Numeri complessi

Rappresentazione algebrica dei numeri complessi:

$z \in \mathbb{C} \qquad z = a + ib \qquad \text{con } \boxed{a, b \in \mathbb{R}}$

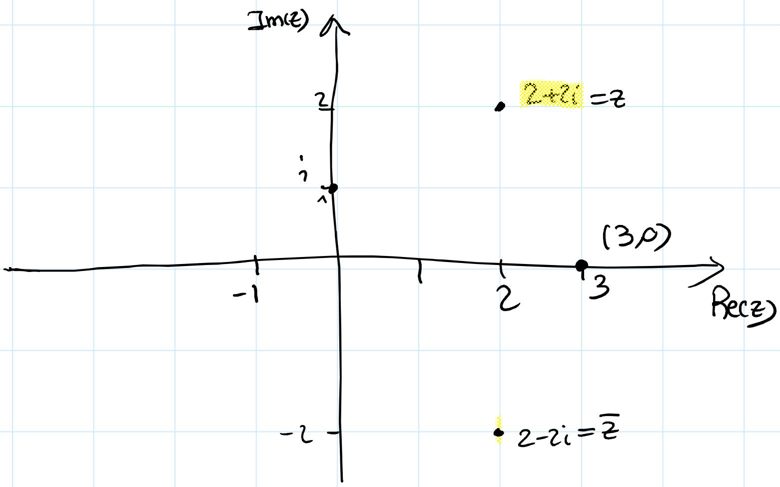
$$z \neq 0 \qquad \frac{1}{z} = \frac{1}{(a+ib)} \frac{a-ib}{(a-ib)} = \frac{a-ib}{a^2+b^2}$$

a parte reale $\quad Re(z) = a$

b parte immaginaria $\quad Im(z) = b$

3 è puramente reale $\quad a=3 \quad b=0 \quad 3 = 3 + i \cdot 0$

$3i$ è puramente immaginario $\quad a=0 \quad b=3 \quad 3i = 0 + i \cdot 3$



Argand-Gauss

$(3,0) = (a,b)$

$i = 0 + i \cdot 1 \quad (0,1)$

$2+2i \quad (2,2)$

Coniugio:

Definizione: dato $z = a+ib \in \mathbb{C}$ si chiama coniugato di z il numero complesso $\bar{z} = a - ib = a + i(-b)$

$z = (a,b)$

$\bar{z} = (a,-b)$

Osservazione: $z \in \mathbb{R} \overset{\text{se e solo se}}{\iff} z = \bar{z}$

$\cancel{a} + ib = \cancel{a} - ib$

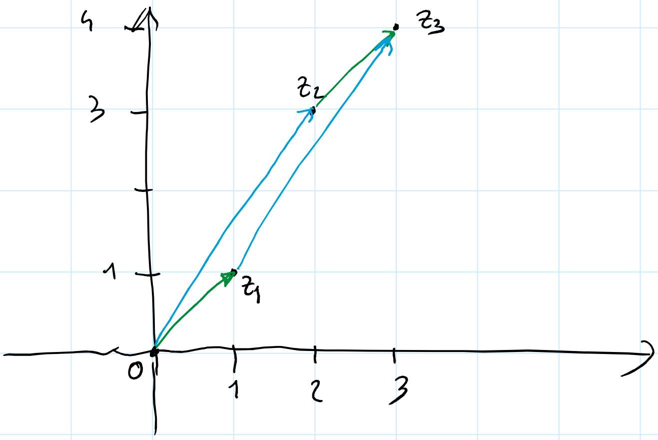
$2ib = 0 \iff b = 0$

$\mathbb{R} \subset \mathbb{C}$

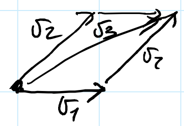
$a \qquad a + i \cdot 0$

Esempio: $z_1 = 1 + i \quad (1,1) \qquad z_3 = z_1 + z_2 = 1 + i + 2 + 3i = 3 + 4i \quad (3,4)$

$z_2 = 2 + 3i \quad (2,3)$



z_3 è la diagonale del parall. costruito con i lati z_1 e z_2



Proprietà del coniugio: $\forall$ per ogni

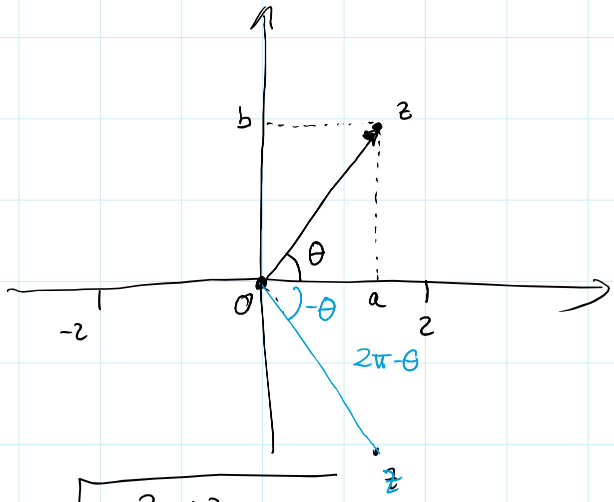
Ⓐ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad \forall\, z_1, z_2 \in \mathbb{C}$

