Calculus II

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Foreword

These notes are based on a second year course of Calculus for Engineering. The main goal is to get students familiar with tools and methods of Calculus for functions of several real variables or, shortly, of vector variable. Topics include bases of Topology of $\mathbb{R}^d$ – mainly focusing on the operation of limit – Differential Calculus – focusing in particular on optimization problems – Vector Fields, Integral Calculus, Differential Equations – focusing on qualitative methods – and Holomorphic functions, that is differentiability for functions of complex variable.

As said, our goal is to help students to familiarize with tools and methods. Therefore, proofs are proposed only when their technical level is not excessive and they provide some insight into what the corresponding statements say. Sometimes, proofs are proposed with extra assumptions than what actually needed just to simplify them and to get quickly to the point. Other times proofs are just "sketch" of proofs, that is not formally rigorous proofs that could be made 100% true proof with some technical work (omitted here). Yet, the goal is to help to understand "why", rather than providing a complete view of the matter. I know this approach is controversial. In my experience, it works better for students who do not have a specific interest in the matter itself and that, nonetheless, need to learn tools to understand their curricular disciplines.

A good number of solved problems is proposed throughout the notes, as well as several exercises (without solution) at the end of each Chapter. The student is encouraged to try to solve problems right after the first few examples have been shown in class. A $*$ legend to distinguish between different levels for examples and exercises:

- ($*$) denotes the **basic level**, that is the minimal and easiest level, where the focus is mostly on the understanding of the definitions, being able to apply them on simple cases without particular technical skills required.
- ($**$) denotes the **intermediate level**, that is the level expected for the majority of the students. Here the focus is on applying the theory to solve complex problems that, however, require the application of standard procedures. A ($**$+) indicates the presence of technical difficulties.
- ($***$) denotes the **advanced level**, that is a level that denotes a deep comprehension of the main ideas behind the theory, including being able to organize an abstract argument (a proof) of a general property.

To help the student with conceptual maps, a number of "checklist" is proposed. They are useful to quickly remind "what to do" to respond to a certain question. The checklists are helpful to reach the intermediate level, but they cannot replace the critical approach which is always required.

A final note. The notes contain a large number of errors of any type (mathematics, english, typos, etc). Each student can participate to make these note better for the next students will come by pointing out these. Thanks, and good luck with your job! May the wind be always at your back!

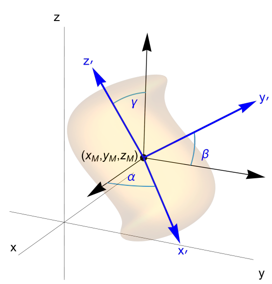
Contents

CHAPTER 1

Euclidean Space $\mathbb{R}^d$

The most important tools of Mathematical Analysis are the operations of *limit*, *derivative* and *integral*. In the first course of Calculus, we introduced these concepts for *numerical functions of numerical variable*, namely for $f = f(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$. The scope of this course is to extend, as much as possible, these operations to functions $f = f(x_1, \ldots, x_d)$ of an array real variables. This Chapter is essentially devoted to extend the operation of limit. Differential and Integral Calculus will be the focus of next Chapters.

There are many good reasons to extend Calculus in this direction. First of all, our physical world is a multidimensional world: to describe a position in space we need 3 coordinates, that is an array of three numbers. Any quantity depending on the position is a function of (at least) three variables. If we add time, then we have functions of four variables (example: temperature and pressure at any point of the atmosphere). The number of variables can be higher.



The configuration of a solid body in three dimensional space can be described using 6 coordinates: the 3 coordinates of the *centre of mass*, say (x_M, y_M, z_M) and the 3 angles (α, β, γ), thus by a unique array $(x_M, y_M, z_M, \alpha, \beta, \gamma)$ of 6 numbers. Other examples can be easily found in common real world situations as, for instance, any business activity depending on N different parameters. If arrays of numbers are a natural way to describe systems, functions that associate to an array a number or another array of numbers are also natural objects, and Calculus tools can be an effective way to tacke several problems. We will illustrate some of them in the following Chapters.

Chapter requirements: basic knowledge of vector spaces, limits for functions of one real variable, and continuity for functions of one real variable.

Learning objectives:

- (basic $*$) norm (what is? how it works?), computing simple limits for sequences of vectors and for functions of vector variable.
- (intermediate, $**$) computing limits, classifying topological properties of sets (open, closed, bounded, compact, connected).

- (advanced, $***$) solving problems and proving properties with non standard arguments.

1.1. Euclidean norm

It is convenient to look at arrays $(x_1, \dots, x_d)$ made of d real numbers, as elements of a set,

$$\mathbb{R}^d := \{(x_1, \dots, x_d) \ : \ x_i \in \mathbb{R}, \ i = 1, \dots, d\}.$$

Elements of $\mathbb{R}^d$ are also called *vectors* and, in this course, they will be denoted as

$$\vec{x} := (x_1, \dots, x_d).$$

There is no universal agreement on this notation. Some authors denote vectors just by normal letters as x, y, z, others use a boldface notation as $\mathbf{x}, \mathbf{y}, \mathbf{z}$. So, pay attention to the notations when you switch from a book to another!

On $\mathbb{R}^d$ two natural algebraic operations are defined: a *sum* and a *product by scalars*, defined as

$$(x_1, \dots, x_d) + (y_1, \dots, y_d) := (x_1 + y_1, \dots, x_d + y_d), \ \ \lambda(x_1, \dots, x_d) := (\lambda x_1, \dots, \lambda x_d).$$

For example

$$(1, 2, 3) + (-1, 2, -5) = (1 + (-1), 2 + 2, 3 + (-5)) = (0, 4, -2),$$

and

$$4(1, 2, 3) = (4, 8, 12), \quad -2(1, 2, 3) = (-2, -4, -6).$$

Notice that $(-1)\vec{x} = -\vec{x}$.

Warning 1.1.1

We cannot sum vectors with different number of components. For example

$$(1, 2, 3) + (-1, 2)$$

does not make any sense.

You may notice that here we are using symbols like $+$ with different meanings: $\vec{x}+\vec{y}$ is the *sum of vector $\vec{x}$ with vector $\vec{y}$*, while the $+$ in the components of vectors is the usual sum of real numbers. In principle, we should use different notations for the two sums, but this would make everything much heavier. Notations naturally suggest of which operation we are talking about. So, $\vec{x} + \vec{y}$ is the sum of vectors, $x + y$ is the sum of numbers.

The set $\mathbb{R}^d$ equipped with sum of vectors and product by scalars is what in Libean Algebra is called a *vector space*. This means that the sum of vectors is

i) *commutative*: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$, $\forall \vec{x}, \vec{y} \in \mathbb{R}^d$,
ii) *associative*: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$, $\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^d$,
iii) there is the *zero* vector $\vec{0} = (0, 0, \dots, 0)$ with the property that $\vec{x} + \vec{0} = \vec{x}$, $\forall \vec{x} \in \mathbb{R}^d$,
iv) every $\vec{x}$ has an *opposite element* $\vec{y}$ such that $\vec{x} + \vec{y} = \vec{0}$ (if $\vec{x} = (x_1, \dots, x_d)$ then $\vec{y} = (-x_1, \dots, -x_d)$). We denote the opposite of $\vec{x}$ by $-\vec{x}$.

In particular, the sum induces a *difference* operation defined as

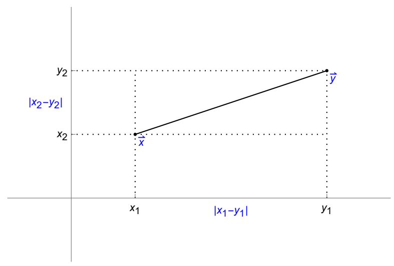
$$\vec{x} - \vec{y} := \vec{x} + (-\vec{y}) \equiv (x_1 - y_1, \dots, x_d - y_d).$$

If now $(\vec{x}_n) \subset \mathbb{R}^d$ and we want to say $\lim_n \vec{x}_n = \vec{\ell} \in \mathbb{R}^d$, we need something analogous to the modulus to measure the distance between $\vec{x}_n$ and $\vec{\ell}$. To grasp the idea, let us consider two vectors

$$\vec{x} = (x_1, x_2), \quad \vec{y} = (y_1, y_2),$$

in $\mathbb{R}^2$. Identifying $\vec{x}$ with a point in the cartesian plane with coordinates x_1 and x_2, we can identify the two vectors $\vec{x}$ and $\vec{y}$ with two points. The distance between these two points can be computed applying the Pythagorean theorem (see figure 1.1), yielding to the value

$$\operatorname{dist}(\vec{x},\vec{y}) = \sqrt{|x_1-y_1|^2+|x_2-y_2|^2} = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}.$$



This idea can be introduced in general on vectors of $\mathbb{R}^d$: given $\vec{x}=(x_1,\ldots,x_d)$ and $\vec{y}=(y_1,\ldots,y_d)$, then

$$\operatorname{dist}(\vec{x},\vec{y}) = \sqrt{\sum_{j=1}^{d}(x_j-y_j)^2}.$$

Remark 1.1.2

If $d=1$, then $\vec{x}=(x)$ and $\vec{y}=(y)$, and

$$\operatorname{dist}(\vec{x},\vec{y}) = \sqrt{(x-y)^2} = |x-y|,$$

boils down to the absolute value of the difference $x-y$, which is the way we comoute distances on the real line $\mathbb{R}$.

The formula we introduced for $\operatorname{dist}(\vec{x},\vec{y})$ depends on differences of coordinates x_j-y_j, that is on the components of $\vec{x}-\vec{y}$. So, it can be reduced to the following fundamental concept:

Definition 1.1.3: Euclidean norm

Given $\vec{x}=(x_1,\ldots,x_d)\in\mathbb{R}$, we call the **euclidean norm of $\vec{x}$** the quantity

$$\|\vec{x}\| := \sqrt{\sum_{j=1}^{d}x_j^2}.$$

Since $\sum_j x_j^2 \geqslant 0$, the norm of $\vec{x}$ is well defined for every $\vec{x}\in\mathbb{R}^d$. The norm plays the same role as the modulus of numbers in $\mathbb{R}$:

Proposition 1.1.4

Euclidean norm fulfils the following properties:

i) **positivity**: $\|\vec{x}\| \geqslant 0, \forall \vec{x} \in \mathbb{R}^d$;
ii) **vanishing**: $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0} := (0, \ldots, 0)$.
iii) **homogeneity**: $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|, \forall \lambda \in \mathbb{R}, \forall \vec{x} \in \mathbb{R}^d$.
iv) **triangular inequality**: $\|\vec{x}+\vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\|, \forall\, \vec{x}, \vec{y} \in \mathbb{R}^d$.

Proof. i) Positivity is evident: the square root of a positive number is always positive. ii) Let us check the vanishing: we have,

$$\|\vec{x}\| = 0, \iff \sum_{j=1}^d x_j^2 = 0, \iff x_j^2 = 0,\ \forall i, \iff x_j = 0,\ \forall j = 1, \ldots, d.$$

iii) Homogeneity:

$$\|\lambda\vec{x}\| = \sqrt{\sum_{j=1}^d (\lambda x_j)^2} = \sqrt{\lambda^2 \sum_{j=1}^d x_j^2} = |\lambda|\sqrt{\sum_{j=1}^d x_j^2} = |\lambda|\|\vec{x}\|.$$

iv) Triangular inequality: for convenience let us start by squaring everything and noticing that

$$\|\vec{x}+\vec{y}\|^2 = \sum_{j=1}^d (x_j+y_j)^2 = \sum_{j=1}^d (x_j^2 + y_j^2 + 2x_j y_j) = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\sum_{j=1}^d x_j y_j =: \|\vec{x}\|^2 + \|\vec{y}\|^2 + \vec{x}\cdot\vec{y},$$

where we denoted by $\vec{x}\cdot\vec{y}$ the *scalar product* of $\vec{x}$ and $\vec{y}$ (Rmk: $\vec{x}\cdot\vec{y}$ is a scalar, that is a number, not a vector!). At this point, we need the

Lemma 1.1.5: Cauchy–Schwarz inequality

$$|\vec{x}\cdot\vec{y}| \leqslant \|\vec{x}\|\|\vec{y}\|,\ \forall \vec{x}, \vec{y} \in \mathbb{R}^d. \tag{1.1.1}$$

Proof. (Lemma) The conclusion is immediate if $\|\vec{x}\| = 0$ or $\|\vec{y}\| = 0$. Indeed, if for example $\|\vec{x}\| = 0$, then (vanishing) $\vec{x} = \vec{0}$. Hence, both $\vec{x}\cdot\vec{y}$ and $\|\vec{x}\|\|\vec{y}\|$ are $= 0$. So, we can assume $\|\vec{x}\|, \|\vec{y}\| \neq 0$. Formula (1.1.1) follows once we prove

$$\sum_{j=1}^d \frac{|x_j|}{\|\vec{x}\|}\frac{|y_j|}{\|\vec{y}\|} \leqslant 1.$$

To prove this, we need an elementary (but important!) inequality:

$$ab \leqslant \frac{1}{2}(a^2+b^2)$$

This inequality follows easily once we notice that it is equivalent to $2ab \leqslant a^2 + b^2$, that is $(a-b)^2 \geqslant 0$. Now, setting $a = \frac{|x_j|}{\|\vec{x}\|}$ and $b = \frac{|y_j|}{\|\vec{y}\|}$, and summing on j, we have

$$\sum_{j=1}^d \frac{|x_j|}{\|\vec{x}\|}\frac{|y_j|}{\|\vec{y}\|} \leqslant \frac{1}{2}\sum_{j=1}^d \left(\frac{x_j^2}{\|\vec{x}\|^2} + \frac{y_j^2}{\|\vec{y}\|^2}\right) = \frac{1}{2}\left(\frac{\sum_{j=1}^d x_j^2}{\|\vec{x}\|^2} + \frac{\sum_{j=1}^d y_j^2}{\|\vec{y}\|^2}\right) = \frac{1}{2}\left(\frac{\|\vec{x}\|^2}{\|\vec{x}\|^2} + \frac{\|\vec{y}\|^2}{\|\vec{y}\|^2}\right) = 1. \qquad \square$$

We can now return to the proof of the triangular inequality. By Cauchy–Schwarz inequality then,

$$\|\vec{x}+\vec{y}\|^2 \leqslant \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\|\vec{x}\|\|\vec{y}\| = (\|\vec{x}\| + \|\vec{y}\|)^2, \iff \|\vec{x}+\vec{y}\| \leqslant \|\vec{x}\| + \|\vec{y}\|. \qquad \square$$

Euclidean norm is not the unique way to assign a length to vectors (see exercise 1.8.2).

1.2. Limit of a sequence

A sequence of $\mathbb{R}^d$ is a sequence of vectors $(\vec{x}_n) \subset \mathbb{R}^d$. Formally, as for numerical sequences, a sequence is a function $\vec{x} = \vec{x}(n) : \mathbb{N} \longrightarrow \mathbb{R}^d$. We will continue use index notation $\vec{x}_n$ instead of function notation $\vec{x}(n)$. We recall that a sequence $(x_n) \subset \mathbb{R}$ has $\lim_{n\to+\infty} x_n = \ell \in \mathbb{R}$ if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ :\ |x_n - \ell| \leqslant \varepsilon,\ \forall n \geqslant N.$$

This property says that the distance $|x_n - \ell|$ between x_n and the limit ℓ becomes arbitrarily small, provided we take the index n sufficiently large. Replacing numbers with vectors and modulus with norm we get the definition of limit for a sequence of vectors.

Definition 1.2.1: Finite limit

Let $(\vec{x}_n) \subset \mathbb{R}^d$. We say that

$$\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \iff \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \ :\ \|\vec{x}_n - \vec{\ell}\| \leqslant \varepsilon,\ \forall n \geqslant N.$$

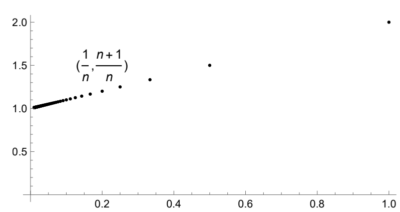
Equivalently,

$$\boxed{\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \iff \lim_n \|\vec{x}_n - \vec{\ell}\| = 0.}$$

(here, $\lim_n \|\vec{x}_n - \vec{\ell}\|$ is the usual numerical limit of the numerical sequence $\|\vec{x}_n - \vec{\ell}\|$)

Example 1.2.2: $(*)$

Q. *Show that* $\left(\frac{1}{n}, \frac{n+1}{n}\right) \longrightarrow (0, 1)$.



A. According to the definition, we have

$$\left\|\left(\frac{1}{n}, \frac{n+1}{n}\right) - (0,1)\right\| = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{n+1}{n} - 1\right)^2} = \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n} \longrightarrow 0.$$

We may look at a sequence of vectors $(\vec{x}_n)$ as a sequence of points moving in space $\mathbb{R}^d$. Each vector has d components, say

$$\vec{x}_n = (x_{n,1}, \ldots, x_{n,d}),$$

so a natural question is: what happens to the components $x_{n,j}$?

Proposition 1.2.3: component-wise convergence

Let $\vec{x}_n = (x_{n,1}, \dots, x_{n,d})$, $n \in \mathbb{N}$. Then

$$\vec{x}_n \longrightarrow \vec{\ell} = (\ell_1, \dots, \ell_d), \iff x_{n,j} \longrightarrow \ell_j, \ \forall j = 1, \dots, d.$$

The proof is left as exercise (see exercises at the end of the Chapter).

Example 1.2.4: $(*)$

Q. *Can* $\vec{x}_n := \left(\frac{1}{n}, \frac{1-n}{n}, \sin n\right)$ *have a finite limit in* $\mathbb{R}^3$*?*

A. The components of $\vec{x}_n$ are $\frac{1}{n} \longrightarrow 0$, $\frac{1-n}{n} = \frac{1}{n} - 1 \longrightarrow -1$ and $\sin n$, which has no limit. We conclude that $\vec{x}_n$ cannot have a finite limit in $\mathbb{R}^3$.

As we know, for numerical sequences infinite limits as $x_n \longrightarrow \pm\infty$ make sense. Now, $\pm\infty$ depends on the particular structure of the real line $\mathbb{R}$ where, informally, $-\infty$ is the left endpoint and $+\infty$ is the right one. In $\mathbb{R}^d$, with $d \geqslant 2$, left and right have no sense, so we need a new definition. The intuition is the following: we say that $\vec{x}_n \longrightarrow \infty_d$ (the infinite of $\mathbb{R}^d$) if the sequence goes at infinite distance from the origin. Formally:

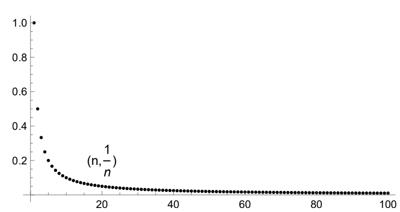
Definition 1.2.5: Infinite limit

Let $(\vec{x}_n) \subset \mathbb{R}^d$ be a sequence of vectors. We say that

$$\vec{x}_n \longrightarrow \infty_d, \iff \|\vec{x}_n\| \longrightarrow +\infty.$$

Example 1.2.6: $(*)$

Q. *Show that* $(n, \frac{1}{n}) \longrightarrow \infty_2$.



A. We have

$$\left\|\left(n, \frac{1}{n}\right)\right\| = \sqrt{n^2 + \frac{1}{n^2}} \longrightarrow +\infty. \quad \square$$

Warning 1.2.7

The last example shows that *differently from the case of finite limit,* $\vec{x}_n \longrightarrow \infty_d$ *does not imply that* $x_{n,j} \longrightarrow \infty$ *for all* $j = 1, \dots, d$.

1.3. Limit of a function

In this section, we extend the operation of limit to the case of *vector valued functions of vector variable*. With this we mean functions

$$\boxed{\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m.}$$

If $m = 1$ (but $d > 1$) we say that $\vec{F}$ is a *numerical function of vector variable* and we use the notation $f(\vec{x})$ for such type of functions.

Example 1.3.1

Some vector valued functions of vector variable with their domains:

- $f(x, y) := x + y$ is a numerical function of vector variable $f : D = \mathbb{R}^2 \longrightarrow \mathbb{R}$;
- $f(x, y) := \sqrt{1 - (x^2 + y^2)}$ is a numerical function of vector variable defined on $D := \{(x, y) \in \mathbb{R}^2 \ : \ 1 - (x^2 + y^2) \geqslant 0\} = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant 1\}$.
- $\vec{F}(x, y) := (x + y, xy)$ is a vector valued function of vector variable $\vec{F} : D = \mathbb{R}^2 \longrightarrow \mathbb{R}^2$;
- $\vec{F}(x, y) := (\sin(x + y), e^{xy}, \log(xy))$ is a vector valued function of vector variable $\vec{F} : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, where $D = \{(x, y) \in \mathbb{R}^2 \ : \ xy > 0\}$. Notice that $xy > 0$ if either $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$, so $D = \{(x, y) \in \mathbb{R}^2 \ : \ x > 0, y > 0 \ \vee x < 0, y < 0\}$ (here $\vee$ =or). □

Our goal is to define

$$\lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}) = \vec{\ell}.$$

This setup is quite general and difficult to visualize with our physical intuition, usually limited to very low dimensional spaces (as $\mathbb{R}^2$ or $\mathbb{R}^3$). However, the definiton is the same no matter what domain $\mathbb{R}^d$ and co-domain $\mathbb{R}^m$ are.

As for the one variable case, a limit makes sense only when the point $\vec{p}$ at which we compute the limit is an *accumulation point* for the domain D. The definition of accumulation point is similar to that in $\mathbb{R}$:

Definition 1.3.2: Accumulation point

Let $D \subset \mathbb{R}^d$. We say that $\vec{p} \in \mathbb{R}^d \cup \{\infty_d\}$ is **accumulation point for** D if

$$\exists (\vec{x}_n) \subset D, \ \vec{x}_n \neq \vec{p}, \ : \ \vec{x}_n \longrightarrow \vec{p}.$$

The set of all accumulation points of D is denoted by $\mathrm{Acc}(D)$.

We are now ready for the

Definition 1.3.3: Limit of a function

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ and $\vec{p} \in \mathrm{Acc}(D)$. We say that $\vec{F}$ **has limit** $\vec{\ell} \in \mathbb{R}^m \cup \{\infty_m\}$ **for** $\vec{x}$ **going to** $\vec{p}$, and we write

$$\lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}) = \vec{\ell}$$

if

$$\vec{F}(\vec{x}_n) \longrightarrow \vec{\ell}, \ \forall (\vec{x}_n) \subset D \backslash \{\vec{p}\}, \ \vec{x}_n \longrightarrow \vec{p}. \tag{1.3.1}$$

This definition has the advantage of covering all the possibilities: limits at a finite point (when $\vec{p} \in \mathbb{R}^d$), at infinite (when $\vec{p} = \infty_d$) and finite limits (when $\vec{\ell} \in \mathbb{R}^m$) or infinite limits ($\vec{\ell} = \infty_m$).

The definition of limit of a function is pivotal in many applications. The first is that it allows to the definition of continuous function:

Definition 1.3.4: Continuity

Let $\vec{F} = \vec{F}(\vec{x}) \, : \, D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{p} \in D \cap \operatorname{Acc}(D)$. We say that $\vec{F}$ **is continuous at** $\vec{p}$ if

$$\lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x}) = \vec{F}(\vec{p}).$$

If $\vec{F}$ is continuous in any point of D we say that $\vec{F}$ **is continuous on** D and we write $\vec{F} \in \mathsf{C}(D)$.

Thus:

- if we know $\vec{F}$ is continuous at point $\vec{p}$, then we can easily compute its limit for $\vec{x} \longrightarrow \vec{p}$: we just compute $\vec{F}(\vec{p})$.
- to check continuity at point $\vec{p}$ we have i) to compute $\lim_{\vec{x} \to \vec{p}} \vec{F}(\vec{x})$ and ii) check if it equals $\vec{F}(\vec{p})$

Proposition 1.3.5: Component-wise continuity

Let $\vec{F} = (f_1, \ldots, f_m) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ (here, each $f_j : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$), and $\vec{p} \in D \cap \operatorname{Acc}(D)$. Then

$$\vec{F} = (f_1, \ldots, f_m) \text{ is continuous at } \vec{p} \iff f_j \text{ is continuous at } \vec{p}, \ \forall j = 1, \ldots, m.$$

Continuity is preserved by manipulating functions through algebraic or composition operations. The unique care is on the formal side. For example, we can sum/subtract functions if they are defined on the same domain but, also, if they take values in the same co-domain (this because we cannot sum vectors of different dimensions). We illustrate some of this properties in the next proposition:

Proposition 1.3.6

The following properties hold:

i) If $\vec{F}, \vec{G} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ are continuous at $\vec{p}$, then

$$\vec{F} \pm \vec{G} \text{ is continuous at } \vec{p}.$$

ii) If $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\vec{G} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ are continuous at $\vec{p}$, then

$$f\vec{G} \text{ is continuous at } \vec{p}.$$

iii) If $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ is continuous at $\vec{p}$ and $\vec{G} : \vec{F}(D) \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$ is continuous at $\vec{F}(\vec{p})$, then

$$\vec{G} \circ \vec{F} \text{ is continuous at } \vec{p}.$$

We omit the proofs (which are similar to the proofs of analogous properties for the continuity of functions of real variable). Let us show some applications of these properties;

Example 1.3.7: polynomials ($**$)

Q. *Show that any polynomial, that is a sum of monomials of type $cx_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}$, is a well defined and continuous functions on $\mathbb{R}^d$.*

A. Clearly each coordinate function $\vec{x} \longmapsto x_j \in \mathsf{C}(\mathbb{R}^d)$, whence $x_j^{k_j}$ as a product of continuous functions. Therefore, also $cx_1^{k_1}\cdots x_d^{k_d} \in \mathsf{C}(\mathbb{R}^d)$, again as a product of continuous functions. From this it follows that a polynomial, being the sum of continuous functions on $\mathbb{R}^d$ is therein continuous as well.

