Wednesday, 1 October 2025 14:29

Example (counting meas.)

$X = \mathbb{N}$ (we can take any count. set)

$\mathcal{F} = \mathcal{P}(\mathbb{N})$

$\mu(E) = \#$ of elements of E

$= \sum_{n=0}^{\infty} \underset{\delta_n(E)}{\underset{\parallel}{1_n(E)}}$

$\Rightarrow \mu$ is a meas on X with r. to $\mathcal{F}$.

$(X, \overset{\text{family of measurable sets}}{\mathcal{F}}, \mu)$ is called measure space

In Prob Th $\mu(E)$ = meas of E

$(\Omega, \underset{\text{family of the possible events}}{\mathcal{F}}, \mathbb{P})$ is called prob. space if it is a meas. with the extra cond $\mathbb{P}(\Omega) = 1$.

$\mathbb{P}(E)$ = prob. of E

Some gen. facts about measures

Rmk: Let $(X, \mathcal{F}, \mu)$ be a meas sp.

count add. of μ $\Rightarrow$ finite add.

$$\mu\left(\bigsqcup_{n=0}^{N} E_n\right) = \sum_{n=0}^{N} \mu(E_n).$$

$$\parallel$$

$$\mu(E_0 \sqcup E_1 \sqcup E_2 \sqcup \dots \sqcup E_N \sqcup \emptyset \sqcup \dots \sqcup \emptyset \sqcup \dots)$$

$$\parallel \text{ count add}$$

$$\mu(E_0) + \mu(E_1) + \dots + \mu(E_N) + \underset{0}{\underset{\parallel}{\mu(\emptyset)}} + \dots$$

$$\parallel$$

$$\mu(E_0) + \mu(E_1) + \dots + \mu(E_N).$$

(for ex: $\mu(E \sqcup F) = \mu(E) + \mu(F)$)

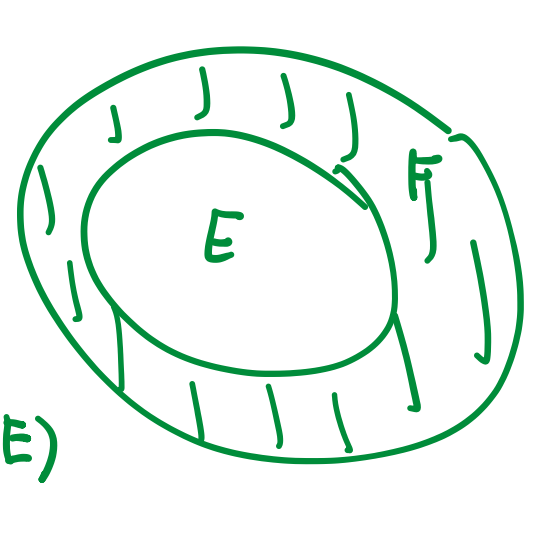
Prop: $(X, \mathcal{F}, \mu)$ meas sp.

i) (monotonicity) $E, F \in \mathcal{F}$, $E \subset F$

$\Rightarrow \mu(E) \leq \mu(F)$

ii) (subtractivity)