Ⓑ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

Ⓒ $\bar{\bar{z}} = z \qquad \forall\, z \in \mathbb{C}$

Ⓓ $z = \bar{z} \iff z \in \mathbb{R}$

Rappresentazione trigonometrica dei numeri complessi



$z = (a,b)$ $\quad a+ib$

$|2| = |-2|$

$r = |z| := \sqrt{a^2+b^2}$ $\quad$ se $b=0$ $\quad \sqrt{a^2} = |a|$

$r = |z| \in \mathbb{R}$ $\quad |z| \geq 0$ $\quad \forall z \in \mathbb{C}$ $\quad (a,b) = (r\cos\theta, r\,\mathrm{sen}\theta)$

Def: la rappresentazione trigonometrica di $z \in \mathbb{C}$ $\quad z \neq 0$

$$\boxed{z = r\cos\theta + i\,r\,\mathrm{sen}\theta = r(\cos\theta + i\,\mathrm{sen}\theta)}$$

$r = |z|$ modulo del numero complesso

θ in radianti argomento del numero complesso

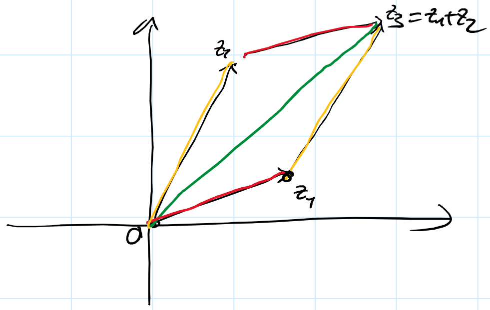
$\theta + k2\pi$ con $k \in \mathbb{Z}$

Modulo di un numero complesso

$|z| = \sqrt{a^2+b^2}$

1. $|z| = 0 \iff z = 0$
2. $|z_1 z_2| = |z_1|\,|z_2| \quad \forall\, z_1, z_2 \in \mathbb{C}$
3. $|z_1+z_2| \leq |z_1| + |z_2|$

$|z_1+z_2|$ $\quad |z_1|$ $\quad |z_2|$



Forma algebrica

$z = \sqrt{3} + i \qquad (\sqrt{3}, 1)$

$a = \sqrt{3} \qquad b = 1$

$z = \boxed{a} + i\boxed{b}$

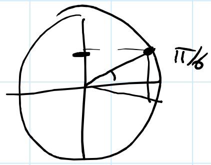
$\uparrow$

Forma trigonometrica

$r \neq 0$

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$

$\begin{cases} \sqrt{3} = 2\cos\theta \\ 1 = 2\operatorname{sen}\theta \end{cases} \qquad \begin{matrix} \cos\theta = \frac{\sqrt{3}}{2} \\ \operatorname{sen}\theta = \frac{1}{2} \end{matrix}$



$z = r(\cos\theta + i\operatorname{sen}\theta) = \boxed{r\cos\theta} + i\boxed{r\operatorname{sen}\theta}$

$\uparrow$

$r = \sqrt{a^2+b^2}$

$\begin{cases} a = r\cos\theta \\ b = r\operatorname{sen}\theta \end{cases} \qquad \begin{cases} \cos\theta = \frac{a}{r} \\ \operatorname{sen}\theta = \frac{b}{r} \end{cases}$

$\boxed{z = \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\operatorname{sen}\frac{\pi}{6}\right)}$

$\boxed{\theta = \frac{\pi}{6} + 2k\pi \quad k \in \mathbb{Z}}$

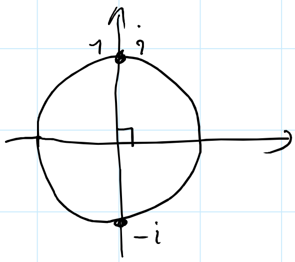
Esempio: $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\operatorname{sen}\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$

$\quad = \sqrt{2}\cdot\frac{\sqrt{2}}{2} + i\sqrt{2}\frac{\sqrt{2}}{2} = 1+i$

Esempi:

$i = \cos\frac{\pi}{2} + i\operatorname{sen}\frac{\pi}{2}$

$-i = 1\left(\cos\frac{3}{2}\pi + i\operatorname{sen}\frac{3}{2}\pi\right)$



Prodotto di numeri complessi in forma trigonometrica:

Proposizione:

Siano z_1 e z_2 due numeri complessi non nulli, allora il prodotto $z_1 z_2$ ha come modulo il prodotto $|z_1|\cdot|z_2|$ e come argomento la somma degli argomenti.