Example 1.3.8: ($**$)

Q. *Show that the Euclidean norm is continuous on $\mathbb{R}^d$.*

A. Remind that

$$\|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_d^2} = g(f(\vec{x})),$$

where $f(\vec{x}) := x_1^2 + \cdots + x_d^2$ is a polynomial, and $g(y) := \sqrt{y}$. Now, by the Example 1.3.7, $f \in \mathsf{C}(\mathbb{R}^d)$ and since $f \geqslant 0$, we have that $g(f(\vec{x}))$ is well defined for every $\vec{x} \in \mathbb{R}^d$. Since $g \in \mathsf{C}([0, +\infty[)$, by the property iii) of Proposition 1.3.6 we conclude that $\|\vec{x}\| = g \circ f(\vec{x}) \in \mathsf{C}(\mathbb{R}^d)$.

Example 1.3.9: ($*$)

Q. *Determine the domain of continuity for the function* $f(x, y) := \log(1 - x^2 - y^2)$.

A. The function is defined on

$$D = \left\{(x, y) \in \mathbb{R}^2 \;:\; 1 - x^2 - y^2 > 0\right\} = \left\{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 < 1\right\}.$$

On D, f is log of a polynomial $1 - x^2 - y^2$, thus it is continuous on its domain.

Example 1.3.10: Linear maps ($**$)

Q. *Let A be an $m \times d$ matrix and define*

$$\vec{F}(\vec{x}) := A\vec{x} \equiv \begin{bmatrix} a_{11} & \ldots & a_{1d} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{md} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \equiv \begin{pmatrix} a_{11}x_1 + \cdots + a_{1d}x_d \\ \vdots \\ a_{m1}x_1 + \cdots + a_{md}x_d \end{pmatrix},$$

($\vec{F}$ is called **linear map** *and we write $\vec{F} \in \mathsf{L}(\mathbb{R}^d, \mathbb{R}^m)$, the set of linear maps from $\mathbb{R}^d$ to $\mathbb{R}^m$). Show that $\vec{F} \in \mathsf{C}(\mathbb{R}^d)$.*

A. Since $\vec{F} = (f_1, \ldots, f_m)$, where $f_j(x_1, \ldots, x_d) = a_{j1}x_1 + \cdots + a_{jd}x_d \in \mathsf{C}(\mathbb{R}^d)$ (polynomial), $j = 1, \ldots, m$, we conclude that $\vec{F} \in \mathsf{C}(\mathbb{R}^d)$.

1.4. Computing limits

Compared to the limits of a single variable, the calculation of limits for vector-valued functions of a vector variable can be extremely complicated. Because of the component-wise property of limits of prop. 1.3, we can focus to the case of *numerical functions of the vector variable*, that is,

$$\boxed{f = f(\vec{x}) \ : \ D \subset \mathbb{R}^d \longrightarrow \mathbb{R}.}$$

So, we consider the problem of computing a limit

$$\lim_{\vec{x} \to \vec{p}} f(\vec{x}).$$

In particular, we will consider the limit at $\vec{p} = \vec{0}$ and $\vec{p} = \infty_d$. Limits when $\vec{x} \to \vec{0}$ are important in many problems and applications (for instance, they are fundamental in the definition of derivative). Furthermore, by a standard change of variable, we can always transform any limit for $\vec{x} \to \vec{p} \in \mathbb{R}^d$ into a limit where the variable goes to $\vec{0}$:

$$\lim_{\vec{x} \to \vec{p}} f(\vec{x}) \overset{\vec{y} := \vec{x} - \vec{p}}{=} \lim_{\vec{y} \to \vec{0}} f(\vec{p} + \vec{y}).$$

This to say that it is not restrictive to consider just the two cases $\vec{p} = \vec{0}$ and $\vec{p} = \infty_d$. As we will see, these two cases are different but they can be handled with similar ideas.

1.4.1. Limits at $\vec{0}$. To fix ideas, let us start with the simplest but non trivial case of

$$\lim_{(x,y) \to (0,0)} f(x, y).$$

We assume that $(0, 0)$ be an accumulation point for the domain D of f. Since

$$(x, y) \longrightarrow (0, 0), \iff \|(x, y) - (0, 0)\| = \|(x, y)\| = \sqrt{x^2 + y^2} \longrightarrow 0,$$

and this can happen iff both $x, y \longrightarrow 0$, we may start wondering if

$$\lim_{(x,y) \to (0,0)} f(x, y) \overset{?}{=} \lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right) = \lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right).$$

If this were the case, we could reduce the calculation of the limit to two one variable limits. To understand whether this is true or not, let us focus on an example.

Example 1.4.1: $(*)$

Q. *Discuss existence and, if any, value of the limit,*

$$\lim_{(x,y) \to 0_2} \frac{xy}{x^2 + y^2}.$$

A. Just for the sake of doing everything with great care at least once, here we have

$$f(x, y) = \frac{xy}{x^2 + y^2}, \ (x, y) \in D = \mathbb{R}^2 \backslash \{(0, 0)\}.$$

Clearly $(0, 0) \in \mathrm{Acc}(D)$ so the problem of computing the limit of f at $(0, 0)$ makes sense. Now,

$$\lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right) = \lim_{y \to 0} \left(\lim_{x \to 0} \frac{xy}{x^2 + y^2} \right).$$

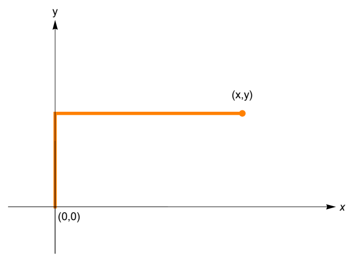
Incidentally, notice that since both $x, y \longrightarrow 0$, in particular, $x, y \neq 0$, so everything is well defined. Moreover,

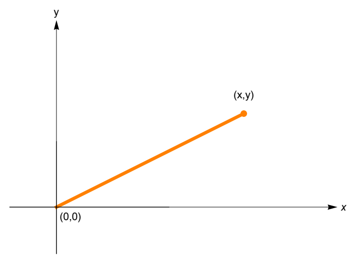
$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{xy}{x^2+y^2}\right) = \lim_{y\to 0}\frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} 0 = 0,$$

and the same happens for the reversed order limit:

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{xy}{x^2+y^2}\right) = 0.$$

(there is no check to be done, flipping x and y nothing change). So fine, can we say that *the limit exists and it equals* 0*?* Quite surprisingly, the answer is **no**! Let us see why!





To understand the issue, let us think about on the meaning of the *iterated limit*

$$\lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right).$$

The innermost limit consider a point (x, y) where y is constant (and different from 0) and $x \longrightarrow 0$. Watching this point in movement, we may imagine a point moving parallel to the x axis directed on the y axis. Then, moving $y \longrightarrow 0$, it is like if we move the point vertically along the $y-$axis to the origin $(0, 0)$. However, there are many other ways to go to $(0, 0)$. For example, we could consider moving the point (x, y) along a straight line through the origin with equation $y = mx$. In this case

$$(x, y) = (x, mx) \longrightarrow (0,0), \iff \|(x, mx)\| = |x|\sqrt{1+m^2} \longrightarrow 0, \iff x \longrightarrow 0.$$

Here we have that

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0}\frac{x\cdot mx}{x^2+(mx)^2} = \lim_{x\to 0}\frac{m}{1+m^2} = \frac{m}{1+m^2}.$$

Here we obtain something weird: depending on which is the way to go to $(0, 0)$, we may have different limits. This should suggest that, in this case, the proposed limit does not exist.

The way we proceeded in the example was an attempt to reduce the calculation of the limit to a one-variable limit. In fact, we may look at point (x, y) as a point in movement along a *curve*.

Definition 1.4.2: Curve

A **curve** is just a function

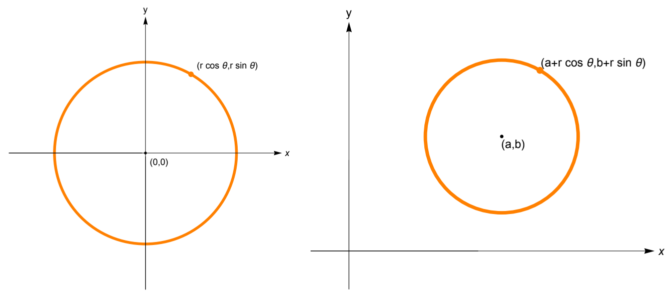
$$\boxed{\vec{\gamma} = \vec{\gamma}(u) \ : \ I \subset \mathbb{R} \longrightarrow \mathbb{R}^d \ (I \text{ interval})}$$

We say that $\gamma \subset D$ if $\gamma(u) \in D$ for every $u \in I$.

Example 1.4.3: circle

A plane circle of radius $r > 0$ centered at $(0,0)$ can be described by

$$\vec{\gamma} = \vec{\gamma}(\theta) := (r\cos\theta, r\sin\theta) : [0, 2\pi] \longrightarrow \mathbb{R}^2.$$



Similarly, if the centre is (a, b) we can use $\vec{\gamma}(\theta) := (a + r\cos\theta, b + r\sin\theta)$, $\theta \in [0, 2\pi]$.

Example 1.4.4: segment

In general, given $\vec{x}, \vec{y} \in \mathbb{R}^d$, the segment joining $\vec{x}$ to $\vec{y}$ can be described through

$$\vec{\gamma}(u) := \vec{x} + u(\vec{y} - \vec{x}) : [0, 1] \longrightarrow \mathbb{R}^d.$$

Given a function $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ and a curve $\vec{\gamma} \subset D$, we call **section of f along $\vec{\gamma}$** the function

$$f \circ \vec{\gamma} : I \subset \mathbb{R} \longrightarrow \mathbb{R}.$$

Intuitively, if $f(\vec{x})$ has a limit ℓ for $\vec{x} \to \vec{0}$, then along any curve $\vec{\gamma}$ that goes to $\vec{0}$ when $u \to u_0$, the corresponding section of f will have limit ℓ. This is the content of the next proposition:

Proposition 1.4.5: sectional property

$$\exists \lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \ell \in \mathbb{R} \cup \{\pm\infty\} \quad \Longrightarrow \quad \lim_{u\to u_0} f(\vec{\gamma}(u)) = \ell,\ \forall \vec{\gamma} \subset D\backslash\{\vec{0}\},\ :\ \lim_{u\to u_0} \vec{\gamma}(u) = \vec{0}.$$

This fact is useful to **disprove** that f has a limit. Indeed, if there are two sections $\vec{\gamma}_1, \vec{\gamma}_2 \subset D\backslash\{\vec{0}\}$, both going to $\vec{0}$ for $u \to u_0$, and moreover

$$\lim_{u\to u_0} f(\vec{\gamma}_1(u)) \neq \lim_{u\to u_0} f(\vec{\gamma}_2(u))$$

then $\lim_{\vec{x}\to 0} f(\vec{x})$ cannot exists. This is precisely what happened with the Example 1.4.1. Let us see some other interesting example.

Example 1.4.6: $(**)$

Q. *Show that*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

does not exist.

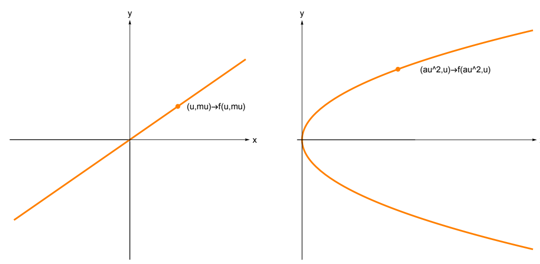
A. Let

$$f(x,y) = \frac{xy^2}{x^2+y^4}, \ (x,y) \in D = \mathbb{R}^2 \backslash \{(0,0)\}.$$

The sections along the axes are $f(u,0) \equiv 0 \longrightarrow 0$ and $f(0,u) \equiv 0 \longrightarrow 0$ when $u \longrightarrow 0$. Along the line $y = mx$,

$$f(u,mu) = \frac{m^2u^3}{u^2+m^2u^4} = \frac{m^2u}{1+m^2u^2} \longrightarrow 0, \ u \longrightarrow 0.$$

So we may think now that the limit exists and it equals 0.



However, when we section f along parabolas through the origin of equation $x = ay^2$, we get

$$f(au^2,u) = \frac{au^2u^2}{a^2u^4+u^4} = \frac{a}{a^2+1} \longrightarrow \frac{a}{a^2+1}, \ u \longrightarrow 0.$$

In particular, this is different from 0 if $a \neq 0$. According to the sectional property the limit cannot exist.

So far we discussed how to use sections to disprove the existence of a limit. Let us now see how to show existence and determine the value of the limit.

Example 1.4.7: $(**)$

Q. *Compute*

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}.$$

A. We start looking at f along standard sections. We have, $f(x,0) \equiv 0 \longrightarrow 0$ for $x \longrightarrow 0$, so if the limit exists it must be 0. This is confirmed by the $y-$axis section since $f(0,y) \equiv 0 \longrightarrow 0$ for $y \longrightarrow 0$. This says that **if** limit exists, **then** it must equals 0. But how to prove this? We know that few sections are not sufficient.

Looking at f, we realize that both numerator and denominator are polynomials going to 0 when $(x,y) \to (0,0)$. Thus, we have an indeterminate form $\frac{0}{0}$. We have to discuss *how fast* are numerator and denominator going to 0. How can we do this? Recall that

$$(x,y) \longrightarrow (0,0), \iff \|(x,y)\| = \sqrt{x^2+y^2} \longrightarrow 0.$$

By writing (x,y) in polar coordinates,

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \end{cases}$$

we can say that

$$\boxed{(x,y)\to(0,0) \iff \rho\longrightarrow 0.}$$

Now,

$$f(x,y) = f(\rho\cos\theta,\rho\sin\theta) = \frac{\rho^3(\cos\theta)(\sin\theta)^2}{\rho^2} = \rho(\cos\theta)(\sin\theta)^2,$$

and since sin, cos are bounded terms, we guess that $f(x,y)\longrightarrow 0$ when $\rho\to 0$. Precisely,

$$|f(\rho\cos\theta,\rho\sin\theta)| \leqslant |\rho(\cos\theta)(\sin\theta)^2| \leqslant \rho,$$

or, equivalently,

$$|f(x,y)| \leqslant \|(x,y)\|.$$

So, by the squeeze theorem, $f(x,y)\longrightarrow 0$ when $(x,y)\longrightarrow(0,0)$, and this completes the proof that $\lim_{(x,y)\to\vec{0}} f = 0$.

In the example, we found a bound like

$$|f(x,y)| \leqslant \|(x,y)\|,$$

which yields $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. More in general, if $\ell\in\mathbb{R}$ is such that

$$|f(x,y)-\ell| \leqslant \|(x,y)\|,$$

which yields $\lim_{(x,y)\to(0,0)} f(x,y) = \ell$. The important point is that at r.h.s. there is a function of $\|(x,y)\|$ that goes to 0 when $(x,y)\to(0,0)$. This extends to the following

Proposition 1.4.8

Let $f = f(\vec{x}) : D\subset\mathbb{R}^d \longrightarrow \mathbb{R}$ with $\vec{0}\in \mathrm{Acc}(D)$. Suppose moreover that:

i) $|f(\vec{x})-\ell| \leqslant g(\|\vec{x}\|)$,

ii) $g(\rho)\longrightarrow 0$ when $\rho\to 0+$.

Then

$$\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \ell.$$

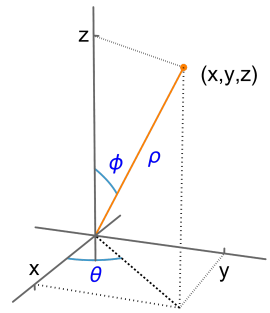
Proof. We apply the definition 1.3: if $(\vec{x}_n)\subset D\backslash\{\vec{0}\}$ is such that $\vec{x}_n\longrightarrow\vec{0}$, then $\|\vec{x}_n\|\longrightarrow 0$ and

$$|f(\vec{x}_n)-\ell| \overset{i)}{\leqslant} g(\|\vec{x}_n\|) \overset{ii)}{\longrightarrow} 0.$$

Therefore, by the squeeze theorem, $f(\vec{x}_n)-\ell\longrightarrow 0$, that is, $f(\vec{x}_n)\longrightarrow\ell$. □

We show an illustration of the previous result on a limit for a function of three variables. To this aim, we need the analogous of polar coordinates for $\mathbb{R}^3$. These are three numbers (ρ,θ,φ) where ρ is the distance of (x,y,z) to $\vec{0}$. The other two are two angles determined as follows:
Given the cartesian coordinates (x,y,z) we denote by ρ the distance to $\vec{0}$, that is

$$\rho = \|(x,y,z)\| = \sqrt{x^2+y^2+z^2}.$$



Then, we denote with $\phi \in [0, \pi]$ the angle made by the vector and the positive $z-$axis (the choice of $z-$axis is conventional, we could indifferently choose the x as well as the y axes). By ordinary definition of sine and cosine we have

$$z = \rho \cos \phi, \iff \cos \phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \iff \phi = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Finally, we project (x, y, z) on the plane xy, that is, we consider point $(x, y, 0)$. Let $\theta \in [0, 2\pi]$ be the angle made by the vector $(x, y, 0)$ and the positive $x-$axis. If r is its length, then

$$x = r \cos \theta, \quad y = r \sin \theta.$$

We notice that $r = \rho \sin \phi$, so

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta,$$

and adding the previous relation for z we get,

$$\begin{cases} x = \rho \sin \phi \cos \theta, \\ y = \rho \sin \phi \sin \theta, \\ z = \rho \cos \phi. \end{cases}$$

Incidentally, here we notice an important example of a vector valued function of vector variable: denoting by $\vec{\omega} := (\rho, \phi, \theta)$ the array of *spherical coordinates*, and by $\vec{p} := (x, y, z)$ the array of *cartesian coordinates*, the previous relation can be written as

$$\vec{p} = \vec{\Psi}(\vec{\omega}), \quad \vec{\Psi}(\rho, \phi, \theta) = \Big(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi\Big).$$

Thus, $\vec{\Psi}$ is a function that allows to pass from *spherical coordinates* to *cartesian coordinates*. In other words, it is a change of variables in $\mathbb{R}^3$. We will return on this function later.

Example 1.4.9: $(**)$

Q. *Compute*

$$\lim_{(x,y,z)\to 0_3} \frac{\sin(xyz)}{x^2 + y^2 + z^2}.$$

A. Let $f(x, y, z) := \frac{\sin(xyz)}{x^2+y^2+z^2}$ defined on its natural domain $D = \mathbb{R}^3 \backslash \{\vec{0}\}$. The sections on the axes $f(x, 0, 0) = f(0, y, 0) = f(0, 0, z) \equiv 0$, so the candidate limit is 0. We may notice that the limit presents an indeterminate form $\frac{0}{0}$. Introducing spherical coordinates, we have

$$f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) = \frac{\sin\left(\rho^3(\cos\theta)(\sin\theta)(\sin\phi)^2(\cos\phi)\right)}{\rho^2}.$$

Here we may notice that, since $(x,y,z) \to \vec{0}$ iff $\rho = \|(x,y,z)\| \to 0$, and reminding that $\sin u \sim_0 u$, the numerator has very likely the magnitude of ρ^3 while the denominator is ρ^2. This suggests that the numerator goes faster to 0 than the denominator, whence the limit should be $= 0$. To make this intuition precise, we may remind that $|\sin u| \leqslant |u|$, $\forall u \in \mathbb{R}$. Therefore

$$\begin{aligned} |f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)| &= \left|\frac{\rho^3(\cos\theta)(\sin\theta)(\sin\phi)^2(\cos\phi)}{\rho^2}\right| \\ &\leqslant \rho|\cos\theta||\sin\theta||\sin\phi|^2|\cos\phi| \leqslant \rho \longrightarrow 0, \end{aligned}$$

when $\rho \longrightarrow 0+$. Therefore, by the Proposition 1.4.1, the limit exists and it is 0.

In previous examples we considered finite limit values for ℓ. What if we want to verify that the limit $\ell = +\infty$ (or $-\infty$)? Of course, we cannot apply the Proposition 1.4.1 with $\ell = \pm\infty$. Nonetheless, we prove a similar test for infinite limits:

Proposition 1.4.10

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ with $\vec{0} \in \mathrm{Acc}(D)$. Suppose moreover that:

i) $f(\vec{x}) \geqslant g(\|\vec{x}\|)$,

ii) $g(\rho) \longrightarrow +\infty$ when $\rho \to 0+$.

Then

$$\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = +\infty.$$

You can prove this statement imitating the argument used in the proof of the Proposition 1.4.1. A similar version holds for $-\infty$ limits (see exercises).

Check List 1.4.11: Computing Limits

To compute a limit for $\vec{x} \to \vec{0}$:

(1) Check sections: these are useful to identify a possible candidate limit and, if you are able to find two sections going at $\vec{0}$ along which f has two different limits, to prove that the limit does not exist.

(2) If you guess that the limit exists equal $\ell \in \mathbb{R}$, then you should try to bound

$$|f(\vec{x}) - \ell|$$

with a function depending on $\|\vec{x}\|$, namely $g(\|\vec{x}\|)$ with the key property that $g(\rho) \longrightarrow 0$ when $\rho \to 0$.

In alternative, if you guess that $\ell = +\infty$ (or $-\infty$), then you should try to bound from below (above) by a $g(\|\vec{x}\|)$ going at $+\infty$ $(-\infty)$ when its argument goes to 0.

1.4.2. Limit at $\vec{p} = \infty_d$. With minor modifications, the Strategy 1.4.11 adapts in a natural way to the case when $\vec{p} = \infty_d$. Let us illustrate the main idea directly on some examples.

Example 1.4.12: $(**)$

Q. *Show that*

$$\lim_{(x,y)\to\infty_2} \left(x^2 + y^2 - 4xy\right)$$

does not exist.

A. Let $f(x,y) := x^2 + y^2 - 4xy$. Sections along the axes are $f(x,0) = x^2$, $f(0,y) = y^2$. Now, since

$$(x,0) \longrightarrow \infty_2, \iff \|(x,0)\| = \sqrt{x^2} = |x| \longrightarrow +\infty,$$

and similarly $(0,y) \longrightarrow \infty_2$ iff $|y| \longrightarrow +\infty$, we deduce that $f(x,0), f(0,y) \to +\infty$ when $|x|, |y| \to +\infty$. So, the candidate limit is $+\infty$. However, along the line $y = x$, the section of f is

$$f(x,x) = x^2 + x^2 - 4x^2 = -2x^2,$$

and since

$$(x,x) \longrightarrow \infty_2, \iff \|(x,x)\| = \sqrt{x^2+x^2} = \sqrt{2}|x| \longrightarrow +\infty, \iff |x| \longrightarrow +\infty,$$

we conclude that $f(x,x) \to -\infty$ when $(x,x) \to \infty_2$. But then, we found at least two sections along which f has two different limits when its argument goes to ∞_2. This implies that the limit of f at ∞_2 cannot exists.

Example 1.4.13: $(***)$

Q. *Compute*

$$\lim_{(x,y)\to\infty_2} \left(x^4 + y^4 - xy\right).$$

A. Looking at the sections along the axes we have $f(x,0) = x^4 \longrightarrow +\infty$ and $f(0,y) = y^4 \longrightarrow +\infty$. So, if the limit exists must be $+\infty$. This seems reasonable because $x^4 + y^4$ should dominate xy. Let us write f in polar coordinates:

$$f(\rho\cos\theta, \rho\sin\theta) = \rho^4(\cos\theta)^4 + \rho^4(\sin\theta)^4 - \rho^2(\cos\theta)(\sin\theta) = \rho^4\left[(\cos\theta)^4 + (\sin\theta)^4\right] - \frac{1}{2}\rho^2\sin(2\theta).$$

Now: notice that the quantity $K(\theta) := (\cos\theta)^4 + (\sin\theta)^4$ is always positive and, by Weierstrass' theorem, it has a minimum as $\theta_* \in [0, 2\pi]$. Therefore $K(\theta) \geqslant K(\theta_*)$ for every $\theta \in [0, 2\pi]$. Since $K(\theta) = 0$ iff $\cos\theta = \sin\theta = 0$, which is impossible, we deduce that $K_0 := K(\theta_*) > 0$. Therefore, being $|\sin(2\theta)| \leqslant 1$, we have

$$f(x,y) \geqslant K_0\rho^4 - \frac{1}{2}\rho^2\sin(2\theta) \geqslant K_0\rho^4 - \frac{1}{2}\rho^2 =: g(\rho) \longrightarrow +\infty,$$

from which we conclude that the limit exists and it is equal to $+\infty$.