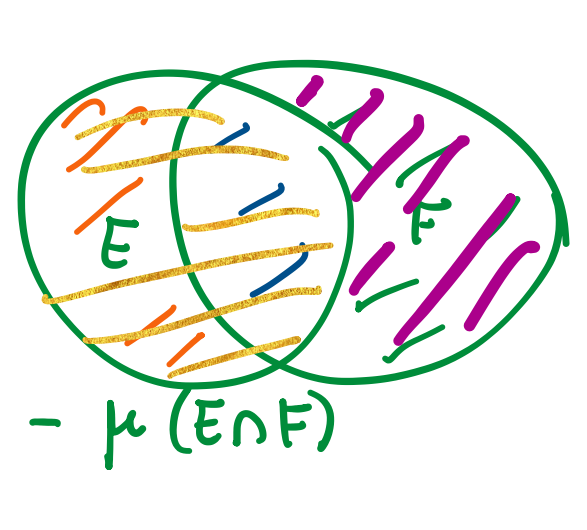
$E, F \in \mathcal{F}$, $E \subset F$



$\mu(F \setminus E) = \mu(F) - \mu(E)$

provided $\boxed{\mu(E) < +\infty}$

iii) $E, F \in \mathcal{F}$

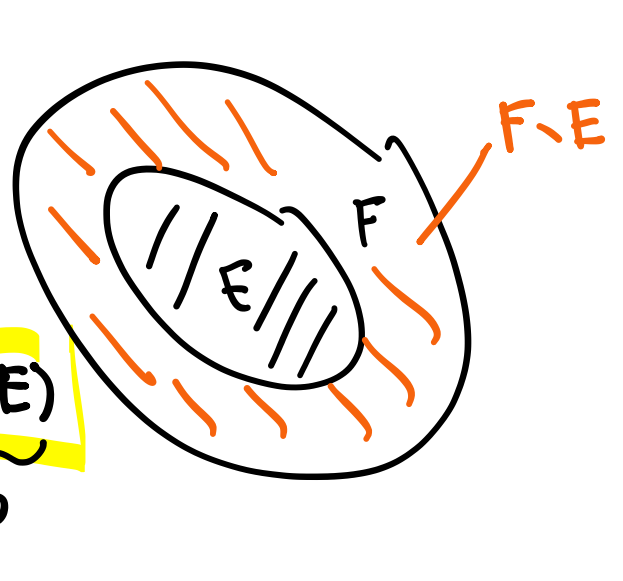


$\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$

provided $\mu(E \cap F) < +\infty$.

Proof:

i) $F = E \sqcup \underbrace{F \setminus E}_{F \cap E^c = (F^c \cup E)^c \in \mathcal{F}}$, $E \overset{\wedge}{\mathcal{F}}$



$\Downarrow$

$\mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0}$

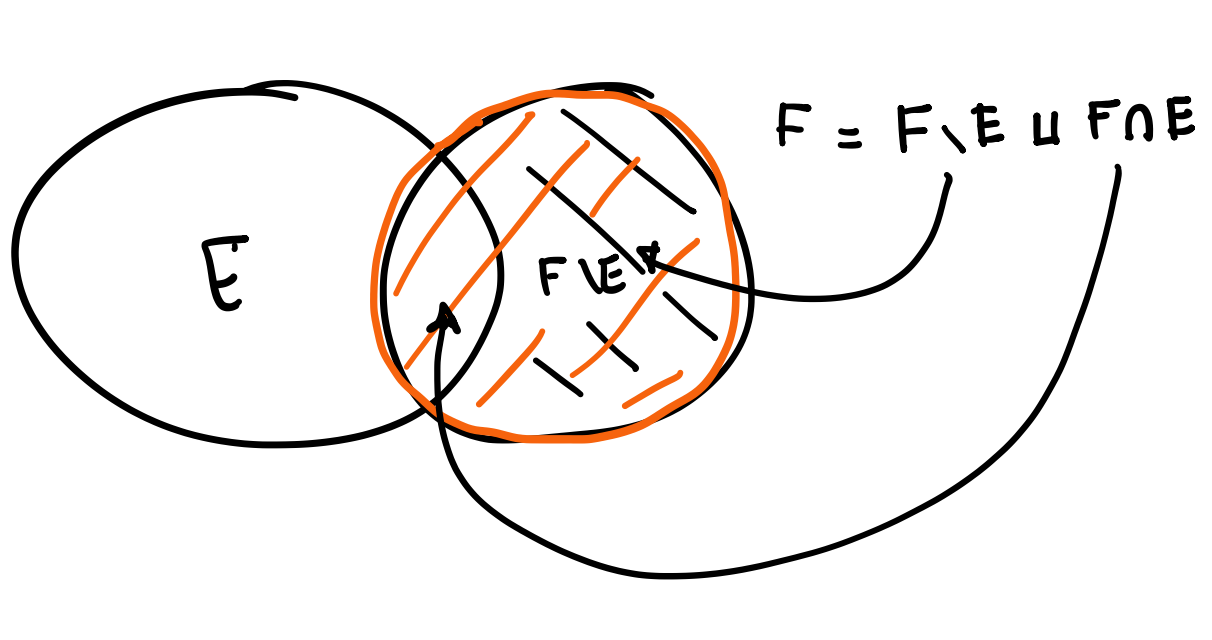
$\geq \mu(E)$.

