_{b} \leq 1$$

Trick: $(a-b)^2 \geq 0$

$$\begin{aligned} &\| \\ &a^2+b^2-2ab \geq 0 \iff \cancel{2}ab \leq \frac{a^2+b^2}{2} \end{aligned}$$

$$\boxed{ab \leq \frac{a^2+b^2}{2}}$$

$$\overset{a}{\frac{|x_j|}{\|\vec{x}\|}} \cdot \overset{b}{\frac{|y_j|}{\|\vec{y}\|}} \leq \frac{1}{2}\left(\frac{x_j^2}{\|\vec{x}\|^2} + \frac{y_j^2}{\|\vec{y}\|^2}\right)$$

$$\sum \quad // \quad \leq \frac{1}{2}\sum_j\left(\frac{x_j^2}{\|\vec{x}\|^2} + \frac{y_j^2}{\|\vec{y}\|^2}\right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{\|\vec{x}\|^2}\cdot\underbrace{\sum x_j^2}_{\|\vec{x}\|^2}}_{=1} + \underbrace{\frac{1}{\|\vec{y}\|^2}\sum y_j^2}_{=1}\right)$$

$= 1$

$\blacksquare$

Def: (Limit for a seq of vects)

Let $(\vec{x}_n)_n \subset \mathbb{R}^d$. We say that

$$\exists \lim_{n \to +\infty} \vec{x}_n = \vec{\ell} \in \mathbb{R}^d$$

if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \quad \|\vec{x}_n - \vec{\ell}\| \leq \varepsilon \quad \forall n \geq N.$$