Example 1.4.14: $(**)$

Q. *Compute*

$$\lim_{(x,y,z)\to\infty_3} \left[(x^2 + y^2 + z^2)^2 - xyz\right].$$

A. A quick check on the sections along the axes show that they tend to $+\infty$. Again: it seems reasonable that the fourth order term $(x^2+y^2+z^2)^2$ dominates on xyz. Passing to spherical coordinates

$$f=(\rho^2)^2-\rho^3(\cos\theta)(\sin\theta)(\sin\phi)^2(\cos\phi)=\rho^4-\frac{1}{4}\rho^3(\sin(2\theta))(\sin(2\phi))(\sin\phi).$$

Now,

$$|(\sin(2\theta))(\sin(2\phi))(\sin\phi)|\leqslant 1,$$

we have

$$f\geqslant\rho^4-\frac{1}{4}\rho^3=:g(\rho)\longrightarrow+\infty.\quad\square$$

Example 1.4.15: (∗∗∗)

Q. Compute

$$\lim_{(x,y,z)\to\infty_3}\left[(x^2+y^2)^2+z^2-xy\right]$$

A. Looking at sections, we have $f(x,0,0)=x^4\longrightarrow+\infty$ when $\|(x,0,0)\|=|x|\longrightarrow+\infty$. Thus, if a limit exists, it must be $=+\infty$. Writing f in polar coordinates, we have

$$\begin{aligned}f(x,y,z)&=\left(\rho^2\sin^2\phi\right)^2+\rho^2\cos^2\phi-\rho^2\cos\theta\sin\theta\sin^2\phi\\&=\rho^4\sin^4\phi+\rho^2\cos^2\phi-\frac{1}{2}\rho^2\sin(2\theta)\sin^2\phi\end{aligned}$$

Remind the goal: bound from below with a function of ρ going at $+\infty$ when $\rho\to+\infty$. In previous expression we have apparently good terms like $\rho^4\sin^4\phi$. This term is good because it goes like ρ^4. However, it is multiplied by $\sin^4\phi$ that could be small if not equal to 0, destroying such a good behavior! That is why we must be clever. We start noticing that

$$\exists C>0\ :\ u^4\geqslant u^2-C.$$

Indeed, if $h(u)=u^4-u^2$, we have $h'(u)=4u^3-2u=2u(2u^2-1)$, from which we easily deduce that h has a global minimum at $u=\pm\frac{1}{\sqrt{2}}$. This means that $h(u)\geqslant h(\pm\frac{1}{\sqrt{2}})=\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}=:-C$. Since in the following arugment the value of C is irrelevant, we will keep just C. In particular

$$\rho^4\sin^4\phi=(\rho\sin\phi)^4\geqslant(\rho\sin\phi)^2-C=\rho^2\sin^2\phi-C.$$

Therefore,

$$\begin{aligned}f(x,y,z)&\geqslant\rho^2\sin^2\phi-C+\rho^2\cos^2\phi-\frac{1}{2}\rho^2\sin(2\theta)\sin^2\phi=\rho^2\Big(1-\frac{1}{2}\underbrace{\sin(2\theta)\sin^2\phi}_{0\leqslant\ \leqslant 1}\Big)-C\\&\geqslant\frac{1}{2}\rho^2-C\longrightarrow+\infty.\quad\square\end{aligned}$$

1.5. Basic Topologial Concepts

Intervals play a special role with functions of real variable real valued, $f=f(x)\ :\ D\subset\mathbb{R}\longrightarrow\mathbb{R}$. They are natural sets and they enter in most of the properties of continuity, differentiability and integrability. Certain specific properties of intervals are implicitly used to prove important results on functions. These are qualitative properties as the fact that *any interval is made by one single piece*, or *an* *interval of type* $[a, b]$ *is bounded and it contains the endpoints*, or, again, *any point of an interval of type* $]a, b[$ *is in the interior of the interval itself*. For example, Weierstrass' theorem concerning the existence of min and max of a continuous function holds on a *closed and bounded interval* (and the type of domain matters).

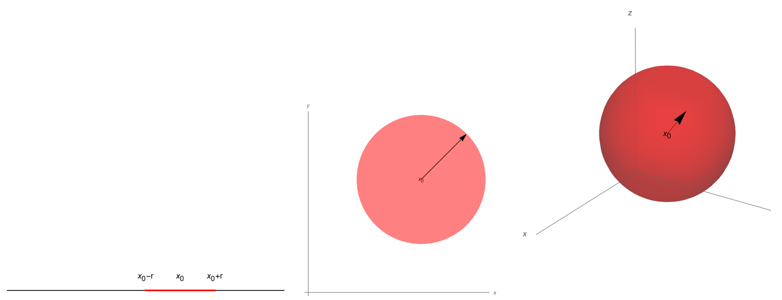
In $\mathbb{R}^d$ there are no such "natural sets". This makes relevant to introduce some key qualitative properties that can be checked for large classes of sets. These are called *topological properties*, where *Topology* literally means *the study of place*. The first step is the following

Definition 1.5.1

Let $\vec{x} \in \mathbb{R}^d$ and $r > 0$. We call **closed ball centered at** $\vec{x}$ **with radius** r the set

$$B(\vec{x}, r] := \left\{ \vec{y} \in \mathbb{R}^d \ : \ \|\vec{y} - \vec{x}\| \leqslant r \right\}.$$

In dimension $d = 1$ a closed ball is just the closed interval $[x - r, x + r]$. In $\mathbb{R}^2$, a closed ball is a plane disk centered at $\vec{x}$ with radius r, while in $\mathbb{R}^3$ it is a sphere centered at $\vec{x}$ and with radius r.



In dimensions higher or equal to 4 it is hard to visualize a ball. Fortunately, for the intuition the case $d = 2$ is sufficient in most of the applications.

Definition 1.5.2

Let $D \subset \mathbb{R}^d$. A point $\vec{x} \in \mathbb{R}^d$ is said to be

- an **interior point of** D if
$$\exists\, B(\vec{x}, r] \subset D.$$
In particular, $\vec{x}$ itself belongs to D. The set of all the interior points of D is denoted by $\mathrm{Int}(D)$ and it is called **interior of** D.
- a **boundary point for** D if every ball centered at $\vec{x}$ contains both points of D and points of D^c, that is
$$B(\vec{x}, r] \cap D \neq \varnothing, \quad B(\vec{x}, r] \cap D^c \neq \varnothing, \quad \forall r > 0.$$
The boundary of D is denoted with ∂D.

Let us see some simple examples. We will not provide detailed justifications in all cases (it might be hard!).

- In $\mathbb{R}$, $\mathrm{Int}([a, b]) =]a, b[$, $\partial[a, b] = \{a, b\}$. Here $D = [a, b]$. Recalling that $B(x, r] = [x - r, x + r]$, it is easy to check that every point $x \in]a, b[$ is contained in $D = [a, b]$ with a suitable interval $[x - r, x + r]$. This is not true for endpoints a, b because, for example $[a - r, a + r]$ contains the sub-interval $[a - r, a[$ which has no points in D, thus $[a - r, a + r] \not\subset D$. About boundary: at a, every ball $[a-r, a+r]$ has point of D and of D^c (recall here $D = [a, b]$) and same for b. If $x \in]a, b[$ and r is small enough, the ball $[x-r, x+r] \subset [a, b]$ then there cannot be points of D^c in such a ball. Same for points out of $[a, b]$. This explains why $\partial[a, b] = \{a, b\}$.

- In $\mathbb{R}^d$,

$$\begin{aligned} \mathrm{Int}B(\vec{x}, r] &= \left\{\vec{y} \in \mathbb{R}^d \ : \ \|\vec{y}-\vec{x}\| < r\right\} =: B(\vec{x}, r[, \\ \partial B(\vec{x}, r] &= \left\{\vec{y} \in \mathbb{R}^d \ : \ \|\vec{y}-\vec{x}\| = r\right\}. \end{aligned}$$

Intuitively clear, some work has to be done to prove details (see exercises).

Definition 1.5.3: open set

A set $D \subset \mathbb{R}^d$ is said to be **open** if $\mathrm{Int}(D) = D$, that is if every point of D lies in D with an entire ball. Empty set $\varnothing$ is considered open by definition.

So, for example, $B(\vec{x}, r[$ (open ball) is open (in the sense of the previous definition) but $B(\vec{x}, r]$ is not open (because points of the *edge* $\{\vec{y} \ : \ \|\vec{y}-\vec{x}\| = r\}$ are not in the interior of $B(\vec{x}, r]$).

Definition 1.5.4: closed set

A set $D \subset \mathbb{R}^d$ is said to be **closed** if D^c is open.

Warning 1.5.5

In common language, open and closed are opposite alternatives. This might lead to the idea that i) every set is either open or closed, and ii) a set cannot be both open and closed. Both these claims are false!

For example, in $\mathbb{R}$ the interval $[a, b[$ is neither open nor closed (check it!). Moreover, $\mathbb{R}^d$ is clearly open and since $(\mathbb{R}^d)^c = \varnothing$ is open, $\mathbb{R}^d$ is also closed. Similarly, $\varnothing$ is (by definition) open, and since $\varnothing^c = \mathbb{R}^d$ is open, $\varnothing$ is also closed. Actually, **it can be shown that $\varnothing$ and $\mathbb{R}^d$ are the unique sets to be both open and closed.**

An important characterization of closed sets is provided by the following

Proposition 1.5.6: Cantor

A set $D \subset \mathbb{R}^d$ is closed if and only if it contains all possible finite limit of convergent sequences in D. Formally:

$$D \text{ closed} \iff \forall (\vec{x}_n) \subset D \ : \ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \text{ then } \vec{\ell} \in D.$$

Proof. $\Longrightarrow$ Assume D closed and let $(\vec{x}_n) \subset D$ be such that $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$. The claim is: $\vec{\ell} \in D$. If false, $\vec{\ell} \in D^c$, and since D^c is open (by assumption, D is closed), then

$$\exists \, B(\vec{\ell}, r] \subset D^c.$$

Since $\vec{x}_n \longrightarrow \vec{\ell}$, according to the definition of limit,

$$\exists N \ : \ \|\vec{x}_n - \vec{\ell}\| \leqslant r, \ \forall n \geqslant N, \implies \vec{x}_n \in B(\vec{\ell}, r] \subset D^c, \ \forall n \geqslant N,$$

and this contradicts $(\vec{x}_n) \subset D$.

$\Longleftarrow$ Assume that D contains all possible finite limit of convergent sequences in D. The goal is to prove that D is closed, that is D^c is open. If $D^c = \varnothing$ there is nothing to prove. So assume $D^c \neq \varnothing$ and pick a point $\vec{\ell} \in D^c$. We have to prove that

$$\exists\, B(\vec{\ell}, r] \subset D^c.$$

Suppose, by contradiction, that ball does not exist. Then,

$$\forall r > 0,\ B(\vec{\ell}, r] \cap D \neq \varnothing.$$

Take $r = \frac{1}{n}$: we have

$$\forall n \in \mathbb{N}\backslash\{0\},\ \exists \vec{x}_n \in D\ :\ \|\vec{x}_n - \vec{\ell}\| \leqslant \frac{1}{n}.$$

Then, $(\vec{x}_n) \subset D$ and $\vec{x}_n \longrightarrow \vec{\ell}$. By assumption, necessarily $\vec{\ell} \in D$ and this contradicts $\vec{\ell} \in D^c$. $\square$

Open and closed sets are important classes of sets as we will see. It is therefore important to have general and easily testable conditions to ensure whether a set D is open or closed. The characteristic properties are not always easy to be used. A common way to define sets in $\mathbb{R}^d$ is through a certain number of equalities or inequalities (strict or large). Consider, for example, the set

$$D := \left\{(x, y, z) \in \mathbb{R}^3\ :\ x^2 + y^2 + z^2 \leqslant 1,\ \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\right\}$$

It turns out that if the equalities/inequalities are described by continuous functions we can automatically get the answer about open/closed nature of the set. Here is a precise statement:

Proposition 1.5.7

Any set defined through a finite number of strict inequalities involving continuous functions is open. Any set defined through a finite number of large inequalities and/or equalities involving continuous functions is closed. Formally, if $g_1, \dots, g_m, h_1, \dots, h_k \in \mathsf{C}(\mathbb{R}^d)$, then

$$D := \left\{\vec{x} \in \mathbb{R}^d\ :\ g_1(\vec{x}) > 0, \dots, g_m(\vec{x}) > 0\right\} \text{ is } \textbf{open},$$

$$D := \left\{\vec{x} \in \mathbb{R}^d\ :\ g_1(\vec{x}) \geqslant 0, \dots, g_m(\vec{x}) \geqslant 0,\ h_1(\vec{x}) = 0, \dots, h_k(\vec{x}) = 0\right\} \text{ is } \textbf{closed}.$$

PROOF. For simplicity, consider $D = \{\vec{x} \in \mathbb{R}^d \ : \ g(\vec{x}) \geqslant 0\}$. We prove that D is closed. If $D = \emptyset$ there is nothing to prove. So, we assume $D \neq \emptyset$. By Cantor's characterization, we have to prove that

$$\text{if } (\vec{x}_n) \subset D, \ : \ \vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d, \ \Longrightarrow \ \vec{\ell} \in D.$$

We know

$$\vec{x}_n \in D, \ \Longrightarrow \ g(\vec{x}_n) \geqslant 0.$$

Since g is continuous and $\vec{x}_n \longrightarrow \vec{\ell} \in \mathbb{R}^d$, then $g(\vec{x}_n) \longrightarrow g(\vec{\ell})$. According to the permanence of sign for numerical sequences, $g(\vec{\ell}) \geqslant 0$, that is $\vec{\ell} \in D$.

The proof that $D := \{\vec{x} \in \mathbb{R}^d \ : \ g(\vec{x}) = 0\}$ is closed is analogous. Finally, if $D = \{g > 0\}$ then $D^c = \{g \leqslant 0\}$, which is closed according to the previous argument, thus D is open. □

Example 1.5.8: $(*)$

Q. *Is* $D := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \leqslant 1, \ \left(x - \frac{1}{2}\right)^2 + y^2 \leqslant \frac{1}{4}\right\}$ *open? Is it closed? Justify your answer.*

A. The set is defined by large inequalities $D = \{g_1(x, y, z) \leqslant 0, \ g_2(x, y, z) \leqslant 0\}$ where $g_1(x, y, z) := x^2 + y^2 + z^2 - 1$ and $g_2(x, y, z) := \left(x - \frac{1}{2}\right)^2 + y^2 - \frac{1}{4}$. Since $g_1, g_2 \in \mathsf{C}(\mathbb{R}^3)$ (they are polynomials), we conclude that D is closed. We recall that, in Warning 1.5.5 we mentioned the fact that the unique both open and closed subset of $\mathbb{R}^d$ are $\mathbb{R}^d$ itself and $\emptyset$. Now, our $D \subsetneq \mathbb{R}^d$ (for example $(10, 0, 0) \notin D$) and $D \neq \emptyset$ (for example $(0, 0, 0) \in D$). Thus, $D \neq \mathbb{R}^3, \emptyset$, so D cannot be open.

1.6. Weierstrass' Theorem

The search for minimum/maximum points of a numerical function is one of the most important problems in many applications. We start introducing the definition of min/max point for f and min/max value of f on a domain D.

Definition 1.6.1: min/max points and values

Let $f = f(\vec{x}) \ : \ D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We say that $\vec{x}_{min} \in D$ is a **global minimum point for f on D** if

$$f(\vec{x}_{min}) \leqslant f(\vec{x}), \ \forall \vec{x} \in D.$$

We call **minimum value of f on D** the value of f at minimum point $\vec{x}_{min}$, that is $f(\vec{x}_{min})$. We write

$$\min_D f \equiv \min_{\vec{x} \in D} f(\vec{x}) := f(\vec{x}_{min}).$$

Similar definitions for **maximum point for f on D** and **maximum value of f on D** (denoted by $\max_D f$ or $\max_{\vec{x} \in D} f(\vec{x})$).

Weierstrass' theorem is a fundamental result concerning existence of min/max points. It states that *every* $f \in \mathsf{C}([a, b])$ *has global minimum/maximum on* $[a, b]$. The conclusion is false if the interval $[a, b]$ is not closed and bounded. We look at an extension of this result to the case of functions of vector variable $f = f(\vec{x}) \ : \ D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We may expect that, under suitable assumptions on D, if $f \in \mathsf{C}(D)$ then f will have global min/max on D. The right general conditions on D are that this must be **closed** and **bounded**:

Definition 1.6.2: bounded set

A set $D \subset \mathbb{R}^d$ is **bounded** if

$$\exists M \ : \ \|\vec{x}\| \leqslant M, \ \forall \vec{x} \in D.$$

We have now all the ingredients for the

Theorem 1.6.3: Weierstrass

Every **continuous function** $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ on a **closed and bounded domain** D has global minimum and global maximum on D.

We skip the proof of this important result. Weierstrass' theorem points out the class of closed and bounded sets.

Definition 1.6.4: compact set

A set $D \subset \mathbb{R}^d$ is **compact** if it is closed and bounded.

Example 1.6.5: $(*)$

Q. *Check that a closed ball $B(\vec{x}, r]$ is compact.*

A. First, the ball is closed being defined by a large inequality involving a continuous function. Indeed

$$\vec{y} = (y_1, \ldots, y_d) \in B(\vec{x}, r], \iff \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2} \leqslant r.$$

Second, the ball is trivially bounded: since $\|\vec{y} - \vec{x}\| \leqslant r$, we have

$$\|\vec{y}\| = \|\vec{y} - \vec{x} + \vec{x}\| \overset{\triangle}{\leqslant} \|\vec{y} - \vec{x}\| + \|\vec{x}\| \leqslant r + \|\vec{x}\| =: M, \ \forall \vec{y} \in B(\vec{x}, r]. \qquad \square$$

Example 1.6.6: $(**)$

Q. *Show that the set $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 = 1, \ y^2 + z^2 \leqslant 1\}$ is compact.*

A. D is closed being defined through large inequalities or equalities involving continuous functions. Let us prove that it is also bounded. If $(x, y, z) \in D$, $y^2 + z^2 \leqslant 1$, then $y^2 \leqslant 1$ and $z^2 \leqslant 1$, so $|y|, |z| \leqslant 1$ (that is, y and z coordinates are bounded). Plugging this information into the first relation we get,

$$x^2 + y^2 = 1 + z^2, \implies x^2 = 1 + z^2 - y^2 \leqslant 1 + z^2 \leqslant 1 + 1 = 2,$$

from which $x^2 \leqslant 2$, that is $|x| \leqslant \sqrt{2}$ (hence, also x is bounded). Therefore

$$\|(x, y, z)\|^2 = x^2 + y^2 + z^2 \leqslant 2 + 1 + 1 = 4 =: M, \ \forall (x, y, z) \in D.$$

This proves that D is also bounded, hence it is compact.

Example 1.6.7: $(**)$

Q.*Discuss whether $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x = yz + 1\}$ is compact or less.*

A. Clearly, D is defined through an equation involving a continuous function, it is therefore closed. About boundedness, notice that D contains points of type $(y^2 + 1, y, y)$ for every $y \in \mathbb{R}$. Now since

$$\|(y^2 + 1, y, y)\|^2 = (y^2 + 1)^2 + 2y^2 \longrightarrow +\infty, \ y \longrightarrow \pm\infty,$$

we conclude that there cannot be a constant M such that $\|(x, y, z)\| \leqslant M$ for every $(x, y, z) \in D$. Therefore, D is not bounded, hence definitely not compact.

When the domain D on which f is continuous is closed and unbounded, Weierstrass' theorem does not apply. In this case, with some extra information on f we can show the existence of an extrema.

Proposition 1.6.8

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be continuous on D closed and unbounded, and such that

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty \ (-\infty).$$

Then f has a global minimum (maximum).

Proof. We do the proof for the minimum, the other case being similar. Assume then that

$$\lim_{\vec{x}\to\infty_d} f(\vec{x}) = +\infty.$$

Pick a point $\vec{p} \in D$ and consider the new domain

$$\widetilde{D} := \{\vec{x} \in D \ : \ f(\vec{x}) \leqslant f(\vec{p}) + 1\}.$$

Notice that, in particular, $\vec{p} \in \widetilde{D}$. Clearly $\widetilde{D} \subset D$ and since D is closed and $\widetilde{D}$ is defined through an inequality involving a continuous function, we have that also $\widetilde{D}$ is closed (this requires a little proof, we accept here). We claim that $\widetilde{D}$ is also bounded. If false,

$$\forall n \in \mathbb{N}, \ \exists \vec{x}_n \in \widetilde{D} \subset D, \ : \ \|\vec{x}_n\| \geqslant n.$$

Then $\vec{x}_n \longrightarrow \infty_d$, thus $f(\vec{x}_n) \longrightarrow +\infty$, and this is impossible since $f(\vec{x}_n) \leqslant f(\vec{x}_0) + 1$ because $\vec{x}_n \in \widetilde{D}$.

Since $\widetilde{D}$ is closed and bounded, that is compact, Weierstrass' theorem applies to f on $\widetilde{D}$: there exists a point $\vec{x}_{min} \in \widetilde{D} \subset D$ such that

$$f(\vec{x}_{min}) \leqslant f(\vec{x}), \ \forall \vec{x} \in \widetilde{D}.$$

If now $\vec{x} \in D\backslash\widetilde{D}$ it means that $f(\vec{x}) \geqslant f(\vec{x}_0) + 1 > f(\vec{x}_0) \geqslant f(\vec{x}_{min})$. We conclude that, no matter where is taken $\vec{x} \in D$, $f(\vec{x}) \geqslant f(\vec{x}_{min})$, that is $\vec{x}_{min}$ is a global minimum for f. □

Example 1.6.9: $(**)$

Q. *Show that the function* $f(x, y) := x^4 + y^4 - xy$ *has global minimum on* $\mathbb{R}^2$. *What about global maximum?*

A. We have $f \in \mathsf{C}(\mathbb{R}^2)$ (because it is a polynomial). Notice that $\mathbb{R}^2$ is closed but unbounded. We have already shown (see Example 1.4.13) that

$$\lim_{(x,y)\to\infty_2} f(x, y) = +\infty.$$

Therefore, by the Corollary of Weierstrass's thm we have that there exists a global minimum for f on $\mathbb{R}^2$. On the other side, since f is upper unbounded (by the limit at ∞_2) the global maximum doesn't exists.

Weiestrass' theorem is a pure *existence* result, it does not provide any concrete method to find global min/max points. This will be provided by Differential Calculus we will develop in the next Chapter.

1.7. Intermediate values theorem

Another important property of continuous functions of one real variable is that, on intervals, if a function takes both positive and negative values, then it must take also value 0 (Bolzano's zeroes theorem). More in general, if the function takes any two values, then it takes all values between these two (intermediate values theorem). Is this somehow still true if $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is function of vector variable? Under which assumptions on D?

Apart for continuity, the key ingredient of the intermediate values theorem is the fact that the domain of the function is an interval. As already pointed out, intervals are common sets in dimension 1, but their analogous (rectangles, parallelepiped, etc) are too specific in $\mathbb{R}^d$. What we really need, however, is the idea of a *set made of one single piece*:

Definition 1.7.1: connected set

We say that D is **connected** if any two points of D are joint by a continuous curve in D, that is

$$\forall \vec{x}, \vec{y} \in D,\ \exists\, \vec{\gamma} = \vec{\gamma}(u) : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^d,\ :\ \vec{\gamma} \subset D,\ \vec{\gamma} \in \mathsf{C}([a,b]),\ :\ \vec{\gamma}(a) = \vec{x},\ \vec{\gamma}(b) = \vec{y}.$$

Despite it is not an easy task to check in practice, with the definition, that a set D is connected, the intuition is clear: an island is a connected set, an archipelago is not.

Theorem 1.7.2

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ continuous on D connected. Then, if f takes any two values, it takes also all other values between these two. Precisely:

$$\alpha, \beta \in f(D), \implies [\alpha, \beta] \subset f(D).$$

In particular, $f(D)$ is an interval

Proof. Let $\alpha, \beta \in f(D)$, that is $\alpha = f(\vec{x})$ and $\beta = f(\vec{y})$ for some $\vec{x}, \vec{y} \in D$. Since D is connected, there exists a continuous curve $\vec{\gamma} \subset D$ such that $\vec{\gamma}(a) = \vec{x}$ and $\vec{\gamma}(b) = \vec{y}$. Define then

$$g : [a,b] \longrightarrow \mathbb{R},\ g(u) := f(\vec{\gamma}(u)).$$

Since g is composition of continuous functions it is itself continuous. Moreover $g(a) = \alpha$ and $g(b) = \beta$. According to the ordinary intermediate values theorem, g takes all values between α and β. This means that

$$\forall c \in [\alpha, \beta],\ \exists u \in [a,b]\ :\ c = g(u) = f(\vec{\gamma}(u)), \implies c \in f(D).$$

Therefore, $[\alpha, \beta] \subset f(D)$. □

Remark 1.7.3

If $f \in \mathsf{C}(D)$ with D compact and connected, then

$$f(D) = [\min_D f, \max_D f].$$

This is because, by Weierstrass' theorem, $\min_D f$ and $\max_D f$ are taken at $\vec{x}_{min}$ and $\vec{x}_{max}$ respectively. Thus $[\min_D f, \max_D f] \subset f(D)$. And because $\min_D f$ and $\max_D f$ are, resp., the minimum and maximum values of f on D, $f(D) = [\min_D f, \max_D f]$.

1.8. Exercises

Exercise 1.8.1 $(**)$**.** By using the definition of norm, prove the *parallelogram identity*:

$$\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2,\quad \forall \vec{x}, \vec{y} \in \mathbb{R}^d.$$

Exercise 1.8.2 $(**)$**.** Define

$$\|\vec{x}\|_1 := \sum_{j=1}^{d} |x_j|,\quad \|\vec{x}\|_\infty := \max_{j=1,\dots,d} |x_j|,\ \forall \vec{x} \in \mathbb{R}^d.$$

Check that $\|\vec{x}\|_1$ and $\|\vec{x}\|_\infty$ verify the same properties of Proposition 1.1.4 verified by the euclidean norm.