ii) If $\mu(E) < +\infty$ $\quad \mu(F) - \mu(E) = \mu(F \setminus E)$.

iii) $E \cup F = E \sqcup (F \setminus E)$

$\Downarrow$

$\rightarrow \mu(E \cup F) = \mu(E) + \boxed{\mu(F \setminus E)}$



$F = F \setminus E \sqcup F \cap E$

$\Rightarrow \mu(F) = \mu(F \setminus E) + \mu(F \cap E)$

If $\mu(E \cap F) < +\infty \Rightarrow$

$\mu(F) - \mu(F \cap E) = \mu(F \setminus E)$

$\Rightarrow \mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$ ∎

Question: what about

$\mu(E \cup F \cup G) \overset{?}{=}$

$E, F, G \in \mathcal{F}$

$\mu(E_1 \cup \dots \cup E_N) = ?$

Continuity properties

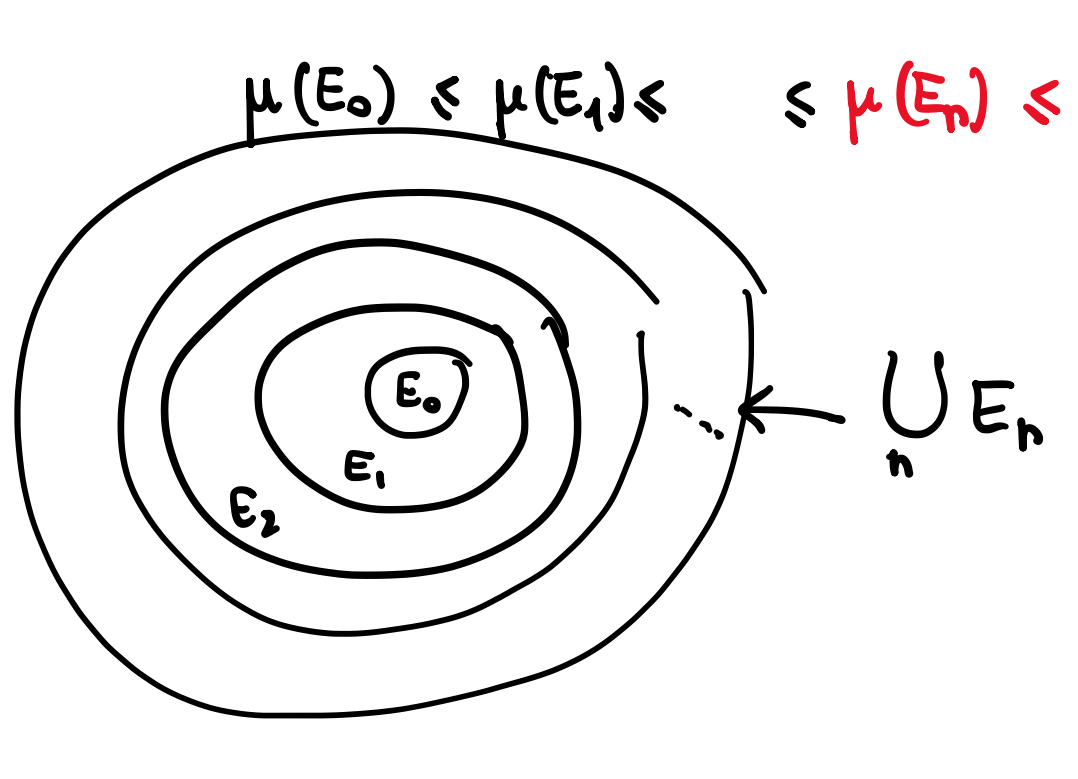
Let $(E_n)_n \subset \mathcal{F}$

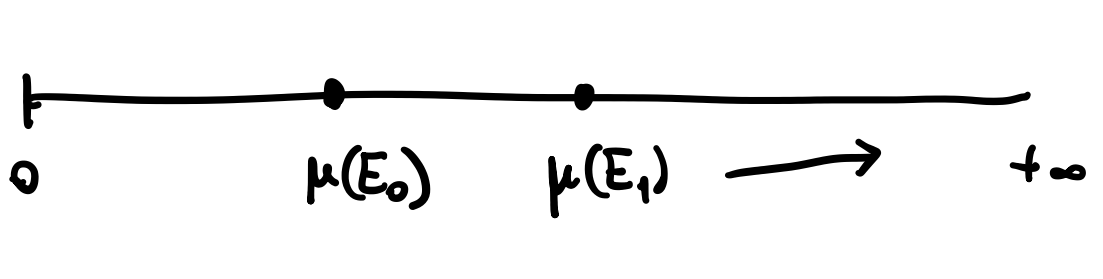
$(E_0, E_1, E_2, \dots\dots)$

s.t.

$E_0 \subset E_1 \subset E_2 \subset \dots \quad E_n \subset E_{n+1} \subset \dots$

$\mu(E_0) \leq \mu(E_1) \leq \quad \leq \mu(E_n) \leq \mu(E_{n+1}) \leq \dots$





Since $(\mu(E_n))_n \subset [0, +\infty]$ is an increasing sequence of real numbers

$\exists \lim_{n \to +\infty} \mu(E_n) \in [0, +\infty]$