Dim:

Ipotesi: $z_1 = r_1 \ (\cos\theta_1 + i \,\mathrm{sen}\,\theta_1)$

$z_2 = r_2 \ (\cos\theta_2 + i\,\mathrm{sen}\,\theta_2)$

Tesi: $\boxed{z_1 z_2 = r_1 r_2 \ (\cos(\theta_1+\theta_2) + i\,\mathrm{sen}\,(\theta_1+\theta_2))}$

Dim: $z_1 z_2 = r_1 (\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)\, r_2 \,(\cos\theta_2 + i\,\mathrm{sen}\,\theta_2) =$

$= r_1 r_2 \left[(\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)\ (\cos\theta_2 + i\,\mathrm{sen}\,\theta_2)\right] =$ $\qquad i^2 = -1$

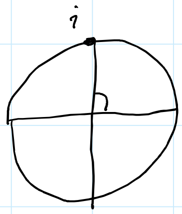
$= r_1 r_2 \left[\underline{\cos\theta_1 \cos\theta_2} + i\,\mathrm{sen}\,\theta_1 \cos\theta_2 + i \cos\theta_1\,\mathrm{sen}\,\theta_2 - \underline{\mathrm{sen}\,\theta_1\,\mathrm{sen}\,\theta_2} \right] =$

$= r_1 r_2 \left[(\cos\theta_1\cos\theta_2 - \mathrm{sen}\,\theta_1\,\mathrm{sen}\,\theta_2) + i(\mathrm{sen}\,\theta_1 \cos\theta_2 + \cos\theta_1\,\mathrm{sen}\,\theta_2)\right] =$

$= r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\,\mathrm{sen}\,(\theta_1+\theta_2)\right).$

Esempi:

$(i)^{10} = \left(1\left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\,\frac{\pi}{2}\right)\right)^{10}$ $\qquad i =$



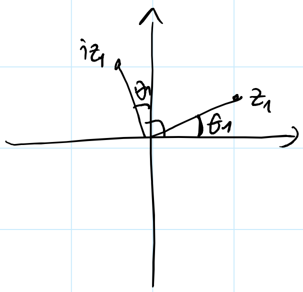
$i = 1\left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\,\frac{\pi}{2}\right)$

$= 1 \left(\cos 5\pi + i\,\mathrm{sen}\,5\pi\right) = -1$

$$\underbrace{\frac{\pi}{2} + \frac{\pi}{2} + \dots + \frac{\pi}{2}}_{10 \text{ volte}} = \cancel{10}^{5} \cdot \frac{\pi}{\cancel{2}} = 5\pi$$

Esercizio: $(\sqrt{3} + i)^5$ $\qquad$ Sugg. $z = \sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\,\mathrm{sen}\,\frac{\pi}{6}\right)$

$$z^5 = 2^5\left(\cos\frac{5}{6}\pi + i\,\mathrm{sen}\,\frac{5}{6}\pi\right)$$



$z_1 = r_1 \ (\cos\theta_1 + i\,\mathrm{sen}\,\theta_1)$

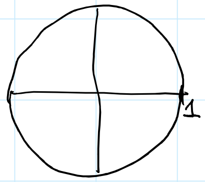
$z_2 = i = 1\left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\,\frac{\pi}{2}\right)$

$z_1 z_2 = z_1 \cdot i$

Corollario: se $z \in \mathbb{C}$ $z \neq 0$ $z = r(\cos\theta + i\,\text{sen}\,\theta)$ allora

$$\boxed{z^{-1} = r^{-1}(\cos(-\theta) + i\,\text{sen}(-\theta))}$$

$$\frac{1}{z} \cdot z = 1 =$$



$$= 1(\cos 0 + i\,\text{sen}\,0)$$

$$z^{-1} = \frac{a - ib}{a^2 + b^2}$$

Potenza: $z = r(\cos\theta + i\,\text{sen}\,\theta)$ $m \in \mathbb{N}$

$$z^m = r^m(\cos(m\theta) + i\,\text{sen}(m\theta))$$

Estrazione di radici: $\boxed{x^2 = 1}$ $x = 1$ oppure $x = -1$

$$x^m = 1$$

Vogliamo risolvere equazioni del tipo $x^m = z$ con $z \in \mathbb{C}$

una soluzione $x \in \mathbb{C}$ di questa equazione viene detta una radice m-esima di z.

Formule di De Moivre

$x^m = z$ $z \neq 0$ $z = r(\cos\theta + i\,\text{sen}\,\theta)$ $r > 0$

$x = s(\cos\alpha + i\,\text{sen}\,\alpha)$ $s > 0$

$$x^m = s^m(\cos(m\alpha) + i\,\text{sen}(m\alpha)) = r(\cos\theta + i\,\text{sen}\,\theta)$$

$s^m = r$ $\boxed{s = \sqrt[m]{r}}$

$\boxed{m\alpha = \theta + 2k\pi}$ con $k \in \mathbb{Z}$