Exercise 1.8.3 $(*)$**.** For each of the following sequences of vectors say whether it is convergent and, if yes, what is the limit for $n \longrightarrow +\infty$:

i) $\vec{x}_n := (e^{-n}, 1)$.
ii) $\vec{x}_n := \big(n, n^2\big)$.
iii) $\vec{x}_n := \left(\frac{1}{n}, \frac{1}{n^2}, \sin\frac{1}{n}\right)$.
iv) $\vec{x}_n := \left(1, 1+\frac{1}{n}, n\right)$.
v) $\vec{x}_n := \left(\tanh n, \frac{\log n}{n}, \frac{\sin n}{n}\right)$.
vi) $\vec{x}_n := \big((-1)^n, (-1)^{n+1}\big)$.

Exercise 1.8.4 $(**)$**.** Do the proof of the Proposition 1.2.3.

Exercise 1.8.5 $(**)$**.** For each of the following statements provide a proof (if true) or a counterexample (if false):

- if $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \vec{0}$ then $x_{n,j} \longrightarrow 0$ for every $j = 1, \dots, d$.
- If $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \infty_d$ then $|x_{n,j}| \longrightarrow +\infty$, $j = 1, \dots, d$.
- if $x_{n,j} \longrightarrow +\infty$ for at least one component j then $\vec{x}_n = (x_{n,1}, \dots, x_{n,d}) \longrightarrow \infty_d$.
- if $\vec{x}_n = (x_{n,1}, \dots, x_{n,d})$ does not have limit, then all components $x_{n,j}$ ($j = 1, \dots, d$) do not have limit as well.

Exercise 1.8.6 $(*)$**.** For each of the following cartesian curves, write a parametric form $\vec{\gamma} = \vec{\gamma}(u)$:

i) $3x + 2y = 1$.
ii) $x = 5$.
iii) $y = x^2$.
iv) $x = y^3$.
v) $x^2 + y^2 = 1$.
vi) $x^2 + 2y^2 = 3$.

Exercise 1.8.7. Looking at suitable sections, prove that the following limits do not exist:

1. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 - y^2}{x^2 + y^2}$.
2. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 + y^3}{x^2 + y^2}$.
3. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{y^2 - xy}{x^2 + y^2}$.
4. $(*)$ $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{x + y^2 + z^3}{\sqrt{x^2 + y^2 + z^2}}$.
5. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy + \sqrt{y^2+1} - 1}{x^2 + y^2}$.
6. $(**)$ $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{xz}{x^4 + y^2 + z^2}$.

Exercise 1.8.8. Compute the following limits:

1. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy}{\sqrt{x^2 + y^2}}$.
2. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^2 y^3}{(x^2 + y^2)^2}$.
3. $(*)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{x^3 - y^3}{x^2 + y^2}$.
4. $(**)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{x\sqrt{|y|}}{\sqrt[3]{x^4 + y^4}}$.
5. $(**)$ $\displaystyle\lim_{(x,y)\to 0_2} \frac{xy}{|x| + |y|}$.

Exercise 1.8.9 $(**)$**.** Determine, if any, the following limits:

1. $\displaystyle\lim_{(x,y)\to 0_2} \frac{e^{4y^3} - \cos(x^2 + y^2)}{x^2 + y^2}$.
2. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{xyz}{x^2 + y^2 + z^2}$.
3. $\displaystyle\lim_{(x,y,z)\to 0_3} \frac{(x^2 + yz)^2}{\sqrt{(x^2 + y^2)^2 + z^4}}$.
4. $\displaystyle\lim_{(x,y)\to 0_2} \frac{\log(1 + 2x^3)}{\sinh(x^2 + y^2)}$.
5. $\displaystyle\lim_{(x,y)\to (0,1)} \frac{x^3 \sinh(y-1)}{x^2 + y^2 - 2y + 1}$.
6. $\displaystyle\lim_{(x,y)\to (1,1)} \frac{(x-1)^2 (y-1)^7}{\big((x-1)^2 + (y-1)^2\big)^{5/2}}$.

Exercise 1.8.10 (∗∗)**.** Determine, if any, the following limits:

1. $\lim_{(x,y)\to\infty_2} \left(x^3 + xy^2 - y^2\right)$.
2. $\lim_{(x,y)\to\infty_2} (x^4 - y^4 + y^2 - x^2)$.
3. $\lim_{(x,y)\to\infty_2} \left(x^2y^2 + x^2 + y^2 - xy\right)$.
4. $\lim_{(x,y,z)\to\infty_3} \left(x^4 + y^4 + z^4 - xyz\right)$.
5. $\lim_{(x,y,z)\to\infty_3} \left(x^2 + y^2 + z^4 - xz\right)$.
6. $\lim_{(x,y,z)\to\infty_3} \left(\sqrt{x^2 + y^2} + z^2 - z\right)$.
7. $\lim_{(x,y,z)\to\infty_3} \left(\sqrt{(x^2 + y^2)^2 + z^4} - xyz\right)$.

Exercise 1.8.11 (∗∗)**.** Prove the Proposition 1.4.10 (hint: adapt the proof of the Proposition 1.4.1). Then, formulate a statement for the case of $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = -\infty$ and prove it.

Exercise 1.8.12 (∗∗)**.** Prove that

- $\mathrm{Int}B(\vec{x}, r] = \{\vec{y} \in \mathbb{R}^d \ : \ \|\vec{y} - \vec{x}\| < r\}$.
- $\partial B(\vec{x}, r] = \{\vec{y} \ : \ \|\vec{y} - \vec{x}\| = r\}$.

Exercise 1.8.13 (∗∗∗)**.** Prove the following characterizations for accumulation points:

i)
$$\vec{x} \in \mathbb{R}^d \cap \mathrm{Acc}(D), \iff \forall r > 0,\ (B(\vec{x}, r] \cap D) \setminus \{\vec{x}\} \neq \emptyset.$$

ii)
$$\infty_d \in \mathrm{Acc}(D), \iff \forall r > 0,\ \left(B(\vec{0}, r]^c \cap D\right) \neq \emptyset.$$

Exercise 1.8.14 (∗∗∗)**.** Prove that:

- the union and the intersection of any two open sets are open sets (that is: if A_1, A_2 open, then $A_1 \cup A_2$ and $A_1 \cap A_2$ are also open).
- the union and the intersection of any two closed sets are closed sets (that is: if C_1, C_2 closed, then $C_1 \cup C_2$ and $C_1 \cap C_2$ are also closed).

Exercise 1.8.15 (∗∗)**.** For each of the following sets say if it is open, closed, bounded, compact.

i) $D := \{(x, y) \in \mathbb{R}^2 \ : \ xy > 0\}$.
ii) $D := \{(x, y) \in \mathbb{R}^2 \ : \ |xy| \leqslant 1\}$.
iii) $D := \{(x, y) \in \mathbb{R}^2 \ : \ |x| + |y| < 1\}$.
iv) $D := \{(x, y) \in \mathbb{R}^2 \ : \ 1 \leqslant xy \leqslant 2,\ x \leqslant y \leqslant 2x\}$.
v) $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + 2y^2 + 3z^4 < 4\}$.
vi) $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ z \geqslant x^2 + y^2,\ x^2 + z^2 \leqslant 1\}$.
vii) $D := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 - y^2 + z^2 \leqslant 1\}$.
viii) $D := \{(x, y, z) \in \mathbb{R}^3 \ : xy = z + 1,\ x^2 + y^2 \leqslant 1\}$.

Exercise 1.8.16 (∗∗∗)**.** Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. Prove that if D is compact, then also $\vec{F}(D)$ is compact. (hint: to prove that $\vec{F}(D)$ is closed use Cantor's characterization; to prove that $\vec{F}(D)$ is bounded argue by contradiction)

Exercise 1.8.17 (∗∗∗)**.** Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. Prove that if D is connected, then also $\vec{F}(D)$ is connected.

CHAPTER 2

Differential Calculus

In this chapter we extend the *Differential Calculus* to the case of *vector-valued functions of several variables*,
$$\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m.$$
The extension to the multi dimensional case of the concept of derivative is not straightforward. Indeed, we cannot just write
$$\vec{F}'(\vec{x}) := \lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x})}{\vec{h}}$$
because the division by a vector is not defined. We will see how to solve this issue to get a proper definition of derivative or, how it is more properly called, of *differential*. In this Chapter, we will illustrate applications to optimization techniques to solve unconstrained and constrained optimization problems.

Chapter requirements: basic knowledge of vector spaces, limits for function of one and several real variable, matrices and their algebraic properties (in particular: rank of a matrix).

Learning objectives:

- (basic $*$) definitions of partial derivative, directional derivative and differential for a function of several variables, gradient vector and jacobian matrix, stationary points and their role in the search of extrema, hessian matrix, being able to solve simple optimization problems.
- (intermediate $**$) solving standard unconstrained and constrained optimization problems for function of several variables, doing proofs with straightforward arguments or adapting other proofs.
- (advanced $***$) solving non standard optimization problems, doing proofs with non standard arguments.

2.1. Directional derivative

The first way to bypass the technical difficulty of dividing by $\vec{h}$ is to assume that $\vec{h} = t\vec{v}$ where $t \in \mathbb{R}$ is variable and $\vec{v} \in \mathbb{R}^d$ is fixed, and considering $t \longrightarrow 0$. This leads to the

Definition 2.1.1: directional derivative

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{x} \in \mathrm{Int}(D)$. We call **directional derivative of $\vec{F}$ at point $\vec{x}$ along $\vec{v} \neq \vec{0}$** the limit
$$\partial_{\vec{v}}\vec{F}(\vec{x}) := \lim_{t\to 0} \frac{\vec{F}(\vec{x}+t\vec{v}) - \vec{F}(\vec{x})}{t} \in \mathbb{R}^m.$$

Example 2.1.2: ($*$)

Q. *Compute $\partial_{(1,1)} f(0,0)$ for $f(x,y) = x\cos y$.*

A. We have
$$\partial_{(1,1)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,t)-f(0,0)}{t} = \lim_{t\to 0} \frac{t\cos t}{t} = \lim_{t\to 0} \cos t = 1.$$

The directional derivative does not provide a satisfactory definition of derivative. This because it may happens that all the $\partial_{\vec{v}}\vec{F}(\vec{x})$ exists but $\vec{F}$ is not even continuous! This phenomenon is well illustrated by the following example.

Example 2.1.3: $(**)$

Q. *Let*

$$f(x,y) := \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y) \neq \vec{0}, \\ \\ 0, & (x,y) = \vec{0}. \end{cases}$$

Show that f admits all the directional derivatives at $\vec{0}$ but it is not therein continuous.

A. **Continuity.** We have $f(x,0) = f(0,y) \equiv 0 \longrightarrow 0$. However, along the section $y = x^2$ we have

$$f(x,x^2) = \frac{x^2x^2}{x^4+x^4} = \frac{1}{2} \longrightarrow \frac{1}{2} \neq f(0,0) = 0.$$

Therefore $\nexists \lim_{(x,y)\to\vec{0}} f(x,y)$ and consequently the function cannot be continuous.

Directional derivatives. Let us prove now that $\exists \partial_{\vec{v}} f(0,0)$ for any $\vec{v}$. Let $\vec{v} = (a,b) \neq \vec{0}$. We have

$$\partial_{\vec{v}} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(a,b)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(ta,tb)}{t} = \lim_{t\to 0} \frac{\frac{t^3a^2b}{t^2(t^2a^4+b^2)}}{t} = \lim_{t\to 0} \frac{a^2b}{t^2a^4+b^2},$$

that is

$$\vec{v} = (a,b), \implies \exists \partial_{\vec{v}} f(0,0) = \begin{cases} 0, & \text{if } b = 0 \text{ (and of course } a \neq 0), \\ \\ \frac{a^2}{b^2}, & \text{if } b \neq 0. \end{cases}$$

Conclusion: we proved that f has all possible directional derivatives, yet it is not continuous.

Even if the directional derivative is not the right definition of derivative, some special directional derivatives have great relevance:

Definition 2.1.4

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, $\vec{x} \in \mathrm{Int}(D)$ and let $\vec{e}_1, \ldots, \vec{e}_d$ be the canonical base of $\mathbb{R}^d$, that is $e_j = (0,\ldots,0,1,0,\ldots,0)$ with 1 in the j−th place. We call **partial derivative of f with respect to the j−th variable at point $\vec{x}$** the

$$\partial_{\vec{e}_j} f(\vec{x}).$$

We will use the notation $\partial_j f(\vec{x})$ (a very popular notation is $\frac{\partial f}{\partial x_j}$).

Partial derivative ∂_j is just an ordinary derivatives w.r.t x_j considering all other variables x_i $i \neq j$ as fixed parameters:

$$\begin{aligned} \partial_j f(x) \quad &= \lim_{t\to 0} \frac{f\big((x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_d) + t(0,\ldots,0,1,0,\ldots,0)\big) - f(x_1,\ldots,x_d)}{t} \\ &= \lim_{t\to 0} \frac{f\left(x_1,\ldots,x_{j-1},x_j+t,x_{j+1},\ldots,x_d\right) - f(x_1,\ldots,x_{j-1},x_j,x_{j+1},\ldots,x_d)}{t} \end{aligned}$$

In low dimensional spaces like $\mathbb{R}^2$ or $\mathbb{R}^3$, we do not use indexes for the coordinates but, rather, just different letters like x, y, z. In this case we denote partial derivatives with symbols like ∂_x, ∂_y and ∂_z. So, for example,

$$\partial_x (y\sin x) = y\cos x, \quad \partial_y (y\sin x) = \sin x.$$

2.2. Differentiability

We already noticed that we cannot use the one dimensional definition of derivative,

$$\vec{F}'(\vec{x}) := \lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x})}{\vec{h}}$$

to define $\vec{F}'(\vec{x})$. For numerical functions, the definition of derivative can be rephrased as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \iff \lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = 0,$$

that is

$$f(x+h) - f(x) - f'(x)h = o(h).$$

Here, $f'(x)$ is a number and $f'(x)h$ is the algebraic product between $f'(x)$ and h. In general, we could write

$$\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h} = o(\vec{h}).$$

We now show that we can give a precise meaning to this relation. The first point is on the interpretation of $\vec{F}'(\vec{x})\vec{h}$. Since $\vec{F}(\vec{x}+\vec{h}), \vec{F}(\vec{x}) \in \mathbb{R}^m$, also $\vec{F}'(\vec{x})\vec{h} \in \mathbb{R}^m$. Moreover, reasonably $\vec{h} \longmapsto \vec{F}'(\vec{x})\vec{h}$ must be a *linear map* transforming $\vec{h} \in \mathbb{R}^d$ into $\vec{F}'(\vec{x})\vec{h} \in \mathbb{R}^m$. In other words

$$\vec{F}'(\vec{x}) \in \mathsf{L}(\mathbb{R}^d, \mathbb{R}^m).$$

Such type of transformation is represented by an $m \times d$ (lines$\times$ columns) matrix. Thus, we expect that $\vec{F}'(\vec{x})$ is a matrix. The second point involves the interpretation of $o(\vec{h})$. We cannot say that $g(\vec{h}) = o(\vec{h})$ if

$$\frac{g(\vec{h})}{\vec{h}} \longrightarrow \vec{0},$$

because we cannot divide per $\vec{h}$. However, since the goal is to say that $g(\vec{h})$ is smaller order than $\vec{h}$, it should be smaller than $\|\vec{h}\|$ as well. This yields the

Definition 2.2.1

Let $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. We say that $\vec{F}$ is **differentiable** at point $\vec{x}$ if there exists an $m \times d$ matrix (denoted by $\vec{F}'(\vec{x})$) such that

$$\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h} = o(\vec{h}),$$

that is,

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h}) - \vec{F}(\vec{x}) - \vec{F}'(\vec{x})\vec{h}}{\|\vec{h}\|} = \vec{0}. \tag{2.2.1}$$

The matrix $\vec{F}'(\vec{x})$ is also called the **jacobian matrix** of $\vec{F}$ at the point $\vec{x}$.

Before we launch in computing differentials, let us notice that, differently from directional differentiability, the definition given above implies continuity.

Proposition 2.2.2

If $\vec{F}$ is differentiable at $\vec{x}$, then it is therein continuous.

Proof. Just notice that $\vec{F}(\vec{y}) = \vec{F}(\vec{x}) + \vec{F}'(\vec{x})(\vec{y}-\vec{x}) + o(\vec{y}-\vec{x}) \longrightarrow \vec{F}(\vec{x})$. □

How do we determine the entries of the Jacobian matrix in practice?

Proposition 2.2.3

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be differentiable at point $\vec{x}$. Then, $\vec{F}$ has all the directional derivatives at point $\vec{x}$ and

$$\partial_{\vec{v}}\vec{F}(\vec{x}) = \vec{F}'(\vec{x})\vec{v},\ \forall \vec{v} \in \mathbb{R}^d. \tag{2.2.2}$$

In particular, if $\vec{F} = (f_1, \dots, f_m)$ then

$$\vec{F}'(\vec{x}) = \begin{bmatrix} \partial_1 f_1(\vec{x}) & \partial_2 f_1(\vec{x}) & \dots & \partial_d f_1(\vec{x}) \\ \vdots & \vdots & & \vdots \\ \partial_1 f_m(\vec{x}) & \partial_2 f_m(\vec{x}) & \dots & \partial_d f_m(\vec{x}) \end{bmatrix}. \tag{2.2.3}$$

Proof. Let us prove (2.2.2). Fix $\vec{v} \neq 0$. Then, since $\vec{F}$ is differentiable at $\vec{x}$,

$$\vec{F}(\vec{x}+t\vec{v}) - \left(\vec{F}(\vec{x}) + \vec{F}'(\vec{x})(tv)\right) = o(t\vec{v}), \implies \frac{\vec{F}(\vec{x}+t\vec{v}) - \vec{F}(\vec{x})}{t} = \vec{F}'(\vec{x})\vec{v} + \frac{o(t\vec{v})}{t}.$$

Now, since

$$\lim_{t\to 0}\left\|\frac{o(t\vec{v})}{t}\right\| = \lim_{t\to 0}\frac{\|o(t\vec{v})\|}{|t|} = \|\vec{v}\|\lim_{t\to 0}\frac{\|o(t\vec{v})\|}{\|t\vec{v}\|} = \|\vec{v}\|\lim_{\vec{h}\to\vec{0}}\frac{\|o(\vec{h})\|}{\|\vec{h}\|} = 0,$$

we conclude that

$$\partial_{\vec{v}}\vec{F}(\vec{x}) = \lim_{t\to 0}\frac{\vec{F}(\vec{x}+t\vec{v}) - \vec{F}(\vec{x})}{t} = \vec{F}'(\vec{x})\vec{v}.$$

Let's prove now the (2.2.3): if we call $\vec{F}'(\vec{x}) = [a_{ij}]$, it is well known by Linear Algebra that

$$\vec{F}'(\vec{x})e_j$$

gives the $j-$th column of the matrix $\vec{F}'(\vec{x})$. So, the element a_{ij} of $\vec{F}'(\vec{x})$ is obtained by taking the $i-$th component of the vector $\vec{F}'(\vec{x})\vec{e}_j$. But: by (2.2.2) we have

$$\vec{F}'(\vec{x})\vec{e}_j = \partial_{\vec{e}_j}\vec{F}(\vec{x}) = \partial_j\vec{F}(\vec{x}) = \left(\partial_j f_1(\vec{x}), \partial_j f_2(\vec{x}), \dots, \partial_j f_m(\vec{x})\right),$$

hence the $i-$th component is $\partial_j f_i(\vec{x})$, and this proves (2.2.3). □

We mention a couple of important particalr cases of the Jacobian matrix:

- $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$: in this case, $f'(\vec{x})$ is a $1 \times d$ matrix, precisely

$$f'(\vec{x}) = \left[\partial_1 f(\vec{x})\ \partial_2 f(\vec{x})\ \dots\ \partial_d f(\vec{x})\right] =: \nabla f(\vec{x}).$$

  The vector $\nabla f(\vec{x})$ is called **gradient of f at point $\vec{x}$**. In this case

$$f'(\vec{x})\vec{h} = \nabla f(\vec{x}) \cdot \vec{h},$$

  where we denoted by $\cdot$ the scalar product of vectors in $\mathbb{R}^d$.
- $\vec{\gamma} = \vec{\gamma}(u) : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}^d$: in this case $\vec{\gamma}'(u)$ is a $d \times 1$ matrix, precisely

$$\vec{\gamma}'(u) = \begin{bmatrix} \gamma_1'(u) \\ \vdots \\ \gamma_d'(u) \end{bmatrix}.$$

Warning 2.2.4

The jacobian matrix (2.2.3) is made of partial derivatives: are these sufficient to say that $\vec{F}$ is differentiable? The answer is **no**. Indeed, Example 2.1.3 shows a function with all the directional derivatives (hence all partial derivatives) which is not continuous. Since, by Proposition 2.2.2, differentiability $\Longrightarrow$ continuity, we conclude that such a function **cannot** be differentiable. The conclusion is: **the existence of partial derivatives does not imply (alone) that the function is differentiable.**

So: *how can we check differentiability?* Let us see an example.

Example 2.2.5: $(**)$

Q. *Discuss the differentiability at* $(0,0)$ *of*

$$f(x,y) := \begin{cases} \frac{x^2y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

A. Here, the jacobian matrix reduces to the gradient vector $\nabla f(\vec{0}) = (\partial_x f(0,0), \partial_y f(0,0))$. For $(x,y) \neq \vec{0}$,

$$\partial_x f(x,y) = \partial_x \frac{x^2y^2}{x^2+y^2} = \frac{2xy^2(x^2+y^2) - x^2y^2 2x}{(x^2+y^2)^2} = \frac{2xy^4}{(x^2+y^2)^2},$$

but we cannot just plug $(x,y) = (0,0)$ here. To compute $\partial_x f(0,0)$ we need to invoke the definition:

$$\partial_x f(0,0) = \partial_{(1,0)} f(0,0) = \lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(t,0)}{t} = \lim_{t\to 0} \frac{0}{t} = 0.$$

Similarly $\partial_y f(0,0) = 0$. Thus, $\nabla f(0,0) = (0,0)$. To check differentiability we have to check that (2.2.1) holds, that is

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0})\cdot\vec{h}}{\|\vec{h}\|} = 0.$$

Set $\vec{h} = (u,v)$, then

$$f(\vec{0}+\vec{h}) - f(\vec{0}) - \nabla f(\vec{0})\cdot\vec{h} = f(u,v) - 0 - (0,0)\cdot(u,v) = f(u,v).$$

Therefore, we have to prove that

$$\lim_{(u,v)\to(0,0)} \frac{f(u,v)}{\|(u,v)\|} = 0. \ (\star)$$

This is a limit in $\mathbb{R}^2$ that we compute by using the methods of the previous chapter. Notice that,

$$\frac{f(u,v)}{\|(u,v)\|} = \frac{\frac{u^2v^2}{u^2+v^2}}{\sqrt{u^2+v^2}} = \frac{u^2v^2}{(u^2+v^2)^{3/2}} \overset{u=\rho\cos\theta,\ v=\rho\sin\theta}{=} \frac{\rho^4(\cos\theta)^2(\sin\theta)^2}{\rho^3} = \rho(\cos\theta)^2(\sin\theta)^2,$$

hence

$$\left|\frac{f(u,v)}{\|(u,v)\|}\right| \leqslant \rho \longrightarrow 0, \text{ as } \rho \longrightarrow 0.$$

We conclude that $(\star)$ holds, whence f is differentiable at $(0,0)$.

A useful test is provided by the following Proposition:

Proposition 2.2.6: differentiability test

Let $\vec{F} = (f_1, \ldots, f_m) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. If

$$\partial_j f_i \in \mathsf{C}(D),\ \forall i, j,\ \Longrightarrow\ \exists f \text{ is differentiable at any } \vec{x} \in D$$

Since the entries of the jacobian matrix $\vec{F}'$ are continuous, $\vec{F}'$ is itself continuous. We write $\vec{F} \in \mathsf{C}^1(D)$.

The proof is not difficult but a bit technical, see Exercises.

Example 2.2.7: $(**)$

Q. *Applying the differentiability test, discuss differentiability for the function of the Example 2.2.5.*

A. At $(x, y) \neq \vec{0}$ we may say that

$$\partial_x f(x, y) = \partial_x \frac{x^2 y^2}{x^2 + y^2} = y^2 \frac{2x(x^2 + y^2) - x^2 \cdot 2x}{(x^2 + y^2)^2} = \frac{2xy^4}{(x^2 + y^2)^2} \in \mathsf{C}(\mathbb{R}^2 \backslash \{\vec{0}\}),$$

and, similarly, $\partial_y f(x, y) \in \mathsf{C}(\mathbb{R}^2 \backslash \{\vec{0}\})$. So far, the differentiability test applies on $D = \mathbb{R}^2 \backslash \{\vec{0}\}$ to conclude that f is therein differentiable.
At $\vec{0}$, we computed above $\partial_x f(0, 0) = \partial_y f(0, 0) = 0$. Thus, to apply the differentiability test we need to check if

$$\lim_{(x,y) \to \vec{0}} \partial_x f(x, y) = \partial_x f(0, 0), \quad \Longleftrightarrow \quad \lim_{(x,y) \to \vec{0}} \frac{2xy^4}{(x^2 + y^2)^2} = 0.$$

Introducing polar coordinates,

$$\left| \frac{2xy^4}{(x^2 + y^2)^2} \right| = \left| \frac{2\rho^5 \cos\theta (\sin\theta)^4}{\rho^4} \right| = 2\rho |\cos\theta| (\sin\theta)^4 \leqslant 4\rho \longrightarrow 0, \text{ when } \rho \longrightarrow 0.$$

Thus $\partial_x f$ is continuous at $\vec{0}$. Similarly $\partial_y f(x, y)$ is continuous at $\vec{0}$. In conclusion: f is differentiable also at $\vec{0}$, whence on the entire $\mathbb{R}^2$.