Q: is this limit $\mu(E)$ for some "limit set" $E \in \mathcal{F}$?

Thm: $(X, \mathcal{F}, \mu)$ meas sp.

$(E_n)_n \subset \mathcal{F}$, $E_n \subset E_{n+1}$ $\forall n \in \mathbb{N}$

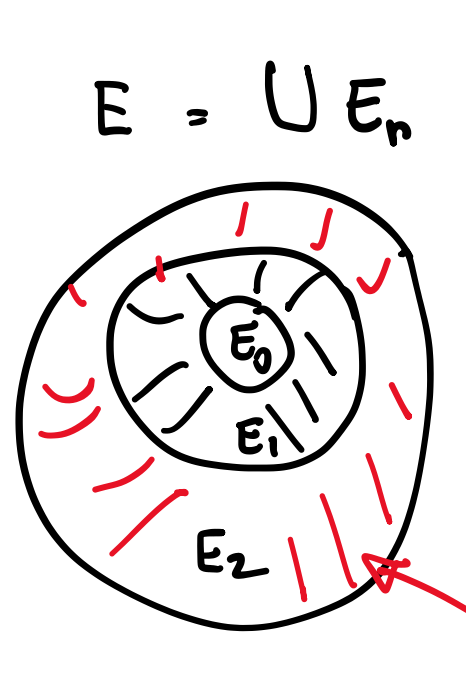
(notation: $E_n \nearrow$). Then, if

$E = \bigcup_n E_n \Rightarrow \mu(E) = \lim_{n \to +\infty} \mu(E_n)$

(not: $E = \overrightarrow{\lim} E_n$) (cont. from below)

Proof: Let's start computing $\mu(E)$

$E = \bigcup E_n$



$F_1 = E_1 \setminus E_0$

$\boxed{E_0 \cup E_1} = \underset{F_0}{E_0} \sqcup F_1$, with $\underset{E_1}{\parallel}$ under $E_0 \cup E_1$

$F_2 = E_2 \setminus E_1$

$E_0 \cup E_1 \cup E_2 = \underset{F_0}{\boxed{E_0}} \sqcup F_1 \sqcup F_2$

$\vdots$

$F_0 = E_0$

$F_n = E_n \setminus E_{n-1}$

$\Rightarrow \bigcup_n E_n = \bigsqcup_n F_n$

$\Rightarrow \mu(E) = \mu\left(\bigsqcup_n F_n\right) \underset{\text{count add}}{=} \sum_n \mu(F_n)$

$= \lim_{N \to +\infty} \boxed{\sum_{n=0}^{N} \mu(F_n)}$, with $\parallel$ $\mu\left(\bigsqcup_{n=0}^{N} F_n\right)$ $\parallel$ E_N underneath

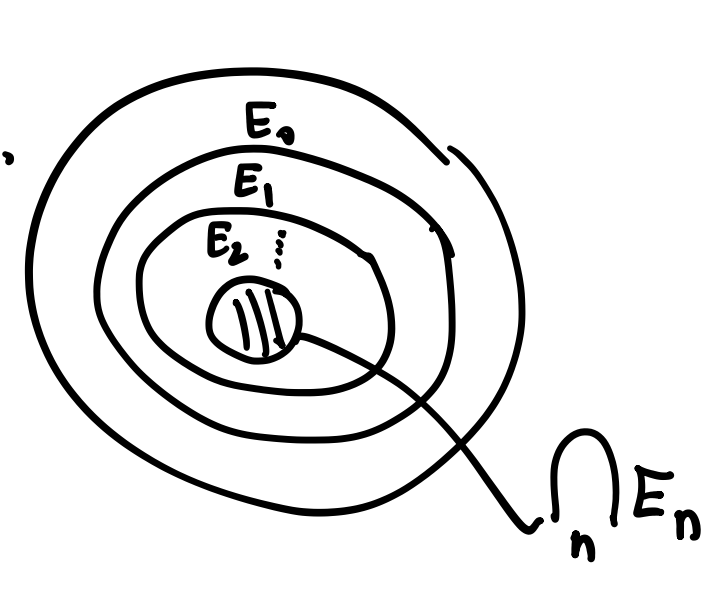
$= \lim_{N \to +\infty} \mu(E_N)$. ∎

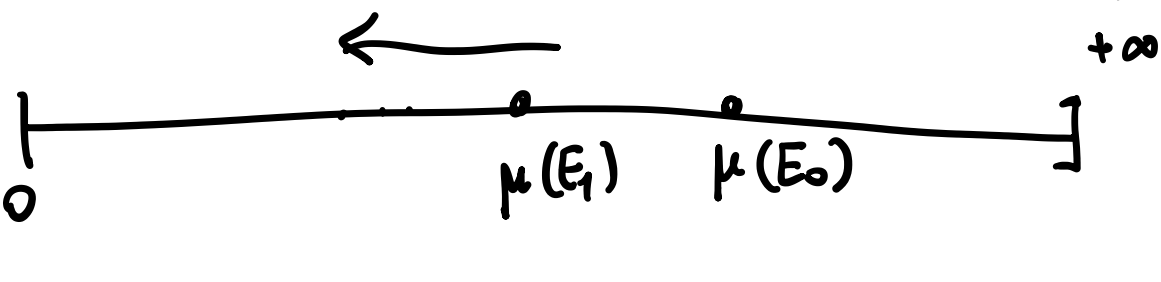
Q: what if $E_n \searrow$?

$E_0 \supset E_1 \supset E_2 \dots\dots \supset E_n \supset E_{n+1} \dots$

$\Downarrow$

$\mu(E_0) \geq \mu(E_1) \geq \mu(E_2) \geq \dots$





$\Rightarrow \exists \lim_{n \to +\infty} \mu(E_n) \in [0, +\infty]$

$\parallel ?$

$\mu\left(\bigcap E_n\right)$

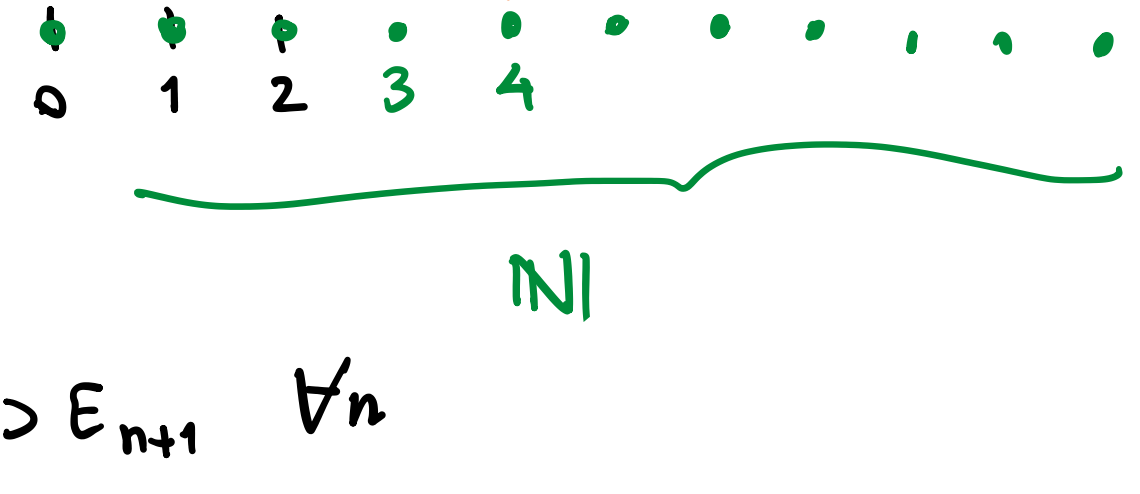
Warning: this is false in general.