Warning 2.2.8

The differentiability test 2.2.6 is only a **sufficient** (but not necessary) condition to ensure differentiability. It may happen that the assumption fails (that is $\partial_j f_i$ are not continuous) but $\vec{F}$ is still differentiable. See Exercise 2.9.4

We can now sketch some strategy on how to check differentiabilty.

Check List 2.2.9: Checking differentiability

To check the differentiability of a function $\vec{F} = (f_1, \ldots, f_m)$:

(1) Compute the jacobian matrix (that is $\partial_j f_i$ for all j, i).
(2) Apply the differentiability test: where all the $\partial_j f_i$ are continuous (so, check continuity of $\partial_j f_i$ if not evident), $\vec{F}$ is differentiable.
(3) If the differentiability test does not apply, the unique possibility is to go back to the definition and check that limit (2.2.1) holds.

Rules of calculus of differentials basically follows the same rules of those of ordinary calculus. For instance

$$(\vec{F}+\vec{G})'(\vec{x}) = \vec{F}'(\vec{x}) + \vec{G}'(\vec{x}).$$

provided $\vec{F}$ and $\vec{G}$ are differentiable at $\vec{x}$. Similarly it holds the important

Proposition 2.2.10: chain rule

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$, $\vec{G} : E \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$, be such that

i) $\exists \vec{F}'(\vec{x})$;

ii) $\exists \vec{G}'(\vec{F}(\vec{x}))$.

Then

$$\exists\ (\vec{G}\circ\vec{F})'(\vec{x}) = \vec{G}'(\vec{F}(\vec{x}))\vec{F}'(\vec{x}). \tag{2.2.4}$$

Warning 2.2.11

While for numerical functions we can write $(g \circ f)'(x) = g'(f(x))f'(x) = f'(x)g'(f(x))$, the same does not hold for vector valued functions of vector variable. Just remind that, since $\vec{F} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$, its jacobian matrix is an $d \times m$ matrix, while $\vec{G} : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ has an $m \times k$ jacobian matrix. Therefore, in general, in the multiplication lines times columns, only $\vec{G}'(\vec{F}(\vec{x}))\vec{F}'(\vec{x})$ makes sense.

A special case of (2.2.4) is the following: suppose that we want to compute

$$\frac{d}{dt} f(\vec{\gamma}(u)),\ \text{where } \vec{\gamma} : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d,\quad f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}.$$

For example, in physics f represents the physical energy of the system, $\vec{\gamma}$ a trajectory of motion. Then $f(\vec{g}(u))$ represents the variation of energy along the trajectory of motion. Calculating its ordinary derivative, we aim to measure the *rate of variation* of this quantity. Assuming that all the required hypotheses are fulfilled, we have the following.

$$\frac{d}{dt} f(\vec{\gamma}(u)) = f'(\vec{\gamma}(u))\vec{\gamma}'(u) = [\partial_1 f(\vec{\gamma}(u))\ \ldots\ \partial_d f(\vec{\gamma}(u))] \begin{bmatrix} \gamma_1'(u) \\ \vdots \\ \gamma_d'(u) \end{bmatrix} = \nabla f(\vec{\gamma}(u)) \cdot \vec{\gamma}'(u). \tag{2.2.5}$$

This quantity is also called **total derivative of f along $\vec{\gamma}$**.

2.3. Extrema

Among the most relevant applications of Differential Calculus there is a method for the search of min/max points of a function. The extension of results of one-variable calculus to the case of a numerical function of vector variable, $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, is not straightforward. The major obstacle is that there is not an ordering of vectors. So, for example, we cannot say that $\nabla f(\vec{x}) \geqslant 0$ implies $f \nearrow$ because none of these properties make sense here. This makes more important here the role of *local extrema*:

Definition 2.3.1: local min/max point

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. We say that a point $\vec{x}_* \in D$ is a **local minimum point for** f if

$$\exists B(\vec{x}_*, r]\ :\ f(\vec{x}_*) \leqslant f(\vec{x}),\ \forall \vec{x} \in B(\vec{x}_*, r] \cap D.$$

Local maximum point for f is defined similarly.

The most important fact of this Section is the

Theorem 2.3.2: Fermat

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\vec{x}_* \in \mathrm{Int}(D)$ be a local min/max. If f is differentiable at $\vec{x}_*$ then $\nabla f(\vec{x}_*) = \vec{0}$.

Proof. We consider the case of $\vec{x}_*$ a local minimum, the case of a local maximum is similar. Thus, we assume that

$$\exists\, B(\vec{x}_*, r],\ :\ f(\vec{x}) \leqslant f(\vec{x}_*),\ \forall \vec{x} \in B(\vec{x}_*, r] \cap D.$$

Since $\vec{x}_* \in \mathrm{Int}(D)$, we may assume directly that $B(\vec{x}_*, r] \subset D$. Now, consider the section of f on a straight line passing by $\vec{x}_*$. Fixed a direction $\vec{v}$, this is described by

$$\vec{\gamma}(t) = \vec{x}_* + t\vec{v}.$$

Notice that

$$\vec{\gamma}(t) \in B(\vec{x}_*, r],\ \iff\ r \geqslant \|(\vec{x}_* + t\vec{v}) - \vec{x}_*\| = \|t\vec{v}\| = |t|\|\vec{v}\|,\ \iff\ |t| \leqslant \frac{r}{\|\vec{v}\|}.$$

Thus, if $g(t) := f(\vec{\gamma}(t))$, we have

$$g(t) = f(\vec{\gamma}(t)) \geqslant f(\vec{x}_*) = f(\vec{\gamma}(0)) = g(0),\ \forall |t| \leqslant \frac{r}{\|\vec{v}\|}.$$

In particular, g ha a minimum at $t = 0$. According to Fermat's Theorem for functions of one real variable, $g'(0) = 0$. Recalling the total derivative formula (2.2.5),

$$g'(t) = \nabla f(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t),$$

so, taking $t = 0$, we have,

$$0 = \nabla f(\vec{x}_*) \cdot \vec{v}.$$

This relation holds for every $\vec{v} \in \mathbb{R}^d$. So, setting $\vec{v} := \nabla f(\vec{x}_0)$ we get

$$0 = \nabla f(\vec{x}_0) \cdot \nabla f(\vec{x}_0) = \|\nabla f(\vec{x}_0)\|^2,\ \implies\ \nabla f(\vec{x}_0) = \vec{0}. \quad \square$$

Definition 2.3.3: stationary point

A point $\vec{x}_*$ where f is differentiable and $\nabla f(\vec{x}_*) = \vec{0}$ is called **stationary point** for f.

Warning 2.3.4

According to Fermat's theorem, a local min/max point which lies in the **interior** of D is a stationary point.

- the vice-versa is false (see Example 2.3.5).
- if the min/max point is not an interior point, then ∇f is not necessarily $= \vec{0}$ (see Example 2.3.6)

Example 2.3.5: $(*)$

Q. *Let $f(x,y) = x^2 - y^2$ on $D = \mathbb{R}^2$. Determine the stationary points of f (if any) and determine if they are min/max points or not.*

A. Clearly, f is differentiable on $\mathbb{R}^2$ and $\nabla f(x,y) = (2x, -2y)$. Therefore, $\nabla f = \vec{0}$ iff $(x,y) = (0,0)$. We easily see that $(0,0)$ is neither a minimum nor a maximum. Indeed, notice that

$$f(x,0) = x^2 \geqslant 0 = f(0,0), \quad f(0,y) = -y^2 \leqslant 0 = f(0,0).$$

Since every ball $B(\vec{0}, r]$ contains points of type $(x,0)$ and $(0,y)$, we see that $(0,0)$ is neither a local min nor local max for f.

Example 2.3.6: $(*)$

Q. *Let $f(x,y) := x + y$ on $D := \{(x,x) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1\}$. Determine all possible min/max points of f on D. Are these points stationary points?*

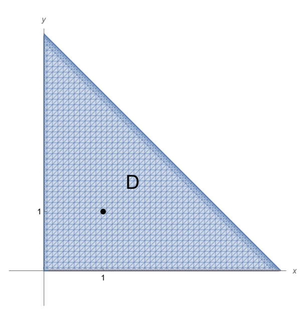
A. The set D is the segment joining $(0,0)$ to $(1,1)$, it is clearly closed and bounded (we can represent D as $\{(x,y) \in \mathbb{R}^2 \ : \ 0 \leqslant x \leqslant 1, \ y = x\}$). We directly see that $(0,0)$ is global minimum for f on D while $(1,1)$ is the global maximum (indeed $f(0,0) = 0 \leqslant 2x = f(x,x) \leqslant 2 = f(1,1)$ for every $(x,x) \in D$, that is for $0 \leqslant x \leqslant 1$). There are no other extrema (local or global). Moreover, f is differentiable and $\nabla f(x,y) \equiv (1,1)$, so none of the extreme points is a stationary point for f.

Despite their partial information, stationary points can still be of some help in the search for extrema. We illustrate how with some examples.

Example 2.3.7: $(*)$

Q. *Determine global min/max for $f(x,y) = x + y - xy$ on $D = \{(x,y) \in \mathbb{R}^2 \ : \ x \geqslant 0, \ 0 \leqslant y \leqslant 3 - x\}$.*

A. Notice first that $f \in \mathsf{C}(D)$ and that D is closed (defined through large inequalities involving continuous functions) and bounded ($x \geqslant 0$ and, by $0 \leqslant 3 - x$ we have also $x \leqslant 3$, thus $0 \leqslant x \leqslant 3$, and from this $0 \leqslant y \leqslant 3$). Therefore, D is compact, and Weierstrass' theorem applies, ensuring for the existence of global min/max for f on D.



Now that we know global min/max points exist, let us determine them. Clearly, global points are also local points, so we may use Fermat's theorem. This however requires some care, because Fermat's theorem applies when min/max points lie in the interior of D. So, we may proceed as follows:

- if $(x,y) \in \mathrm{Int}(D) = \{x > 0, 0 < y < 4 - x\}$, then according to Fermat's theorem, $\nabla f(x,y) = \vec{0}$. Since $\nabla f(x,y) = (1-y, 1-x)$, we have

$$\nabla f(x,y) = (0,0), \iff \begin{cases} 1 - y = 0, \\ \\ 1 - x = 0, \end{cases} \iff (x,y) = (1,1).$$

  Since $(1,1) \in D$, this point is a possible candidate to be min/max point for f.
- if $(x,y) \notin \mathrm{Int}(D)$, that is $(x,y) \in \partial D$, the condition $\nabla f(x,y) = \vec{0}$ is of no help. We notice that

$$\begin{aligned} \partial D &= \{(x,0) : 0 \leqslant x \leqslant 4\} \cup \{(0,y) : 0 \leqslant y \leqslant 4\} \cup \{(x, 4-x) : 0 \leqslant x \leqslant 4\} \\ \\ &=: \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{aligned}$$

  On Γ_1: $f(x,0) = x$ with $x \in [0,4]$, so the min point of f on Γ_1 is achieved at $x = 0$ (point $(0,0)$) while max point of f on Γ_1 is achieved at $x = 4$ (point $(4,0)$). These points are, for now, min/max for f just on Γ_1.
  On Γ_2 we have a similar discussion because $f(0,y) = y$ with $y \in [0,4]$. Thus $(0,0)$ is a min for f on Γ_2 while $(0,4)$ is a max for f on Γ_2.
  On Γ_3, $f(x, 4-x) = x + (4-x) - x(4-x) = 4 - 4x + x^2 = (x-2)^2$, and this quantity is minimum for $x = 2$ (thus f has a min on Γ_3 at point $(2,2)$), maximum at $x = 0, 4$ (thus f has a maximum on Γ_3 at points $(4,0)$ and $(0,4)$).

Therefore

- max points: the candidates are $(1,1)$ (stationary point in $\mathrm{Int}(D)$), $(4,0)$, $(0,4)$. Since

$$f(1,1) = 1, \quad f(4,0) = f(0,4) = 4,$$

  we conclude that $(4,0)$ and $(0,4)$ are max points for f on D.
- min points: the candidates are $(1,1)$, $(0,0)$ and $(2,2)$. Since

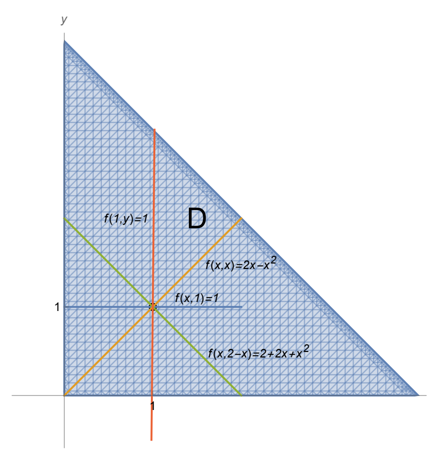
$$f(1,1) = 1, \ f(0,0) = 0, \ f(2,2) = 0,$$

  we conclude that $(0,0)$ and $(2,2)$ are minimum points for f on D.

Someone may ask: *what is, then, the point* $(1,1)$ *for* f*?* Let us give a look at few sections of f passing through point $(1,1)$. We will consider the following ones:

$$y = 1, \quad x = 1, \quad y = x, \quad y = 2 - x.$$

We have $f(x,1) = x + 1 - x \cdot 1 = 1$, that is f is constant, and similarly $f(1,y) \equiv 1$. On $y = x$, we have $f(x,x) = 2x - x^2 =: g(x)$. Since $g'(x) = 2 - 2x \geqslant 0$ iff $x \leqslant 1$, we see that $x = 1$ is a max point for g, thus $(1,1)$ is max for f along curve $y = x$. This might lead to think that $(1,1)$ is perhaps a (local) maximum. However, when we consider $y = 2 - x$, we see that $f(x, 2-x) = x + (2-x) - x(2-x) = 2 - 2x + x^2 = 1 + (x-1)^2$ from which we clearly see that $x = 1$ is a minimum for this quantity. That is, $(1,1)$ is a min point for f along $y = 2 - x$. Since we have two sections along which the same point $(1,1)$ is in one case a minimum, in the other a maximum for f, we conclude that $(1,1)$ is neither a local minimum nor a local maximum for f. $\square$



The example suggests a definition:

Definition 2.3.8: saddle point

Let $\vec{x}_*$ be a stationary point for f. If there exist two different curves $\vec{\gamma}_1, \vec{\gamma}_2 : I \subset \mathbb{R} \longrightarrow \mathbb{R}^d$ such that

- both pass through point $\vec{x}_*$ (that is $\vec{\gamma}_1(t_0) = \vec{\gamma}_2(t_0) = \vec{x}_*$ for some $t_0 \in I$)
- $\vec{x}_*$ is max point for f along $\vec{\gamma}_1$ and min point for f along $\vec{\gamma}_2$, that is

$$f(\vec{\gamma}_1(t)) < f(\vec{x}_0) < f(\vec{\gamma}_2(t)), \; \forall t \in I \backslash \{t_0\},$$

then we say that $\vec{x}_*$ is a **saddle point** for f.

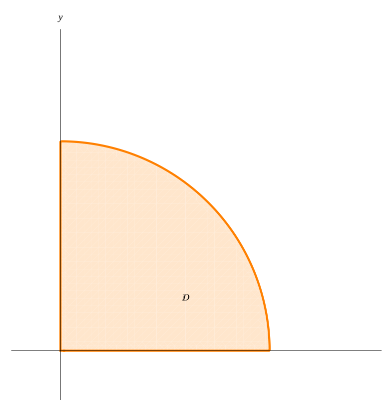
Example 2.3.9: $(**)$

Q. *Let*

$$f(x,y) = xye^{x^2+y^2}, \quad (x,y) \in D := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0, \; y \geqslant 0, \; x^2+y^2 \leqslant 1\}.$$

Is D open? closed? compact? connected? Determine global min/max points of f on D and the image $f(D)$ of D through f.

A. D is defined by large inequalities, thus it is closed. It is not open because the boundary points are in D but not in $\mathrm{Int}(D)$. Clearly if $(x,y) \in D$ we have $\|(x,y)\| \leqslant 1$, so D is bounded, hence compact.
Existence min/max. Since f is continuous and D compact, Weierstrass's theorem applies, and f has both global min/max.
Determination. Let $(x,y) \in D$ be a min/max point. Then, either $(x,y) \in \mathrm{Int}(D)$ (hence, according to Fermat's theorem, $\nabla f(x,y) = \vec{0}$) or $(x,y) \in \partial D$ (and, in this case, $\nabla f(x,y)$ is not necessarily null).



Now,

$$\nabla f(x,y) = \left(e^{x^2+y^2}(y+2x^2y), e^{x^2+y^2}(x+2y^2x)\right) = 0, \iff \begin{cases} y(1+2x^2) = 0, \\ x(1+2y^2) = 0, \end{cases} \iff x = y = 0.$$

The unique stationary point for f is $(0,0)$ which is not in $\mathrm{Int}(D)$. This means that *there are not stationary points for f in* $\mathrm{Int}(D)$. In particular, min/max points are definitely in ∂D. Notice that

$$\partial D = \{(x,0) \; : \; 0 \leqslant x \leqslant 1\} \cup \{(0,y) \; : \; 0 \leqslant y \leqslant 1\} \cup \left\{(x,y) \; : \; x^2+y^2 = 1, \; x \geqslant 0, \; y \geqslant 0\right\} =: \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.$$

On Γ_1 we have $f(x,0) = 0$, so f is constant; on Γ_2 we have the same $f(0,y) = 0$. For Γ_3 it is convenient to represent points as

$$\Gamma_3 = \left\{(\cos\theta, \sin\theta) \; : \; \theta \in \left[0, \frac{\pi}{2}\right]\right\},$$

so that,

$$f(\cos\theta, \sin\theta) = (\cos\theta)(\sin\theta)e^1 = \frac{e}{2}\sin(2\theta).$$

Clearly, this quantity is maximum when $2\theta = \frac{\pi}{2}$, that is $\theta = \frac{\pi}{4}$, that is at point $\frac{1}{\sqrt{2}}(1,1)$, and minimum when $\theta = 0, \frac{\pi}{2}$, that is at points $(1,0)$ and $(0,1)$.

We can now draw the conclusion. Candidates min points are points $(x,0)$ ($x \in [0,1]$), $(0,y)$ ($y \in [0,1]$) where f is constantly equal to 0, hence all these points are minimum points. Candidates max points are, again, points $(x,0)$, $(0,y)$ (for $x, y \in [0,1]$) and $\frac{1}{\sqrt{2}}(1,1)$, where f takes value $\frac{e}{2}$. We conclude that there is a unique max point and it is $\frac{1}{\sqrt{2}}(1,1)$.
Finally: D is connected then, according to the intermediate values theorem, $f(D)$ is an interval, and precisely $f(D) = [0, \frac{e}{2}]$.

Example 2.3.10: ($**$)

Q. *Determine min/max (if any) of* $f(x,y,z) := x^2 + y^2 + z^2 - xy + z$ *on* $D := \mathbb{R}^3$. *What about* $f(D)$?

A. Here $D = \mathbb{R}^3$ is clearly closed and unbounded, therefore Weierstrass' theorem does not apply. Clearly $f \in \mathsf{C}(\mathbb{R}^3)$. Let us check if the limit at ∞_3 exists. Notice that $f(x,0,0) = x^2 \longrightarrow +\infty$ if $\|(x,0,0)\| \longrightarrow +\infty$, thus if $\lim_{(x,y,z)\to\infty_3} f(x,y,z)$ exists it must be $= +\infty$. To prove this, let us pass to spherical coordinates,

$$\begin{cases} x = \rho \sin\phi \cos\theta, \\ y = \rho \sin\phi \sin\theta, \\ z = \rho \cos\phi. \end{cases}$$

We have

$$f(x,y,z) = \rho^2 - \rho^2 \cos\theta \sin\theta \sin^2\phi + \rho\cos\phi = \rho^2 \left(1 - \frac{1}{2}\sin(2\theta)\sin^2\phi\right) + \rho\cos\phi.$$

Therefore,

$$f(x,y,z) \geqslant \rho^2 \left(1 - \frac{1}{2}\right) - \rho = \frac{\rho^2}{2} - \rho \longrightarrow +\infty, \text{ when } \rho \longrightarrow +\infty.$$

This proves that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. According to Proposition 1.6, f has global minimum on $\mathbb{R}^3$ (but, of course, there is no global maximum being f upper unbounded). This proves existence, let us now pass to the determination.
Let $(x,y,z) \in \mathbb{R}^3$ be a min point for f. Since clearly $(x,y,z) \in \mathrm{Int}(\mathbb{R}^3) = \mathbb{R}^3$, according to Fermat's theorem $\nabla f(x,y,z) = \vec{0}$. We have

$$\nabla f(x,y,z) = (2x - y, 2y - x, 2z - 1) = \vec{0}, \iff \begin{cases} 2x - y = 0, \\ 2y - x = 0, \\ 2z - 1 = 0, \end{cases} \iff \begin{cases} x = 0, \\ y = 0, \\ z = \frac{1}{2}. \end{cases}$$

Thus, the unique minimum is $(0,0,\frac{1}{2})$. Finally, since $D = \mathbb{R}^3$ is connected and $f \in \mathsf{C}$, $f(D) = [f(0,0,\frac{1}{2}), +\infty[= \left[\frac{1}{2}, +\infty\right[$.

Check List 2.3.11: Serching for extrema

To search for extrema of $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ differentiable,

(1) Existence: is D compact? If D is only closed and unbouded check, if possible, $\lim_{\vec{x}\to\infty_d} f(\vec{x})$, draw existence applying Weierstrass' theorem or its consequences.
(2) Determine (if any) stationary points in the interior $\mathrm{Int}(D)$.
(3) Determine (if any) min/max points on the boundary ∂D.
(4) With candidates of Step 2 & 3, determine min/max by direct comparison of values taken on such points.

2.4. Taylor's formula

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be differentiable at a point $\vec{p}$. According to the characterization (2.2.1), we can write

$$f(\vec{p} + \vec{h}) - f(\vec{p}) - \nabla f(\vec{p}) \cdot \vec{h} = o(\|\vec{h}\|),$$

or, equivalently, setting $\vec{x} = \vec{p} + \vec{h}$,

$$f(\vec{x}) = f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}) + o(\|\vec{x} - \vec{p}\|). \tag{2.4.1}$$

This formula is called **first order Taylor's formula of f centered at $\vec{p}$**. By analogy, the hyperplane of equation

$$y = f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}),$$

is called **tanget hyperplane** to f at $\vec{p}$.

In first year Calculus, we have shown that by adding some requirement on the regularity of the function, Taylor's formula can be improved by an approximation through a polynomial of higher degree and with a small remainder. For example, Peano's theorem states that if $f = f(x) : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable in a neighborhood of $p \in D$ and $\exists f''(p)$, then

$$f(x) = f(p) + f'(p)(x - p) + \frac{f''(p)}{2}(x - p)^2 + o((x - p)^2).$$

Our goal here is to extend this formula to the case of a function $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. The first step is to understand what is f'' in this case. Recalling that the derivative of f is identified with a vector valued function $\nabla f = \nabla f(\vec{x}) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, we have the following Definition.

Definition 2.4.1: second derivative

We say that $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ is **twice differentiable at $\vec{p} \in D$** if

- f is differentiable on $B(\vec{p}, r] \subset D$, for some $r > 0$.
- $\nabla f = (\partial_1 f, \ldots, \partial_d f)$ is differentiable at $\vec{p}$.

The jacobian matrix of ∇f is denoted by $\nabla^2 f(\vec{p})$, and it is called **hessian matrix**. We have:

$$\nabla^2 f(\vec{p}) = \begin{bmatrix} \nabla(\partial_1 f) \\ \nabla(\partial_2 f) \\ \vdots \\ \nabla(\partial_d f) \end{bmatrix}(\vec{p}) \equiv \begin{bmatrix} \partial_1(\partial_1 f) & \partial_2(\partial_1 f) & \ldots & \partial_d(\partial_1 f) \\ \partial_1(\partial_2 f) & \partial_2(\partial_2 f) & \ldots & \partial_d(\partial_2 f) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1(\partial_d f) & \partial_2(\partial_d f) & \ldots & \partial_d(\partial_d f) \end{bmatrix}(\vec{p}) \tag{2.4.2}$$

Example 2.4.2: $(*)$

Q. *Compute the hessian matrix of $f(x, y) = x^2 - y^2$ at a generic $(x, y) \in \mathbb{R}^2$.*

A. Since f is a polynomial, f is differentiable on $\mathbb{R}^2$ and

$$\partial_x f(x, y) = 2x, \quad \partial_y f(x, y) = 2y, \quad \Longrightarrow \quad \nabla f(x, y) = (2x, 2y).$$

It is now evident that also ∇f is differentiable on $\mathbb{R}^2$ and

$$\partial_x(\partial_x f) = 2, \;\; \partial_y(\partial_x f) = 0, \;\; \partial_y(\partial_y f) = 2, \;\; \partial_x(\partial_y f) = 0,$$

so

$$\nabla^2 f(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The entries of the Hessian matrix are second derivatives $\partial_j(\partial_i f)$. If $i \neq j$ we may think that doing first the derivative w.r.t. x_i and then w.r.t. x_j yields the same result than doing first the derivative w.r.t. x_j and then w.r.t. x_i.