$X = \mathbb{N}$

$\mathcal{F} = \mathcal{P}(\mathbb{N})$

μ = counting meas.

$E_n = \{n, n+1, n+2, \dots\dots\}$



$E_n \supset E_{n+1}$ $\forall n$

$\mu(E_n) = +\infty$ $\forall n \Rightarrow \lim_n \mu(E_n) = +\infty$.

$E = \bigcap E_n = \emptyset \quad \Rightarrow \mu(E) = 0$. ∎

However, if we add an assumption, cont. from above becomes true:

Corollary:

$(X, \mathcal{F}, \mu)$ meas sp

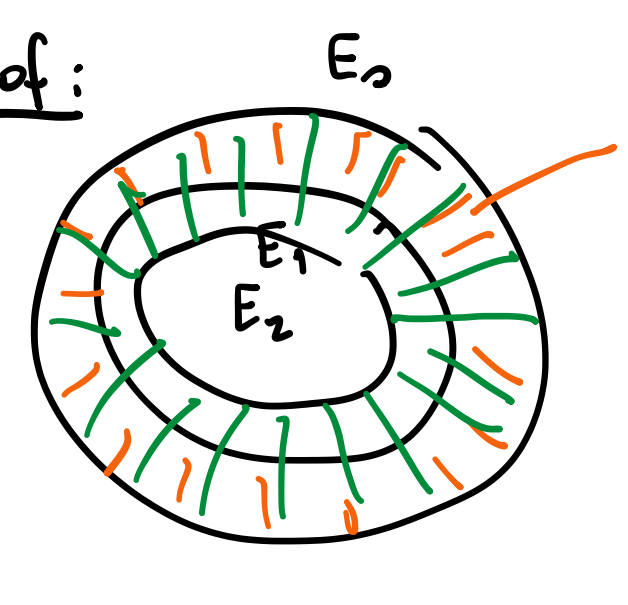
$(E_n)_n \subset \mathcal{F}$ s.t. $E_n \searrow$ $(E_n \supset E_{n+1}$ $\forall n)$

Suppose moreover $\boxed{\mu(E_0) < +\infty}$

$\Rightarrow$

$\underset{\underset{\lim E_n}{\leadsto}}{\mu\left(\bigcap E_n\right)} = \lim_n \mu(E_n)$.

Proof:



$E_0 \setminus E_1 = F_1$

$\cap$

$E_0 \setminus E_2 = F_2$

$\vdots$

$\cap$

$E_0 \setminus E_n = F_n$

$\Rightarrow (F_n)_n \subset \mathcal{F}$, $F_n \nearrow$

$\Rightarrow$ (cont from below)

$\lim_n \mu(F_n) = \mu\left(\bigcup F_n\right) \quad (*)$

$\parallel$

$\mu(E_0 \setminus E_n) \overset{\text{subtr.}}{\underset{\Uparrow}{=}} \mu(E_0) - \mu(E_n)$

(by ass $\mu(E_0) < +\infty$, $\vee$ $\mu(E_n)$)

$\lim_n (\mu(E_0) - \mu(E_n))$

$\parallel$

$\mu(E_0) - \lim \mu(E_n)$ (lhs of $(*)$)

On rhs

$\mu\left(\bigcup F_n\right) = \mu\left(\bigcup \underbrace{E_0 \setminus E_n}_{\cap E_n^c}\right)$

$= \mu\left(E_0 \cap \underbrace{\bigcup E_n^c}_{(\bigcap E_n)^c}\right)$, with $\parallel$ E under $\bigcap E_n$

$= \mu(E_0 \setminus E)$

$\overset{\text{subtr}}{=} \mu(E_0) - \mu(E)$

$\Rightarrow \mu(E) = \lim \mu(E_n)$ ∎

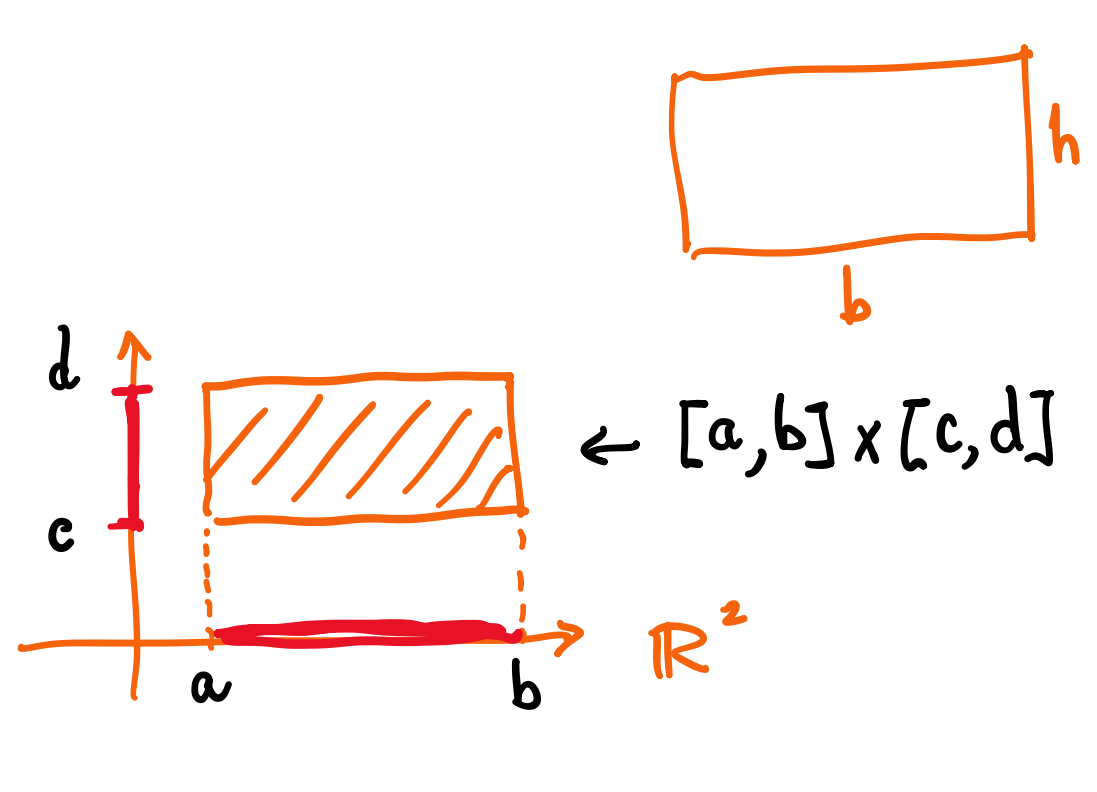
Do 1.4.7, 1.4.9 (***)

Lebesgue Measure

$X = \mathbb{R}^m$

Goal: define a measure on subsets of $\mathbb{R}^m$ s.t. the measure provides an ext of length $(m=1)$, area $(m=2)$, vol $(m=3)$.