This because x_i and x_j are "independent" variables. A bit surprisingly, this is not necessarily true and it requires some additional hypothesis (see also Exercise 2.9.5).

Theorem 2.4.3: Schwarz

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$. Then,

$$\partial_j(\partial_i f) \in \mathsf{C}(D),\ \forall j, i = 1, \dots, d, \implies \partial_j(\partial_i f) \overset{on\ D}{\equiv} \partial_i(\partial_j f),\ \forall j, i.$$

In this case, we pose $\partial_{ji} f := \partial_j(\partial_i f)$. In particular, the hessian matrix is symmetric.

We have now all the ingredients to extend first order Taylor's formula (2.4.1) to second order:

Theorem 2.4.4: Second order Taylor formula

Let $f : D \subset \mathbb{R}^d \subset \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable at $\vec{p} \in D$. Then,

$$f(\vec{x}) = f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}) + \frac{1}{2}\nabla^2 f(\vec{p})(\vec{x} - \vec{p}) \cdot (\vec{x} - \vec{p}) + o\left(\|\vec{x} - \vec{p}\|^2\right), \tag{2.4.3}$$

Proof. If $\vec{x} = \vec{p}$, formula (2.4.3) becomes a trivial identity. So, let us assume $\vec{x} \neq \vec{p}$ and set $\vec{u} := \frac{\vec{x}-\vec{p}}{\|\vec{x}-\vec{p}\|}$ (in this way $\|\vec{u}\| = 1$). Define also the auxiliary function

$$g(t) := f(\vec{p} + t\vec{u}).$$

We notice that $g(0) = f(\vec{p})$ and $g(\|\vec{x} - \vec{p}\|) = f(\vec{p} + (\vec{x} - \vec{p})) = f(\vec{x})$. Moreover, by one variable Calculus Taylor's formula for g at $t = 0$ we have

$$g(t) = g(0) + g'(0)t + \frac{1}{2}g''(0)t^2 + o(t^2). \tag{2.4.4}$$

Notice first that $g(0) = f(\vec{p})$. Then, by the *total derivative rule*,

$$g'(t) = \nabla f(\vec{p} + t\vec{u}) \cdot \vec{u}, \implies g'(0) = \nabla f(\vec{p}) \cdot \vec{u}.$$

Now, notice that

$$g'(t) = \nabla f(\vec{p} + t\vec{u}) \cdot \vec{u} = \sum_{i=1}^{d} \partial_i f(\vec{p} + t\vec{u}) u_i,$$

so, again by the total derivative rule,

$$g''(t) = \sum_{i=1}^{d} \left(\nabla(\partial_i f)(\vec{p} + t\vec{u}) \cdot \vec{u}\right) u_i = \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \partial_j(\partial_i f)(\vec{p} + t\vec{u}) u_j \right) u_i,$$

from which,

$$g''(0) = \sum_{i,j=1}^{d} \partial_j(\partial_i f)(\vec{p}) u_i u_j = \nabla^2 f(\vec{p})\vec{u} \cdot \vec{u}.$$

Therefore, setting $t = \|\vec{x} - \vec{p}\|$, from formula (2.4.4) we get

$$f(\vec{x}) = f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}) + \frac{1}{2}\nabla^2 f(\vec{p})(\vec{x} - \vec{p}) \cdot (\vec{x} - \vec{p}) + o\left(\|\vec{x} - \vec{p}\|^2\right),$$

which is the conclusion. □

Remark 2.4.5

What is the interpretation of

$$\nabla^2 f(\vec{p})(\vec{x}-\vec{p}) \cdot (\vec{x}-\vec{p}) \ ?$$

This the product lines by columns of the matrix $\nabla^2 f(\vec{x})$ with vector $\vec{x}-\vec{p}$, obtaining a vector. Then, we do the scalar product of this vector with $\vec{x}-\vec{p}$. Sometimes, this operation is denoted by

$$\nabla^2 f(\vec{p})(\vec{x}-\vec{p})^2$$

even if $(\vec{x}-\vec{p})^2$ does not make any sense!

2.5. Classification of stationary points

One of the most important applications of Taylor's formula is a test to classify stationary points. The key idea is in the following informal argument. If f is twice differentiable at a stationary point $\vec{p}$, then, by Taylor's formula (2.4.3), we have

$$f(\vec{x}) - f(\vec{p}) = \frac{1}{2}\nabla^2 f(\vec{p})(\vec{x}-\vec{p}) \cdot (\vec{x}-\vec{p}) + o\left(\|\vec{x}-\vec{p}\|^2\right).$$

Now, as we might expect, the remainder term is negligible w.r.t. the quadratic term, so

$$f(\vec{x}) - f(\vec{p}) \approx_{\vec{x}\to\vec{p}} \frac{1}{2}\nabla^2 f(\vec{p})(\vec{x}-\vec{p}) \cdot (\vec{x}-\vec{p}).$$

Thus, the sign of $f(\vec{x}) - f(\vec{p})$ depends on the sign of $\nabla^2 f(\vec{p})\vec{v} \cdot \vec{v}$. So, the idea is that if this sign is positive, $f(\vec{x}) > f(\vec{p})$, that is $\vec{p}$ is a minimum point, if sign is negative we have a maximum point while if sign can be both positive and negative then we have a saddle point. The scope of this section is to make this intuition precise. We need, here, to refresh some concepts from Linear Algebra.

2.5.1. Aside on positive/negative/indefinite matrices.

Let us start recalling the following definition:

Definition 2.5.1

Let M be a $d \times d$ matrix. We say that

- M is **strictly positive definite** (notation $M > 0$) if
$$M\vec{v} \cdot \vec{v} > 0, \ \forall \vec{v} \in \mathbb{R}^d \backslash \{\vec{0}\}.$$
- M is **strictly negative definite** ($M < 0$) if $-M > 0$.
- M is **indefinite** if
$$\exists \vec{u}, \vec{v} \neq 0, \ : \ M\vec{u} \cdot \vec{u} > 0 > M\vec{v} \cdot \vec{v}.$$

In general, it is not easy to determine whether a matrix M is > 0, < 0 or it is indefinite. There is, however, a sufficient condition in a particular case important for us:

Proposition 2.5.2

Let M be a $d \times d$ **symmetric** matrix and let $\lambda_1, \dots, \lambda_d$ its eigenvalues. Then:

i) $M > 0$ iff $\text{sgn}(\lambda_j) = +1$ for every $j = 1, \dots, d$;
ii) $M < 0$ iff $\text{sgn}(\lambda_j) = -1$ for every $j = 1, \dots, d$;
iii) M is indefinite if there exist $i \neq j$ such that $\text{sgn}(\lambda_i) = +1$ and $\text{sgn}(\lambda_j) = -1$.

Proof. Since M is symmetric, it is diagonalizable: there exists a an orthogonal matrix T (that is, $TT^{\perp} = \mathbb{I}$) such that

$$M = T^{\perp}DT, \quad \text{where } D = \operatorname{diag}(\lambda_1, \dots, \lambda_d) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_d \end{bmatrix}$$

Then,

$$M\vec{v} \cdot \vec{v} = T^{\perp}DT\vec{v} \cdot \vec{v} = D(T\vec{v}) \cdot (T\vec{v}) \overset{\vec{u}:=T\vec{v}}{=} D\vec{u} \cdot \vec{u} = \sum_{j=1}^{d} \lambda_j u_j^2.$$

If $\vec{v}_j = T^{-1}\vec{e}_j = T^{\perp}\vec{e}_j$, then $M\vec{v}_j \cdot \vec{v}_j = \lambda_j$ and from this we see that i), ii) and iii) immediately follow. □

Remark 2.5.3

If $M > 0$ and $\lambda_* := \min \lambda_j$, then setting $\vec{u} = T\vec{v}$ (where $M + T^{\perp}DT$)

$$M\vec{v} \cdot \vec{v} = \sum_{j=1}^{d} \lambda_j u_j^2 \geqslant \lambda_* \sum_{j=1}^{d} u_j^2 = \lambda_* \|\vec{u}\|^2 = \lambda_* \|T\vec{v}\|^2.$$

Now, since $\|T\vec{v}\|^2 = T\vec{v} \cdot T\vec{v} = T^{\perp}T\vec{v} \cdot \vec{v} \overset{T^{\perp}T=\mathbb{I}}{=} \vec{v} \cdot \vec{v} = \|\vec{v}\|^2$, we get the bound

$$M\vec{v} \cdot \vec{v} \geqslant \lambda_* \|\vec{v}\|^2, \ \forall \vec{v} \in \mathbb{R}^d. \tag{2.5.1}$$

Similarly, if $M < 0$,

$$M\vec{v} \cdot \vec{v} \leqslant \lambda^* \|\vec{v}\|^2, \ \forall \vec{v} \in \mathbb{R}^d, \tag{2.5.2}$$

where now $\lambda^* := \max \lambda_j$.

The condition presented in the previous Proposition is a concrete way to test positivity/negativity/indefinitness. We do not need to determine eigenvalues but only their sign. To this aim, the following criterion is much direct:

Proposition 2.5.4: Sylverster's criterion

Let M be a $d \times d$ symmetric matrix, and let M_k be the $k \times k$ submatrix of M made of the elements of first k lines and columns. Then

- $M > 0$ iff $\operatorname{sgn} \det M_k = +1$, $\forall k = 1, \dots, d$.
- $M < 0$ iff $\operatorname{sgn} \det M_k = (-1)^k$, $\forall k = 1, \dots, d$.

Proof. Define

$$D_k := \operatorname{diag}(\lambda_1, \dots, \lambda_k), \ k = 1, \dots, d.$$

By the Proposition 2.5.2

- $M > 0$ iff $\operatorname{sgn} \det D_k = +1$ for all $k = 1, \dots, d$;
- $M < 0$ iff $\operatorname{sgn} \det D_k = (-1)^k$ for all $k = 1, \dots, d$;

Now, since $M = T^{\perp}DT$, we have $M_k = T_k^{\perp}D_kT_k$, so

$$\det M_k = \det T_k^{\perp} \det D_k \det T_k = (\det T_k)^2 \det D_k, \implies \operatorname{sgn} \det M_k = \operatorname{sgn} \det D_k.$$

From this the conclusion follows. □

Example 2.5.5: $(*)$

Q. *Determine the signature of the following matrices:*

$$1.\ \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad 2.\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad 3.\ \begin{bmatrix} -3 & 1 & 0 \\ 1 & -4 & 2 \\ 0 & 2 & -5 \end{bmatrix}$$

A. 1. We have $\det M_1 = 2$, $\det M_2 = -1$, the matrix is indefinite. 2. We have $\det M_1 = 2$, $\det M_2 = 4 - 1 = 3 > 0$, the matrix is positive definite. 3. We have $\det M_1 = -3$, $\det M_2 = 11$, and

$$\det M_3 = -3 \cdot \begin{bmatrix} -4 & 2 \\ 2 & -5 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 2 & -5 \end{bmatrix} + 0 \cdot \det \ldots = (-3) \cdot 16 - 1 \cdot (-5) = +43.$$

2.5.2. Test for local extrema. It is well known that, in general, a stationary point is not a local extrema. The next Proposition provides a sufficient condition to ensure this happens:

Proposition 2.5.6

Let f be twice differentiable at **stationary point** $\vec{x}_*$. We have:

i) if $\nabla^2 f(\vec{p}) > 0$ (all eigenvalues positive), then $\vec{p}$ is a local minimum point for f.
ii) if $\nabla^2 f(\vec{p}) < 0$ (all eigenvalues negative), then $\vec{p}$ is a local maximum point for f.
iii) if $\nabla^2 f(\vec{p})$ has both positive and negative eigenvalues, then $\vec{p}$ is a saddle point.

Proof. i) Since $\nabla f(\vec{p}) = \vec{0}$, by Taylor's formula

$$f(\vec{x}) = f(\vec{p}) + \frac{1}{2}\nabla^2 f(\vec{p})(\vec{x} - \vec{p}) \cdot (\vec{x} - \vec{p}) + o(\|\vec{x} - \vec{p}\|^2).$$

Thus, being $\nabla^2 f(\vec{p}) > 0$, by the bound (2.5.1) we get

$$\begin{aligned} f(\vec{x}) - f(\vec{p}) &= \frac{1}{2}\nabla^2 f(\vec{p})(\vec{x} - \vec{p}) \cdot (\vec{x} - \vec{p}) + o(\|\vec{x} - \vec{p}\|^2) \geqslant \lambda_* \|\vec{x} - \vec{p}\|^2 + o(\|\vec{x} - \vec{p}\|^2) \\ &= \|\vec{x} - \vec{p}\|^2 \left(\lambda_* + \frac{o(\|\vec{x} - \vec{p}\|^2)}{\|\vec{x} - \vec{p}\|^2} \right). \end{aligned}$$

Now, since

$$\lim_{\vec{x} \to \vec{p}} \left(\lambda_* + \frac{o(\|\vec{x} - \vec{p}\|^2)}{\|\vec{x} - \vec{p}\|^2} \right) = \lambda_* > 0,$$

we can say that there is $r > 0$ such that

$$\lambda_* + \frac{o(\|\vec{x} - \vec{p}\|^2)}{\|\vec{x} - \vec{p}\|^2} > 0,\ \forall \vec{x} \in B(\vec{p}, r] \backslash \{\vec{p}\}.$$

Therefore,

$$f(\vec{x}) - f(\vec{p}) \geqslant 0,\ \forall \vec{x} \in B(\vec{p}, r],$$

which is the conclusion. The proof of ii) is analogous.

iii) Suppose that $\lambda_i > 0 > \lambda_j$ are opposite sign eigenvalues. Let us denote by $\vec{v}_i, \vec{v}_j$ corresponding eigenvectors. Then, moving along the straight line $\vec{x} = \vec{p} + t\vec{v}_i$ we have

$$f(\vec{p} + t\vec{v}_i) - f(\vec{p}) = \frac{1}{2}\nabla^2 f(\vec{p})(t\vec{v}_i) \cdot (t\vec{v}_i) + o(t^2\|\vec{v}_i\|^2) = t^2 \left(\frac{1}{2}\lambda_i \|\vec{v}_i\|^2 + \frac{o(t^2)}{t^2} \right) \geqslant 0,$$

when $t \longrightarrow 0$, that is $f(\vec{p}+t\vec{v}) \geq f(\vec{p})$, and f has a minimum at $\vec{p}$ along this section. By the same argument

$$f(\vec{p}+t\vec{v}_j) - f(\vec{p}) = t^2\left(\frac{1}{2}\lambda_j\|\vec{v}_j\|^2 + \frac{o(t^2)}{t^2}\right) \leqslant 0,$$

when $t \longrightarrow 0$. Thus, along line $\vec{p}+t\vec{v}_j$ f has a maximum at $t=0$. Since we found two sections passing through $\vec{p}$ along which f has, respectively, a minimum and a maximum, we conclude that $\vec{p}$ is a saddle point for f. □

Warning 2.5.7

In general, if $\nabla^2 f(\vec{p})$ is not strictly positive (or negative), nothing can be said about the nature of the stationary point.

Example 2.5.8: $(*)$

Q. *Find and classify all critical points of*

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x,y) = x^4 - y^4.$$

We have

$$\nabla f(x,y) = (4x^3, -4y^3),$$

from which we see that $\partial_x f, \partial_y f$ are both continuous, thus f is differentiable. (x,y) is a stationary point for f iff $\nabla f(x,y) = \vec{0}$ iff $(x,y) = \vec{0}$. The Hessian matrix is

$$\nabla^2 f(x,y) = \begin{bmatrix} 12x^2 & 0 \\ 0 & -12y^2 \end{bmatrix}, \quad \nabla^2 f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The first shows that second partial derivatives are continuous, thus, at once, ∇f is differentiable (so f is twice differentiable) and $\nabla^2 f$ is symmetric (Schwarz's theorem). In particular, $\nabla^2 f(0,0) \not> 0$, so we cannot draw any conclusion from it. However, $f(x,0) = x^4$, so $(0,0)$ is a strict minimum for the $x-$section, while $f(0,y) = -y^4$, so $(0,0)$ is a strict maximum for the $y-$section. Hence $(0,0)$ is not an extrema, but it is a saddle point. □

Example 2.5.9: $(**)$

Q. *Let*

$$f(x,y) := x(x^4+y^2) - y^2x^3, \ (x,y) \in \mathbb{R}^2.$$

Find and classify the stationary points of f and discuss the existence of local/global extrema. Find $f(\mathbb{R}^2)$.

We have

$$\partial_x f(x,y) = 5x^4 + y^2 - 3x^2y^2, \quad \partial_y f(x,y) = 2xy - 2yx^3,$$

hence $\partial_x f, \partial_y \in \mathsf{C}(\mathbb{R}^2)$ and f is differentiable on the whole $\mathbb{R}^2$. (x,y) is a stationary point iff

$$\nabla f(x,y) = \vec{0}, \iff \begin{cases} 5x^4 + y^2 - 3x^2y^2 = 0, \\ 2xy - 2yx^3 = 0, \end{cases} \iff \begin{cases} 5x^4 + y^2 - 3x^2y^2 = 0, \\ xy(1-x^2) = 0, \end{cases}$$

The second equation poses three cases: $x = 0$, $y = 0$ and $x^2 = 1$.

$$\begin{cases} y^2 = 0, \\ x = 0, \end{cases} \iff (x, y) = (0, 0), \quad \begin{cases} 5x^4 = 0, \\ y = 0, \end{cases} \iff (x, y) = (0, 0),$$

and

$$\begin{cases} 5 + y^2 - 3y^2 = 0, \\ x^2 = 1, \end{cases} \iff \begin{cases} y^2 = \frac{5}{2}, \\ x^2 = 1, \end{cases} \iff (x, y) = \left(\pm 1, \pm\sqrt{\frac{5}{2}}\right),$$

with all possible combinations on sings. Therefore, there are five stationary points. To classify them we look at the Hessian matrix. Notice that

$$\nabla^f(x, y) = \begin{bmatrix} 20x^3 - 6xy^2 & 2y - 6x^2 y \\ 2y - 6x^2 y & 2x - 2x^3 \end{bmatrix}.$$

from which we see that the entries are continuous, thus f is twice differentiable. At $(0, 0)$, $\nabla^2 f$ vanishes, to the hessian test does not apply. However, $f(x, 0) = x^5$ says immediately that $(0, 0)$ is not an extrema. To see if it is a saddle point is more difficult. By looking at sections like $x = y^2$ we have

$$f(y^2, y) = y^2(x^8 + y^2) - y^2 y^6 = y^{10} + y^4 - y^8 \sim_{y \to 0} y^4,$$

that is, along $x = y^2$ f has a minimum at $(0, 0)$. Taking $x = -y^2$, we have the reversed situation

$$f(-y^2, y) = -y^2(y^8 + y^2) + y^2 y^6 = -y^{10} - y^4 + y^8 \sim_{y \to 0} -y^4,$$

that is, along $x = -y^2$ f has a maximum. We deduce that $(0, 0)$ is a saddle point.
For the other points we have

$$\nabla^2 f\left(1, \sqrt{\frac{5}{2}}\right) = \begin{bmatrix} 20 - 6\frac{5}{2} & \sqrt{\frac{5}{2}}(2-6) \\ \sqrt{\frac{5}{2}}(2-6) & 2 - 2 \end{bmatrix} = \begin{bmatrix} 15 & -4\sqrt{\frac{5}{2}} \\ -4\sqrt{\frac{5}{2}} & 0 \end{bmatrix}.$$

The sub-determinants are $15 > 0$ and $-16\frac{5}{2} = -40 < 0$, therefore $\left(1, \sqrt{\frac{5}{2}}\right)$ turns out to be a saddle point. The same happens for all the other points. Therefore, f has not local extrema points, hence even global extrema. About $f(\mathbb{R}^2)$, finally, because $\mathbb{R}^2$ is connected $f(\mathbb{R}^2)$ is an interval. Moreover f is unbounded, just check the x section $f(x, 0) = x^5$. Therefore $f(\mathbb{R}^2) =]-\infty, +\infty[$. □

2.6. Convexity

Convexity is a very important qualitative property of a numerical function. One of the most important reasons is that is ensures that a stationary point can be a global minimum/maximum point. Convexity and concavity are found in many important contexts, as in Economy, Finance, Operations Management.

We start with the definition of **convex** set. The intuitive idea is simple: D is convex if for any two points $\vec{x}, \vec{y}$ in D, the segment joining the two points falls entirely in D.

Definition 2.6.1: convex set

We say that $D \subset \mathbb{R}^d$ is **convex** if

$$\forall \vec{x}, \vec{y} \in D, \implies t\vec{x} + (1-t)\vec{y} \in D, \ \forall t \in [0, 1].$$

Definition 2.6.2: convex/concave function

Let $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$, D convex. We say that f is **convex** on D if

$$f(t\vec{x} + (1-t)\vec{y}) \geqslant t f(\vec{x}) + (1-t) f(\vec{y}),\ \forall \vec{x}, \vec{y} \in D,\ \forall t \in [0,1]. \tag{2.6.1}$$

f is said to be **concave** if $-f$ is convex.

As for differentiable functions of one real variable, convexity has a geometric interpretation in relation with tangent hyperplanes.

Proposition 2.6.3

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be differentiable on D open and convex. The following properties are equivalent:

i) f is convex on D.
ii) f is above each tangent hyperplane, that is

$$f(\vec{x}) \geqslant f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}),\ \forall \vec{x}, \vec{p} \in D.$$

iii) if f is also twice differentiable, then $\nabla^2 f(\vec{p}) \geqslant 0$, for every $\vec{p} \in D$.

We skip the proof of this result, but we focus on a major consequence.

Corollary 2.6.4

Let $f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be differentiable and convex on D open and convex. If $\vec{p} \in D$ is a stationary point, then $\vec{p}$ is a global minimum for f on D.

Proof. Just notice that, according to the previous characterization of convexity at point $\vec{p}$, we have

$$f(\vec{x}) \geqslant f(\vec{p}) + \nabla f(\vec{p}) \cdot (\vec{x} - \vec{p}) \overset{\nabla f(\vec{p}) = \vec{0}}{=} f(\vec{p}),\ \forall \vec{x} \in D,$$

and this is the conclusion. □

This corollary enlighten why convexity/concavity is important in many optimization problems: it ensures that the equation

$$\nabla f(\vec{x}) = \vec{0},$$

determines min/max points. This equation is also named **First Order Condition**. Furthermore, the fact the target function f is convex/concave boosts numerical algorithms to solve F.O.C.

2.7. Constrained Optimization

Many applied problems can be formalized as the maximization/minimization of a certain quantity (function) f of several variables over certain *constraints* on the variables. For instance, consider the problem of searching for the parallelepiped with maximum volume among those with fixed surface S. Formalizing this we have

$$\max_{x,y,z>0\ :\ 2(xy+yz+xz)=S} xyz.$$

A general form for this problem is

$$\min/\max_D f(x_1, \dots, x_d),\quad \text{on } D = \{(x_1, \dots, x_d) \in \mathbb{R}^d\ :\ g_1(x_1, \dots, x_d) = 0, \dots, g_m(x_1, \dots, x_d) = 0\}.$$

The main feature of this setup is that the domain D has no interior points in general.

Example 2.7.1

Let $D := \{(x, y) \in \mathbb{R}^2 \ : \ ax + by + c = 0\}$, with $(a, b) \neq (0, 0)$. Then, D is a straight line in $\mathbb{R}^2$, hence $\mathrm{Int}(D) = \emptyset$.

In particular, a min/max point for f won't be a stationary point for f in general. Thus, the equation $\nabla f(\vec{x}) = \vec{0}$ might be of no interest for the search of extrema. We have already met this type of situations, as in Examples 2.3.7 and 2.3.9), where we had to search for min/max points on parts of a domain where there are no interior points by definition (the boundary of the assigned domain). So far we illustrated a solution found "by hands" based on a simple idea: *use the constraint to reduce the number of variables*.

To understand the strategy, let us consider a "downsized" version of the problem posed above.

Example 2.7.2: $(*)$

Q. *Determine, among all the rectangle with fixed perimeter P, that or those (if any) having maximum area.*

A. Formally, we want to determine

$$\max_{(x,y)\in]0,+\infty[\ :\ 2(x+y)=P} xy.$$

Here, the optimization domain is $D = \{(x, y) \in \mathbb{R}^2 \ : \ y = \frac{P}{2} - x\}$. Since $y = \frac{P}{2} - x$, we have that

$$xy = x\left(\frac{P}{2} - x\right).$$

Thus, maximizing xy is the same of maximizing $x\left(\frac{P}{2} - x\right)$ where $x \in]0, +\infty[$). In other words

$$\max_{(x,y)\in]0,+\infty[\ :\ 2(x+y)=P} xy = \max_{x\in]0,\frac{P}{2}[} x\left(\frac{P}{2} - x\right),$$

which is an elementary problem. We don't even need to invoke the Differential Calculus to solve it. Notice that

$$x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2 = -\left(\frac{P}{4} - x\right)^2 + \frac{P^2}{4},$$

and this makes clear that the maximum is achieved at $x = \frac{P}{4}$. Thus, the rectangle with fixed perimeter P and maximum area has sides $x = \frac{P}{4}$ and $y = \frac{P}{4}$, thus it is a square. Incidentally, notice that $\nabla f(x, y) = (y, x)$, so $\nabla f(\frac{P}{4}, \frac{P}{4}) = (\frac{P}{4}, \frac{P}{4}) \neq \vec{0}$. This confirms that constrained min/max points are not necessarily stationary points for f.

The key idea of the previous example suggests a general one. Consider the proble

$$\min_{g(x,y)=0}/\max f(x, y).$$

Suppose that we can use the equation $g(x, y) = 0$ to "extract" $y = \phi(x)$ or, which is the same, $x = \psi(y)$. These functions are called **implicit functions defined by** $g = 0$. Then,

$$\min_{g(x,y)=0}/\max f(x, y) = \min_x/\max f(x, \phi(x)) = \min_y/\max f(\psi(y), y).$$

The problem is that, in general, we might not be able to explicit y or x from an equation $g(x, y) = 0$. This might be very complicate so that to explicit one of the variables as a function of the other might be an impossible task. We will now illustrate an informal argument (that can be made a rigorous proof modulo technicalities) that yield a condition to be verified at constrained min/max points, condition that does not involve any implicit function.

Suppose (x^*, y^*) is a min/max point for f on $g = 0$ and assume that

$$g(x, y) = 0, \iff y = \phi(x).$$

Then,

$$f(x, \phi(x)) \text{ has a min/max point at } x = x^*.$$

By Fermat's theorem,

$$\frac{d}{dx} f(x, \phi(x))\Big|_{x=x^*} = 0. \tag{2.7.1}$$

Now, by the chain rule,

$$\frac{d}{dx} f(x, \phi(x)) = \partial_x f(x, \phi(x)) + \partial_y f(x, \phi(x))\phi'(x),$$

so, setting $x = x^*$, by the condition (2.7.1) we get

$$\boxed{\partial_x f(x^*, y^*) + \partial_y f(x^*, y^*)\phi'(x^*) = 0.} \tag{2.7.2}$$

We now compute $\phi'(x)$. Since

$$g(x, \phi(x)) \equiv 0, \implies 0 = \frac{d}{dx} g(x, \phi(x)) = \partial_x g(x, \phi(x)) + \partial_y g(x, \phi(x))\phi'(x).$$

Assuming that $\partial_y g(x, y) \neq 0$ *when* $(x, y) \in \{g = 0\}$, we get

$$\phi'(x) = -\frac{\partial_x g(x, \phi(x))}{\partial_y g(x, \phi(x))}, \implies \phi'(x^*) = -\frac{\partial_x g(x^*, y^*)}{\partial_y g(x^*, y^*)},$$

Plugging this last relation into equation (2.7.2) we obtain

$$0 = \partial_x f(x^*, y^*) + \partial_y f(x^*, y^*)\left(-\frac{\partial_x g(x^*, y^*)}{\partial_y g(x^*, y^*)}\right),$$

that is, by rearranging terms,

$$\boxed{\partial_x f(x^*, y^*)\partial_y g(x^*, y^*) - \partial_y f(x^*, y^*)\partial_x g(x^*, y^*) = 0.} \tag{2.7.3}$$

Now, this condition has a nice geometrical interpretation. We recognize that it can be viewed as a scalar product, precisely,

$$(\partial_x f, \partial_y f) \cdot (\partial_y g, -\partial_x g) = 0,$$

where everything is evaluated at (x^*, y^*). This means that, at point (x^*, y^*),

$$\nabla f \perp (\partial_y g, -\partial_x g), \iff \nabla f \parallel (\partial_x g, \partial_y g) = \nabla g, \iff \nabla f = \lambda \nabla g,$$

for some coefficient λ. The same conclusion can be obtained if from $g(x, y) = 0$ we can explicit $x = \psi(y)$, provided $\partial_x g \neq 0$. Thus, conclusion $\nabla f = \lambda \nabla g$ holds provided that $\partial_x g, \partial_y g$ are never both equal to 0 on the constraint set $\{g = 0\}$. We have informally proved the following

Theorem 2.7.3: Lagrange

Let $g : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be differentiable and such that $\nabla g \neq \vec{0}$ on $D := \{g = 0\}$. Then, if (x^*, y^*) is a min/max point for f differentiable on D we have that

$$\exists \lambda \in \mathbb{R} : \nabla f(x^*, y^*) = \lambda \nabla g(x^*, y^*). \tag{2.7.4}$$

A point (x^*, y^*) verifying the condition (2.7.4) is called **constrained stationary point for** f.

Example 2.7.4: (∗)

Q. *Determine*

$$\min_{x^2+y^2=1}/\max(x+y).$$

A. **Existence.** Let $f(x,y)=x+y$ and $D=\{x^2+y^2=1\}=\{x^2+y^2-1=0\}=:\{g=0\}$. Clearly, D is closed and bounded, hence compact, $f\in C$, the existence of global min/max points for f on D is ensured by Weierstrass' theorem.

Determination. To determine these points, we apply Lagrange's theorem. We first check that $\nabla g\neq\vec{0}$ on D. Indeed,

$$\nabla g=(2x,2y)=\vec{0},\quad\Longleftrightarrow\quad (x,y)=(0,0)\notin D,$$

thus $\nabla g\neq\vec{0}$ on D. Now, if (x,y) is a min/max point, we have

$$\nabla f(x,y)=\lambda\nabla g(x,y).$$

Since $\nabla f=(1,1)$, this means

$$(1,1)=\lambda(2x,2y),\quad\Longleftrightarrow\quad\begin{cases}1=2\lambda x,\\ 1=2\lambda y,\end{cases}\quad\Longleftrightarrow\quad (x,y)=\left(\frac{1}{2\lambda},\frac{1}{2\lambda}\right).$$

This point must belong to D, that is

$$\left(\frac{1}{2\lambda}\right)^2+\left(\frac{1}{2\lambda}\right)^2=1,\quad\Longleftrightarrow\quad 2\lambda^2=1,\quad\Longleftrightarrow\quad \lambda=\pm\frac{1}{\sqrt{2}},$$

that leads to points $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. These are candidates to be min/max points. It is now sufficient to notice that

$$f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\sqrt{2},\quad f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)=-\sqrt{2},$$

from which we see that the first point is a max point, the latter is a min point. □

Warning 2.7.5

The condition (2.7.4) stands to Lagrange's theorem as the condition $\nabla f=\vec{0}$ stands to Fermat's one. As in this last case, to be a stationary point does not necessarily imply to be an extrema, also for Lagrange's theorem to be a constrained stationary point does not imply to be a min/max point.

Definition 2.7.6

We say that $g=g(x,y)$ is a **submersion** on $S\subset\mathbb{R}^2$ if $\nabla g(x,y)\neq\vec{0}$ for every $(x,y)\in S$.

Remark 2.7.7: (practical hint)

Lagrange multiplier λ is unnecessary to determine points where $\nabla f=\lambda\nabla g$. Indeed:

$$\nabla f(x,y)=\lambda\nabla g(x,y),\quad\Longleftrightarrow\quad \operatorname{rank}\begin{bmatrix}\nabla f(x,y)\\ \nabla g(x,y)\end{bmatrix}<2,\quad\Longleftrightarrow\quad \det\begin{bmatrix}\nabla f(x,y)\\ \nabla g(x,y)\end{bmatrix}=0.$$

Example 2.7.8: (∗∗)

Q. *Find points of the ellipse* $x^2 + 2y^2 - xy = 9$ *at min/max distance to the origin.*

A. We have to minimize/maximize the distance to the origin, that is the function

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Because of the properties of the root, to minimize this function or just $x^2 + y^2$ is the same (it produces the same points but of course not the same values!) being $\sqrt{x^2 + y^2}$ min/max iff $x^2 + y^2$ it is, we replace the previous f with

$$f(x, y) = x^2 + y^2,$$

which is easier to be managed. Let also $g(x, y) := x^2 + 2y^2 - xy - 9$ in such a way we have to maximize/minimize f on $\{g = 0\}$.

Existence. Clearly $f \in \mathsf{C}(\mathbb{R}^2)$. The optimization domain is $D := \{g = 0\}$ which is clearly closed being defined by en equality involving a continuous function $g \in \mathsf{C}$. Let us check that D is also bounded. Representing (x, y) with "adapted polar coordinates", here $(x, y) = (\rho \cos\theta, \frac{1}{\sqrt{2}}\rho \sin\theta)$ (with this choice $x^2 + 2y^2 = \rho^2 \cos^2\theta + 2\frac{1}{2}\rho^2 \sin^2\theta = \rho^2$), we have

$$(x, y) \in D, \iff \rho^2 - \frac{1}{\sqrt{2}}\rho^2 \cos\theta \sin\theta - 9 = 0, \iff \rho^2 = \frac{9}{1 - \frac{1}{2\sqrt{2}} \sin(2\theta)}.$$

Now, since $1 - \frac{1}{2\sqrt{2}} \sin(2\theta) \geqslant 1 - \frac{1}{2\sqrt{2}} > 0$, we have

$$(x, y) \in D, \implies \rho^2 \leqslant \frac{9}{1 - \frac{1}{2\sqrt{2}}} =: M.$$

Since $\rho^2 = x^2 + 2y^2$ is bounded, also $\|(x, y)\|^2 = x^2 + y^2 \leqslant x^2 + 2y^2 = \rho^2$ is bounded, thus D is bounded. According to Weierstrass' theorem, f admits both min/max on $\{g = 0\}$.

Determination. We just proved that min/max points of f on D exist. To determine them, our aim is to apply Lagrange's theorem. First, we need to check whether g is a submersion on D. Now, g is not a submersion at points (x, y) such that

$$\nabla g = 0, \iff (2x - y, 4y - x) = (0, 0), \iff \begin{cases} 2x - y = 0, \\ \\ 4y - x = 0, \end{cases} \iff x = y = 0.$$

Since $g(0, 0) = -9 \neq 0$ we conclude that $(0, 0) \notin \{g = 0\}$, so g is a submersion on $\{g = 0\}$. Therefore, Lagrange's theorem applies and at a min/max point we must have $\nabla f = \lambda \nabla g$ or, as noticed above,

$$0 = \det \begin{bmatrix} \nabla f \\ \nabla g \end{bmatrix} = \det \begin{bmatrix} 2x & 2y \\ 2x - y & 4y - x \end{bmatrix} = 2x(4y - x) - 2y(2x - y) = 2(y^2 - x^2) + 4xy.$$

This can be rewritten as

$$(x + y)^2 - 2x^2 = 0, \iff (x + y)^2 = 2x^2, \iff x + y = \pm\sqrt{2}x, \iff y = (\pm\sqrt{2} - 1)x.$$

Therefore we have points $(x, (\pm\sqrt{2} - 1)x)$. Now,

$$(x, (\sqrt{2} - 1)x) \in \{g = 0\}, \iff x^2 + 2(\sqrt{2} - 1)^2 x^2 - (\sqrt{2} - 1)x^2 = 9, \iff (8 - 5\sqrt{2})x^2 = 9,$$

from which $x = \pm\frac{3}{\sqrt{8-5\sqrt{2}}}$. This produces points $\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right)$ (same sign, 2 points). Similarly

$$(x, (-\sqrt{2}-1)x) \in \mathscr{M}, \iff x^2 + 2(-\sqrt{2}-1)^2x^2 - (-\sqrt{2}-1)x^2 = 9, \iff (8+5\sqrt{2})x^2 = 9,$$

from which $x = \pm\frac{3}{\sqrt{8+5\sqrt{2}}}$. This produces points $\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right)$ (same sign, two points). Now, being

$$f\left(\pm\frac{3}{\sqrt{8-5\sqrt{2}}}, \pm\frac{3(\sqrt{2}-1)}{\sqrt{8-5\sqrt{2}}}\right) = \frac{36-18\sqrt{2}}{8-5\sqrt{2}} > f\left(\pm\frac{3}{\sqrt{8+5\sqrt{2}}}, \pm\frac{3(-\sqrt{2}-1)}{\sqrt{8+5\sqrt{2}}}\right) = \frac{36+18\sqrt{2}}{8+5\sqrt{2}}$$

we have that the first points are max for f, the latter are min.

Lagrange's theorem extends to the more general problem

$$\min/\max_{g(x_1,\dots,x_d)=0} f(x_1,\dots,x_d).$$

Theorem 2.7.9: Lagrange

Let $g : E \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ be a differentiable submersion on $D := \{g = 0\}$ (that is, $\nabla g \neq \vec{0}$ on D). Then, if $\vec{x}_*$ is a min/max point for f differentiable on D, we have that

$$\exists \lambda \in \mathbb{R} \,:\, \nabla f(\vec{x}_*) = \lambda \nabla g(\vec{x}_*). \tag{2.7.5}$$

Remark 2.7.10

$$\nabla f = \lambda \nabla g, \iff \operatorname{rank}\begin{bmatrix} \nabla f \\ \\ \nabla g \end{bmatrix} < 2.$$

Now, since this matrix is a $2 \times d$ matrix, its rank is < 2 iff *all* 2×2 *sub-determinants are* $= 0$.

Example 2.7.11: $(***)$

Q. *Solve the isoperimetric problem*

$$\max_{(x,y,z)\in\mathbb{R}^3:\ x,y,z\geqslant 0,\ 2(xy+yz+xz)=S} xyz.$$

A. **Existence** $(***)$ Let $f(x,y,z) := xyz$, clearly $f \in \mathsf{C}$, and $D := \{(x,y,z) \in [0,+\infty[^3 \colon 2(xy+yz+xz) = S\}$ is closed. Unfortunately, D is unbounded (for example, $(0,y,z) \in D$ provided $yz = \frac{A}{2}$, which is an hyperbola, so $(0, y, \frac{A}{2y}) \in D$ for every $y > 0$). However, we can easily prove that the maximum must be sought on a bounded part of D. We first notice that $(x,y,z) \in D$ iff $x = \sqrt{S/6}$. The corresponding volume is $xyz = \left(\frac{S}{6}\right)^{3/2}$. Now, fix $x > 0$. Then

$$xy, xz \leqslant \frac{S}{2}, \implies y, z \leqslant \frac{S}{2x}, \implies xyz \leqslant x\frac{S^2}{4x^2} = \frac{S^2}{4x} \leqslant \left(\frac{S}{6}\right)^{3/2},$$

provided $x > \frac{6^{3/2}}{4}\sqrt{S}$. So, any maximum point must verify $0 \leqslant x \leqslant \frac{6^{3/2}}{4}\sqrt{S}$. Similarly, $0 \leqslant y, z \leqslant \frac{6^{3/2}}{4}\sqrt{S}$, whence

$$(x, y, z) \in D \cap [0, \frac{6^{3/2}}{4}\sqrt{S}]^3.$$

This set is now closed and bounded, hence compact. Since $f \in \mathsf{C}$, max f exists according to Weierstrass' theorem.

Search $(**)$**.** We apply the Lagrange's theorem. First, let us check that $g = 2(xy + yz + xz) - S$ is a submersion on D. We have

$$\nabla g = (2(y+z), 2(x+z), 2(x+y)) = \vec{0}, \iff \begin{cases} y + z = 0, \\ x + z = 0, \\ x + y = 0, \end{cases} \iff (x, y, z) = (0, 0, 0) \notin D.$$

Thus $\nabla g \neq \vec{0}$ on D. Let now (x, y, z) be a max point for f. According to Lagrange's theorem,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \iff \operatorname{rank} \begin{bmatrix} yz & xz & xy \\ 2(y+z) & 2(x+z) & 2(x+y) \end{bmatrix} < 2.$$

This happens iff all the 2×2 sub-determinants of this matrix vanish, that is

$$\begin{cases} yz(x+z) - xz(y+z) = 0, \\ yz(x+y) - xy(y+z) = 0, \\ xz(x+y) - xy(x+z) = 0, \end{cases} \iff \begin{cases} z^2(y-x) = 0, \\ y^2(z-x) = 0, \\ x^2(z-y) = 0. \end{cases}$$

The first poses either $z = 0$ or $y = x$. In the first case, the remaining equations reduces to $xy = 0$, that is either $x = 0$ or $y = 0$. This means points $(0, y, 0)$ and $(x, 0, 0)$, none of which can be maximum point since in both cases volume=0. In the second case, $y = x$, second and third equations reduce to $x^2(z - x) = 0$, that is either $x = 0$ (then solutions $(0, 0, z)$, none of which is maximum) or $z = x$. Thus the unique possible candidates are (x, x, x). Imposing the perimeter condition, $3x^2 = \frac{S}{2}$, $x = \sqrt{\frac{S}{6}}$, and we conclude that *the volume is maximum when the parallelepiped is a cube of side* $x = \sqrt{\frac{S}{6}}$.

Example 2.7.12: $(***)$

Q. *A segment of length L is divided into n parts of lengths $x_1, \dots, x_n$. Find the maximum of $x_1 \cdots x_n$. By this, deduce the classical inequality between aritmetic and geometric means:*

$$\sqrt[n]{x_1 \cdots x_n} \leqslant \frac{x_1 + \cdots + x_n}{n}, \quad \forall x_1, \dots, x_n \geqslant 0.$$

A. The problem consists in solving for

$$\max_{x_1+\cdots+x_n=L,\ x_1,\dots,x_n\geqslant 0} x_1 \cdots x_n.$$

Existence. Let $D := \{x_1, \dots, x_n \geqslant 0 \ : \ x_1 + \cdots + x_n = L\}$. D is closed and bounded (because $0 \leqslant x_j \leqslant L$ for $j = 1, \dots, n$). Since $f(x_1, \dots, x_n) = x_1 \cdots x_n$ is clearly continuous, existence of $\max_D f$ is ensured by Weierstrass' theorem.

Search. Let $g = x_1 + \cdots + x_n - L$. Clearly g is differentiable, and since

$$\nabla g = (1, \dots, 1) \neq 0,$$

g is a submersion on D. According to Lagrange's theorem, any min/max point for f verifies

$$\operatorname{rank}\begin{bmatrix} \nabla f(x_1,\ldots,x_n) \\ \\ \nabla g(x_1,\ldots,x_n) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} x_2\cdots x_n & x_1x_3\cdots x_n & \cdots & x_1\cdots x_{n-1} \\ \\ 1 & 1 & \cdots & 1 \end{bmatrix} < 2.$$

This is possible iff all the 2×2 sub determinants vanish. Choosing column i and j respectively we have

$$\det\begin{bmatrix} x_1\cdots x_{i-1}x_{i+1}\cdots x_n & x_1\cdots x_{j-1}x_{j+1}\cdots x_n \\ \\ 1 & 1 \end{bmatrix} = x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_n(x_j - x_i).$$

Therefore, $(x_1,\ldots,x_n)\in D$ is constrained stationary point for f on D iff

$$x_1\cdots x_{i-1}x_{i+1}\cdots x_{j-1}x_{j+1}\cdots x_n(x_j - x_i) = 0,\ \forall i \neq j = 1,\ldots,n.$$

This produces points where a coordinate is null (hence $f=0$) and, if $x_j > 0$ for any j, $x_i - x_j = 0$ for all i,j, and this means that $(x_1,\ldots,x_n) = (\alpha,\alpha,\ldots,\alpha)$. Imposing that this point belongs to D we find the point $(\frac{L}{n},\ldots,\frac{L}{n})$ as the maximum point! Therefore,

$$\max_{x_1+\ldots+x_n=L,\ x_1,\ldots,x_n>0} x_1\cdots x_n = \left(\frac{L}{n}\right)^n.$$

In particular, recalling that $x_1+\ldots+x_n = L$, this conclusion can be written as

$$x_1\cdots x_n \leqslant \left(\frac{x_1+\ldots+x_n}{n}\right)^n,\ \iff\ \sqrt[n]{x_1\cdots x_n} \leqslant \frac{x_1+\ldots+x_n}{n}.$$

Example 2.7.13: $(**)$

Q. *Among all the convex polygons inscribed into a circle, find those of maximum perimeter.*

A. Let $r>0$ be the radius of the circle, $\theta_1,\ldots,\theta_n$ the subsequent angles formed by the vertexes of the polygon. Then

$$\text{perimeter} \ = \ P(\theta_1,\ldots,\theta_n) = \sum_{j=1}^n 2r\sin\frac{\theta_j}{2}.$$

We have $0<\theta_j<2\pi$ and $\theta_1+\cdots+\theta_n = 2\pi$. Thus, we have to determine

$$\max_{\theta_1+\ldots+\theta_n=2\pi,\ 0\leqslant\theta_j\leqslant 2\pi,\ j=1,\ldots,n} \sum_{j=1}^n 2r\sin\frac{\theta_j}{2}.$$

Let

$$D := \{(\theta_1,\ldots,\theta_n)\in[0,2\pi]^n\ :\ \theta_1+\cdots+\theta_n = 2\pi\}.$$

Clearly D is closed and bounded (compact), and being $P\in \mathsf{C}$, $\max_D P$ exists in virtue of Weierstrass' theorem. Arguing as in the previous example, we easily find stationary points of P on D: they must fulfill

$$\operatorname{rank}\begin{bmatrix} r\cos\frac{\theta_1}{2} & \cdots & r\cos\frac{\theta_n}{2} \\ \\ 1 & \cdots & 1 \end{bmatrix} < 2,\ \iff\ r\cos\frac{\theta_i}{2} = r\cos\frac{\theta_j}{2},\ \forall i,j,\ \iff\ \theta_i = \theta_j,\ \forall i,j.$$

Therefore, the polygon with maximum perimeter has $\theta_1=\theta_2=\ldots=\theta_n=\frac{2\pi}{n}$, thus it is a regular polygon.

2.8. General Lagrange's multipliers theorem

Lagrange's theorem extends to a more general setup where several constraint are involved, that is in a problem like

$$\min/\max_{g_1(\vec{x})=0,\ldots,g_m(\vec{x})=0} f(\vec{x}).$$

We start with the extension of the definition of submersion map:

Definition 2.8.1

Let $\vec{G} := (g_1,\ldots,g_m) : E \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. We say that $\vec{G}$ is a **submersion** on S if

(2.8.1) $$\nabla g_1(\vec{x}),\ldots,\nabla g_m(\vec{x}) \text{ are linearly independent, } \forall \vec{x} \in S.$$

Since a vector is linearly independent iff it is different from zero, the previous definition encompasses the definition of submersion given in the previous section.