- Meas rect = b · h



$\leftarrow [a,b] \times [c,d]$

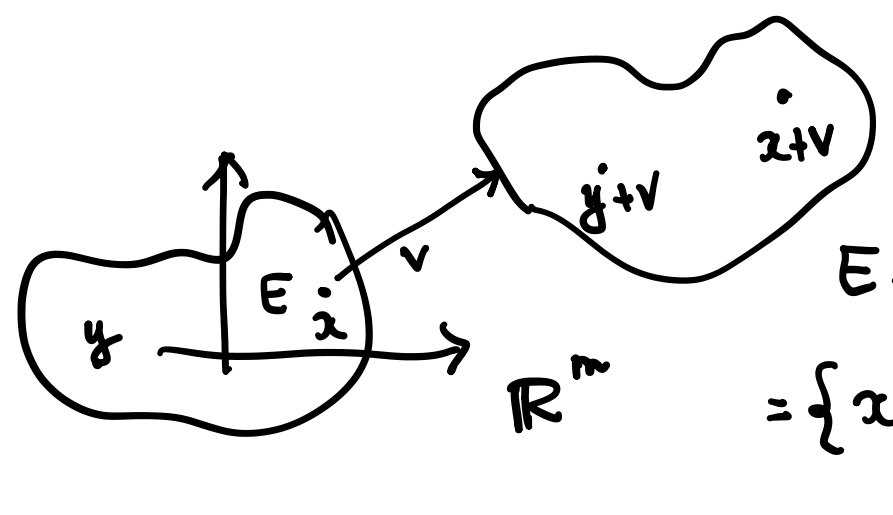
Def:

$I = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_m, b_m]$

is called interval.

Goal #1: Meas $I = \underbrace{(b_1 - a_1)(b_2 - a_2) \dots (b_m - a_m)}_{|I| \text{ (size/meas of } I)}$

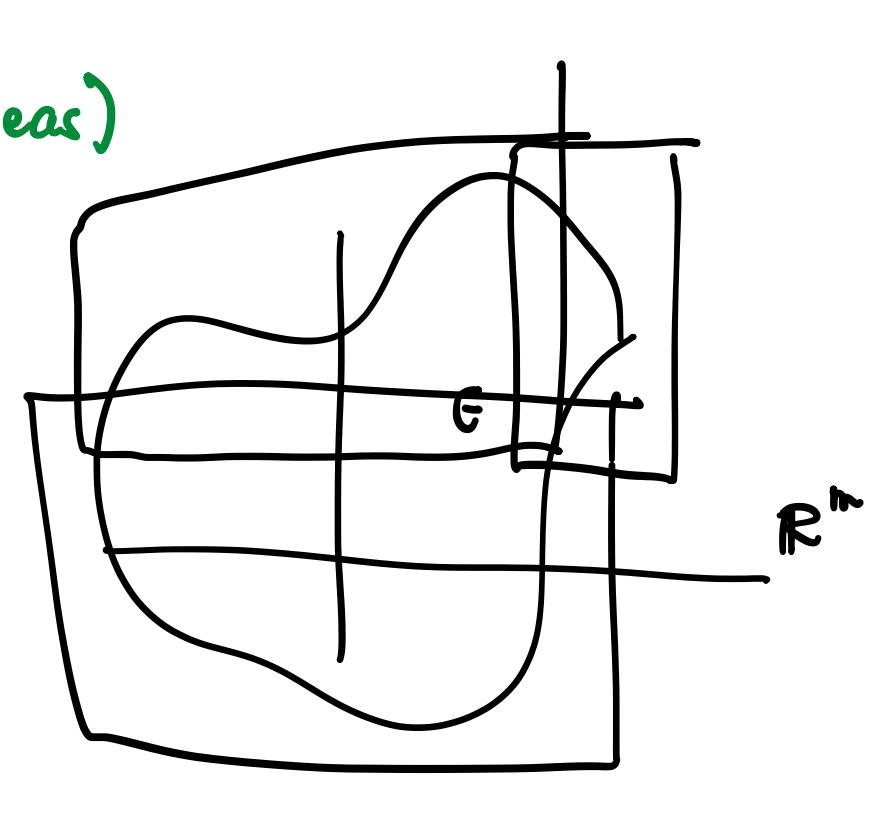
Goal #2: Meas $(E + v)$ = Meas(E).



$E + v$

$= \{x + v : x \in E\}$

Def (Outer meas)



Covering: $(I_n)_{n \in \mathbb{N}}$ of intervals s.t.

$E \subset \bigcup_n I_n$

$\lambda^*(E) := \underset{\text{best lower bound}}{\inf} \left\{ \sum |I_n| : (I_n)_n \text{ cover } E \right\}$

outer meas of E.