Remark 2.8.2

Since $\nabla g_1(\vec{x}),\ldots,\nabla g_m(\vec{x})$ are m vectors of $\mathbb{R}^d$, they can be linearly independent only if $m \leqslant d$. A practical way to check condition (2.8.1) is then the following:

$$\operatorname{rank}\begin{bmatrix} \nabla g_1(\vec{x}) \\ \nabla g_2(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = \operatorname{rank} \vec{G}'(\vec{x}) = m.$$

Since jacobian matrix $\vec{G}'(\vec{x})$ is an $m \times d$ matrix (with $m \leqslant d$ as noticed above), rank $\vec{G}'(\vec{x}) = m$ iff *at least one $m \times m$ sub-determinant of $\vec{G}'(\vec{x})$ is not zero.* In practice, it is convenient to test when $\vec{G}$ **is not** a submersion: this means that rank $\vec{G}'(x) < m$, that is, is not maximum. This happens *iff each $m \times m$ submatrix of $\vec{G}'(\vec{x})$ has null determinant.*

Theorem 2.8.3: Lagrange's multipliers

Let $\vec{G} = (g_1,\ldots,g_m) : E \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be a submersion on $D := \{g_1 = 0,\ldots,g_m = 0\}$. Then, if $\vec{x} \in D$ is a min/max point for f differentiable on D,

(2.8.2) $$\boxed{\exists \lambda_1,\ldots,\lambda_m \in \mathbb{R} \ : \ \nabla f(\vec{x}) = \lambda_1 \nabla g_1(\vec{x}) + \cdots + \lambda_d \nabla g_m(\vec{x}).}$$

Remark 2.8.4

The condition (2.8.2) says that $\nabla f(\vec{x})$ is linearly dependent of $\nabla g_1(\vec{x}),\ldots,\nabla g_m(\vec{x})$ and it involves "multipliers" $\lambda_1,\ldots,\lambda_m$ which, however, are unnecessary. In fact, since $\vec{G} = (g_1,\ldots,g_m)$ is a submersion in $\vec{x}$,

$$(2.8.2) \iff \operatorname{rank}\begin{bmatrix} \nabla f(\vec{x}) \\ \nabla g_1(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \nabla g_1(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} = m.$$

The left matrix is an $(m+1) \times d$ matrix whose rank cannot be $m+1$. Therefore

$$(2.8.3) \qquad (2.8.2) \iff \text{all } (m+1)\times(m+1) \text{ subdeterminants of } \begin{bmatrix} \nabla f(\vec{x}) \\ \nabla g_1(\vec{x}) \\ \vdots \\ \nabla g_m(\vec{x}) \end{bmatrix} \text{ equals } 0.$$

Example 2.8.5

Q. *Let* $D := \{(x,y,z) \in \mathbb{R}^3 \; : \; xy + z^2 = 1, \; x^2 + y^2 = 1\}$. *i) Show that S is the non-empty zero set of a submersion on D. ii) Is D compact? iii) Find, if any, the points of D at the min/max distance to the origin.*

A. i) Let us check that $D \neq \emptyset$. We have

$$(x,x,z) \in D, \iff \begin{cases} x^2 + z^2 = 1, \\ 2x^2 = 1, \end{cases} \iff \begin{cases} x = \pm\frac{1}{\sqrt{2}}, \\ z = \pm\frac{1}{\sqrt{2}}. \end{cases}$$

Therefore $(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}) \in \mathscr{M}$ (all combinations of signs, the first two coordinates with same sign). Define now $\vec{G} \equiv (g_1, g_2) := (xy + z^2 - 1, x^2 + y^2 - 1)$ in such a way that $D = \{g_1 = 0, \; g_2 = 0\}$. We must discuss if $\vec{G}$ is a submersion on D. $\vec{G}$ **is not** a submersion iff

$$\operatorname{rank} \vec{G}'(x,y,z) < 2, \iff \operatorname{rank} \begin{bmatrix} y & x & 2z \\ 2x & 2y & 0 \end{bmatrix} < 2, \iff \begin{cases} 2(y^2 - x^2) = 0, \\ -4xz = 0, \\ -4yz = 0. \end{cases}$$

The second equation yields two cases:

$$\begin{cases} x = 0, \\ y^2 = 0, \iff y = 0, \\ z \in \mathbb{R} \end{cases} \quad \text{or} \quad \begin{cases} z = 0, \\ x^2 - y^2 = 0, \iff y = x, \; \vee \; y = -x. \end{cases}$$

Therefore, $\vec{G}$ is not a submersion at points $(0,0,z)$, $z \in \mathbb{R}$ and $(x,x,0)$, $(x,-x,0)$, $x \in \mathbb{R}$. Are these points in D? Clearly $(0,0,z) \notin D$. Moreover,

$$(x,x,0) \in D, \iff \begin{cases} x^2 = 1, \\ 2x^2 = 1, \end{cases} \text{(impossible)}, \quad (x,-x,0) \in D, \iff \begin{cases} -x^2 = 1, \\ 2x^2 = 1, \end{cases} \text{(impossible)}.$$

Thus, no points where $\vec{G}$ is not a submersion are in D, this means that $\vec{G}$ is a submersion on D.
ii) Since $D = \{g_1 = 0, g_2 = 0\}$ and $g_1, g_2 \in \mathsf{C}$, D is closed. It is also bounded because, by the second constraint, $x^2 + y^2 = 1$ we deduce $|x|, |y| \leqslant 1$, and by the first

$$z^2 = 1 - xy \leqslant 2, \implies |z| \leqslant \sqrt{2}.$$

Therefore, D is compact.
iii) **Existence.** The distance from (x,y,z) to $(0,0,0)$ is $\sqrt{x^2 + y^2 + z^2}$. Because this quantity is min/max when $f(x,y,z) = x^2 + y^2 + z^2$ it is, we use this last function to find min/max points. Since D is compact and f is continuous, min/max points exist by Weierstrass' theorem.
Search. By i), Lagrange's theorem applies. Since f is clearly differentiable, if (x,y,z) is a min/max point

$$\operatorname{rank} \begin{bmatrix} \nabla f(x,y,z) \\ \nabla g_1(x,y,z) \\ \nabla g_2(x,y,z) \end{bmatrix} < 3, \iff \det \begin{bmatrix} y & x & 2z \\ 2x & 2y & 0 \\ 2x & 2y & 2z \end{bmatrix} = 0,$$

that is,
$$2z(2y^2 - 2x^2) = 0, \iff z(y-x)(y+x) = 0.$$
Candidates are therefore the points $(x, y, 0)$, $x, y \in \mathbb{R}$, (x, x, z), $(x, -x, z)$, with $x, z \in \mathbb{R}$. Now
$$(x, y, 0) \in D, \iff \begin{cases} x^2 = 1, \\ \\ x^2 + y^2 = 1, \end{cases} \iff (x, y, 0) = (\pm 1, 0, 0).$$
Similarly
$$(x, x, z) \in D, \iff \begin{cases} x^2 + z^2 = 1, \\ \\ 2x^2 = 1, \end{cases} \iff \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right);$$
$$(x, -x, z) \in D, \iff \begin{cases} -x^2 + z^2 = 1, \\ \\ 2x^2 = 1, \end{cases} \iff \left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{\sqrt{2}}\right);$$
By computing distance, we see that $(\pm 1, 0, 0)$ are the points at min distance while $\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{\sqrt{2}}\right)$ are points at max distance.

Here below, we summarize and algorithm for the search constrained extrema. In some cases it may happens that the constraint $\vec{G}$ is not a submersion on the set $\{\vec{G} = \vec{0}\}$. On points where $\vec{G}$ is not a submersion, Lagrange's theorem cannot be applied. You can always consider these possible bad points as possible candidates min or max points and add them to the list of candidates to check with direct evaluation of the maximizing/minimizing function.

Check List 2.8.6: Searching for constrained extrema

To solve
$$\min_{g_1(\vec{x})=0,\dots,g_m(\vec{x})=0}/\max \quad f(\vec{x}).$$

(1) Determine if $\vec{G} := (g_1, \dots, g_m)$ is a submersion on $D := \{\vec{G} = \vec{0}\} = \{g_1 = 0, \dots, g_m = 0\}$. To this aim
$$\vec{G} \text{ not submersion} \iff \text{rank } \vec{G}'(\vec{x}) < m, \iff \det[\vec{G}']_{m\times m} = 0, \ \forall [\vec{G}']_{m\times m}.$$
Output: $\widetilde{D} \subset D$, on which $\vec{G}'$ is a submersion.

(2) Discuss compactness of D

(3) Existence: if $f \in \mathsf{C}$ and D is compact $\Longrightarrow$ Weierstrass' theorem; if $f \in \mathsf{C}$ and D is closed and unbounded, $\Longrightarrow$ compute $\lim_{\vec{x}\to\infty_d} f(\vec{x})$, if $+\infty$ $(-\infty)$ $\Longrightarrow$ existence of min (max).

(4) Search: apply Lagrange's multipliers theorem on $\widetilde{D}$, solving for
$$\begin{cases} \vec{x} \in \widetilde{D} \subset D, \\ \text{rank} \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \vdots \\ \nabla g_m \end{bmatrix} < m+1, \iff \forall \det \begin{bmatrix} \nabla f \\ \nabla g_1 \\ \vdots \\ \nabla g_m \end{bmatrix}_{m+1\times m+1} = 0. \end{cases}$$
Output: constrained stationary points of f contained in $\widetilde{D} \Longrightarrow$ compare values of f on stationary points and on any other point in $D\backslash\widetilde{D}$ (if any) and determine where f attains min and where it attains max values.

2.9. Exercises

Exercise 2.9.1 ($*$)**.** Compute the following directional derivatives:

i) $\partial_{(\sqrt{3},1)} f(1,1)$ where $f(x,y) := \log(1+xy)$.
ii) $\partial_{(2,2)} f(1,0)$ where $f(x,y) := \arctan(x+y)$.
iii) $\partial_{(1,1)} f(0,0)$ where $f(x,y) := \frac{x^2 y}{|x|+y^2}$ for $(x,y) \neq \vec{0}$ and $f(0,0)=0$.
iv) $\partial_{(-1,1)} f(0,0)$ where $f(x,y) = \frac{xy}{x^2+y^4}$, for $(x,y) \neq \vec{0}$ and $f(0,0)=0$.
v) $\partial_{(-1,-2)} f(0,0)$, where $f(x,y) := \frac{y(e^x-1)}{\sqrt{x^2+y^2}}$ for $(x,y) \neq \vec{0}$ and $f(0,0)=0$.

Exercise 2.9.2 ($**$)**.** For each of the following functions say whether f is continuous at point $(0,0)$, there exist $\partial_x f(0,0)$, $\partial_y f(0,0)$, and f is differentiable in $(0,0)$.

1. $f(x,y) := \begin{cases} \frac{x^3}{x^2+y^2}, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$

2. $f(x,y) := \begin{cases} \frac{x^2+y^2}{|x|+|y|}, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$

3. $f(x,y) := \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2}, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$

4. $f(x,y) := \begin{cases} \frac{x^2y}{x^2+y^2} + x - y, & (x,y) \neq 0_2, \\ 0, & (x,y) = 0_2. \end{cases}$

Exercise 2.9.3 ($**$)**.** Show that the function $f(x,y) = x\sqrt{x^2+y^2}$, $(x,y) \in \mathbb{R}^2$ is differentiable on $\mathbb{R}^2$.

Exercise 2.9.4 ($**$)**.** Show that the function

$$f(x,y) := \begin{cases} x^2 \sin \frac{1}{y}, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

is differentiable at $\vec{0}$ but $\partial_y f$ is not continuous at $\vec{0}$.

Exercise 2.9.5 ($**$)**.** Let

$$f(x,y) := \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{if } (x,y) \neq 0_2, \\ 0, & \text{if } (x,y) = 0_2. \end{cases}$$

Then $\exists \partial_{xy} f(0,0), \partial_{yx} f(0,0)$ but $\partial_{xy} f(0,0) \neq \partial_{yx} f(0,0)$.

Exercise 2.9.6 ($*$)**.** Determine and classify the stationary points.

i) $f(x,y) := xy(x+1)$.
ii) $f(x,y) := x^2+y^2+xy$.
iii) $f(x,y) := x^3+y^3+2x^2+2y^2+x+y$.
iv) $f(x,y) := xe^y + ye^x$.
v) ($**$) $f(x,y,z) := (x^3-3x-y^2)z^2+z^3$.

Exercise 2.9.7 ($*$)**.** Find the value of the parameter $\lambda \in \mathbb{R}$ such that the function $f(x,y) := x^2+\lambda y^2-4x+2y$ has a stationary point in $(2,-1)$. What kind of point is this?

Exercise 2.9.8 ($**$)**.** Let$f(x,y) := x^2(1-y)$ on $D := \{(x,y) \in \mathbb{R}^2 \ : \ x^2+|y| \leqslant 4\}$. Study the sign of f, determine its eventual stationary points on D and min/max of f on D. Determine $f(D)$.

Exercise 2.9.9 ($*$)**.** Discuss min/max of $f(x,y) := xye^{-xy}$ on $D = \{(x,y) \in \mathbb{R}^2 \ : \ 1 \leqslant x \leqslant 4, \ y \geqslant 0, \ |xy| \leqslant 1\}$.

Exercise 2.9.10 ($**$)**.** For each of the following functions a) find the stationary points, b) find any min/max in the domain, c) find the image of the domain.

i) $f(x,y) = x^4 + y^4 - xy$, on $D = \mathbb{R}^2$.
ii) $f(x,y) = x\left((\log x)^2 + y^2\right)$ on $D =]0,+\infty[\times\mathbb{R}$.
iii) $f(x,y) = xy(x+y)$, on $D = \mathbb{R}^2$.
iv) $f(x,y,z) = x^2 + y^2 + z^2 - 2xy + 2xz$ on $D = \mathbb{R}^3$.
v) $f(x,y,z) = x^4 + y^4 + z^4 - xyz$, on $D = \mathbb{R}^3$.

Exercise 2.9.11 $(**)$. Let
$$f(x,y) = (x^2+y^2)^2 - 3x^2y,\ (x,y) \in \mathbb{R}^2.$$
i) Determine $\lim_{(x,y)\to\infty_2} f(x,y)$ (if any). ii) Find stationary points of f. iii) Find eventual global min/max of f on $\mathbb{R}^2$ and find $f(\mathbb{R}^2)$.

Exercise 2.9.12 $(**)$. Let
$$f(x,y,z) := (x^2+y^2+z^2)^2 - xyz,\ (x,y,z) \in \mathbb{R}^3.$$
i) Show that $\lim_{(x,y,z)\to\infty_3} f(x,y,z) = +\infty$. ii) Find stationary points of f. iii) Show that f has global minimum on $\mathbb{R}^3$ and find $f(\mathbb{R}^3)$.

Exercise 2.9.13 $(**)$. Let $f(x,y) := x^2\left(y^2 - (x-1)^2\right)$, $(x,y) \in \mathbb{R}^2$. i) Does it exists $\lim_{(x,y)\to\infty_2} f(x,y)$? If yes, compute it. ii) Find and classify the stationary points of f on $\mathbb{R}^2$. iii) What about extrema of f on $\mathbb{R}^2$? Determine $f(\mathbb{R}^2)$. iv) Show that f has min/max on $D := \{(x,y) \in \mathbb{R}^2\ :\ y \leqslant 0,\ 0 \leqslant x \leqslant y+1\}$ and find them. What is $f(D)$?

Exercise 2.9.14 $(**)$. Consider the function $f(x,y) := x^4+y^4-8(x^2+y^2)$ on $\mathbb{R}^2$. i) Compute $\lim_{(x,y)\to\infty_2} f(x,y)$. ii) Find and classify the stationary points of f. What can you say about global min/max points of f? What about $f(\mathbb{R}^2)$? iii) Find the min/max points of f on the domain $D := \{(x,y) \in \mathbb{R}^2\ :\ x^2+y^2 \leqslant 9\}$.

Exercise 2.9.15 $(*)$. Determine min/max of f on the set D in the following cases:
i) $f = x+y$, $D = \{(x,y)\ :\ x^2+y^2 = 1\}$;
ii) $f = 2x^2+y^2-x$, $D = \{(x,y)\ :\ x^2+y^2 = 1\}$;
iii) $f = xy$, $D = \{(x,y)\ :\ x^2+y^2+xy-1 = 0\}$;
iv) $f = x^2+5y^2-\frac{1}{2}xy$, $D = \{(x,y)\ :\ x^2+4y^2 = 4\}$.

Exercise 2.9.16 $(*)$. Determine min/max of f on the set D in the following cases:
i) $f = x-2y+2z$, $D = \{(x,y,z)\ :\ x^2+y^2+z^2 = 9\}$;
ii) $f = z^2e^{xy}$, $D = \{(x,y,z)\ :\ x^2+y^2+z^2 = 1\}$.

Exercise 2.9.17 $(**)$. Let $D := \{(x,y,z) \in \mathbb{R}^3\ :\ z^2 = x^2+y^2+1,\ z = 2x^2+y^2\}$. Show that i) $D \neq \emptyset$ is the zero set of a submersion on D, ii) D is compact. iii) $\mathscr{M}$ has points of maximum quote: find them.

Exercise 2.9.18 $(**)$. Let $D := \{(x,y,z) \in \mathbb{R}^3\ :\ z^2 = xy+1\}$. Show that i) $D \neq \emptyset$ is the zero set of a submersion on D,ii) D is not compact. iii) Show that there exists points of D at minimum distance to the origin and find them.

Exercise 2.9.19 $(**)$. Let $D := \{(x,y,z) \in \mathbb{R}^3\ :\ (x^2+y^2+z^2)^2 - xyz = 1\}$. i) Show that $D \neq \emptyset$ is the zero set of a submersion on D. ii) Say if D is compact or not. iii) Determine, if they exists, points on D at maximum distance to the origin.

Exercise 2.9.20 $(**)$. Let $D := \left\{(x,y,z) \in \mathbb{R}^3\ :\ x^2+y^2-z^2 = 0,\ x^2-y^2 = 1\right\}$. i) Show that $D \neq \emptyset$ is the zero set of a submersion on D. ii) Say if D is compact or less. iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 2.9.21 $(**)$. Let $D := \left\{(x,y,z) \in \mathbb{R}^3\ :\ x^2-xy+y^2-z^2 = 1,\ x^2+y^2 = 1\right\}$. i) Show that $D \neq \emptyset$ is the zero set of a submersion on D. ii) Show that D is compact. iii) Find stationary points of $f(x,y,z) = xyz$ on D. What can you say about the problem to find extrema of f on D?

Exercise 2.9.22 $(**)$. Let $D := \{(x,y,z) \in \mathbb{R}^3\ :\ x^2+y^2-z^2 = 1\}$. i) Show that $D \neq \emptyset$ is the zero set of a submersion on D. ii) Is D compact? iii) Determine, if any, points of D at min/max distance to $\vec{0}$.

Exercise 2.9.23 ($*$)**.** Find the stationary points of $f(x,y,z) := xyz$, $(x,y,z) \in \mathbb{R}^3$ on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (here $a, b, c > 0$). Deduce min/max of f on the ellipsoid.

Exercise 2.9.24 ($**$)**.** Determine min/max of $f(x,y,z) = xy + yz + zx$ on the plane $x + y + z = 3$.

Exercise 2.9.25 ($**$)**.** Compute the min/max distance of the point $(0,1,0)$ to the following subset of $\mathbb{R}^3$:

$$\begin{cases} x^2 + y^2 + z^2 = 1, \\ \\ x^2 + y^2 = x. \end{cases}$$

Exercise 2.9.26 ($**$)**.** Consider the set $D := \left\{(x,y,z) \in \mathbb{R}^3 \ : \ z = x^2 + y^2, \ x + y + z = 0\right\}$. Discuss the problem of determining points of D at min/max z.

Exercise 2.9.27 ($**$)**.** Let $D := \{(x,y,z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1, \ 2z - 3x = 0\}$ and $f(x,y,z) := xz$. i) Show that $D \neq \emptyset$ is the zero set of a submersion on D. ii) Show that D is compact. iii) Find extrema of f on D.

Exercise 2.9.28 ($**$)**.** Let $D := \{(x,y,z) \in \mathbb{R}^3 \ : \ 2x^2 + 2y^2 - z^2 = 1, \ (x-y)^2 + z = 2\}$. i) Show that $D \neq \emptyset$ is the zero set of a submersion on D. ii) Show that $\mathscr{M}$ is not compact. iii) Find stationary points of $f(x,y,z) := z$ on D.

Exercise 2.9.29 ($***$)**.** A function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is said to be homogeneous of degree $\alpha \geqslant 1$ if

$$f(t\vec{x}) = t^\alpha f(\vec{x}), \ \forall t > 0, \ \forall \vec{x} \in \mathbb{R}^d.$$

Show that the following identity holds:

$$\vec{x} \cdot \nabla f(\vec{x}) = \alpha f(\vec{x}), \ \forall \vec{x} \in \mathbb{R}^d.$$

Exercise 2.9.30 ($***$)**.** Let $\vec{F} : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ be differentiable and invertible with inverse $\vec{F}^{-1} : \mathbb{R}^m \longrightarrow \mathbb{R}^d$ differentiable. Prove that, necessarily, $d = m$. (hint: differentiate $\vec{F}^{-1}\left(\vec{F}(\vec{x})\right) = \vec{x}, \forall \vec{x} \in \mathbb{R}^d \dots$)

Exercise 2.9.31 ($***$)**.** *Let $\vec{a} \in \mathbb{R}^d \backslash \{\vec{0}\}$. Determine*

$$\min_{\vec{x} \cdot \vec{a} = 1} \|\vec{x}\|.$$

Exercise 2.9.32 ($***+$)**.** The scope of the exercise is to prove the differentiability test 2.2.6. For simplicity, we assume that $f = f(x,y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, D open, have $\partial_x f, \partial_y f \in \mathsf{C}(D)$. The goal is to prove that

$$f(x+h, y+k) - f(x,y) = \nabla f(x,y) \cdot (h,k) + o(\|(h,k)\|).$$

The idea is to write

$$f(x+h, y+k) - f(x,y) = \Big(f(x+h, y+k) - f(x, y+k)\Big) + \Big(f(x, y+k) - f(x,y)\Big),$$

and use the classical one-variable Lagrange's theorem to write

$$f(x+h, y+k) - f(x, y+k) = \partial_x f(x_{h,k}, y+k) h,$$

where $x \leqslant x_{h,k} \leqslant x + h$ (if for instance $h > 0$). Here $x_{h,k}$ actually depends also on k (why?). Do the same for the other increment. Then, writing

$$\partial_x f(x_{h,k}, y+k) = \partial_x f(x, y+k) + \Delta_{h,k}, \ \text{where } \Delta_{h,k} = \partial_x f(x_{h,k}, y+k) - \partial_x f(x, y+k),$$

try to conclude the proof.

CHAPTER 3

Vector fields

Consider a function $f = f(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}$ which is differentiable on D. Then

$$\nabla f : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d.$$

In general, a function $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$, is called *vector field on* $\mathbb{R}^d$. Vector fields are very important entities in Physics. They are used to describe forces (*force field*) or other entities (like the electromagnetic field, or the velocity field). For example, the *gravitational field* induced by a point mass m positioned at point $\vec{p} \in \mathbb{R}^3$ is

$$\vec{F}(\vec{x}) = -Gm\frac{\vec{x}-\vec{p}}{\|\vec{x}-\vec{p}\|^3}, \;\; \vec{x} \in \mathbb{R}^3 \backslash \{\vec{p}\} =: D,$$

where G is the *universal gravitational constant*.

An important concept related to a vector field is its *potential*, namely a function f, if any, such that $\nabla f = \vec{F}$. For example, in the case of the gravitational field it is easy to check that

$$f(\vec{x}) = Gm\frac{1}{\|\vec{x}-\vec{p}\|}, \; \vec{x} \in D,$$

is a potential for $\vec{F}$ (do this check). The main scope of this Chapter is to understand how the problem of determining a potential of a vector field can be solved in general.

In dimension $d = 1$, this problem is well known and it consists in finding a *primitive* for function of one real variable: given F, determine f such that $f' = F$. By the Fundamental Theorem of Integral Calculus, we know that if $f \in \mathsf{C}$ then the solution always exists and it is given by

$$f(x) = \int_c^x F(y)\, dy.$$

The multidimensional version of this fact is much more involved and even good fields do not have any potential.

Chapter requirements: differential calculus for functions of vector variables, one variable integration and primitive calculus.

Learning objectives:

- (basic $*$) concept of vector field and potential of a vector field, conservative and irrotational fields, determining potentials of simple fields.
- (intermediate $**$) discussing and solving the problem of determining a potential for a vector field.
- (advanced $***$) discussing non standard problems, abstract reasoning (proofs).

3.1. Definitions

We start formalizing the ideas introduced above.

Definition 3.1.1

A continuous function $\vec{F} = \vec{F}(\vec{x}) : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ on D open set is called **vector field on** D. A vector field $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is said to be **conservative on** D if there exists $f : D \longrightarrow \mathbb{R}$ such that

$$\nabla f(\vec{x}) = \vec{F}(\vec{x}),\ \forall \vec{x} \in D.$$

A function f verifying this property is called **potential of** $\vec{F}$ **on** D.

Apart for the case $d = 1$, even for $d = 2$ it is not immediate to visualize a vector field. As a function, the "graph" of a vector field is the set $\{(\vec{x}, \vec{F}(\vec{x}))\ :\ \vec{x} \in D\} \subset \mathbb{R}^d \times \mathbb{R}^d$ (so, if $d = 2$ we have a subset of $\mathbb{R}^2 \times \mathbb{R}^2 \equiv \mathbb{R}^4$). Instead of plotting impossible graph, an interesting representation is provided by a figure where, at each point $\vec{x} \in D$ we draw the vector $\vec{F}(\vec{x})$. This makes representable planar and even (in some cases) space vector fields.

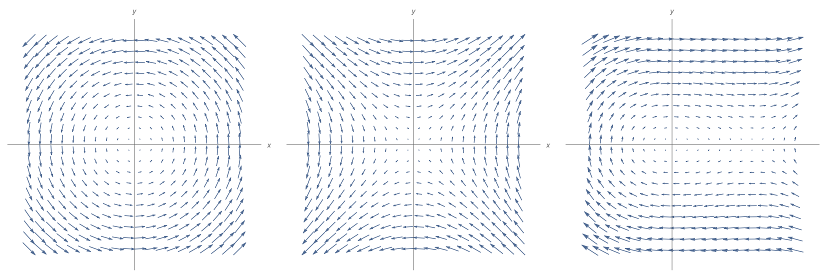


Figure 1. Left to right: vector fields $(-y, x)$, (x, y) and $(y, x^2 - x)$.

Remark 3.1.2

If $\vec{F} = (f_1, \dots, f_d)$ then

$$\nabla f = \vec{F},\ \text{on } D, \iff \begin{cases} \partial_1 f(\vec{x}) = f_1(\vec{x}), \\ \vdots & \forall \vec{x} \in D. \\ \partial_d f(\vec{x}) = f_d(\vec{x}), \end{cases}$$

Clearly, if f is a potential for $\vec{F}$, also $f + c$, where c is a real constant, is a potential for $\vec{F}$ because $\nabla(f + c) = \nabla f + \nabla c = \vec{F} + \vec{0} = \vec{F}$. In dimension 1, potentials are the primitives of F. We know that if the domain is an **interval**, then all the potentials differ by an additive constant. In higher dimension this remains true if the domain D is made of one piece, that is if it is connected. The proof is similar and it is based on the following

Lemma 3.1.3

If $\nabla h = \vec{0}$ on D connected set, then h is constant.

Proof. Let $\vec{x}, \vec{y} \in D$. We show $h(\vec{x}) = h(\vec{y})$. Since D is connected, there exists a curve $\vec{\gamma} = \vec{\gamma}(t)$ in D joining $\vec{x}$ to $\vec{y}$, that is $\vec{\gamma} : [a, b] \longrightarrow \mathbb{R}^d$ such that $\vec{\gamma}(a) = \vec{x}$, $\vec{\gamma}(b) = \vec{y}$. Consider $\phi(t) := h(\vec{\gamma}(t))$. We may assume $\vec{\gamma}$ is regular, that is $\exists \vec{\gamma}'(t)$ for every $t \in [a, b]$. Since $\vec{\gamma}(t) \in D$ for every t, we have

$$\phi'(t) = \nabla h(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t) = 0,\ \forall t \in [a, b],\ \implies\ \phi(t) \equiv C,$$

that is, in particular, $\phi(a) = \phi(b)$. But $\phi(a) = h(\vec{\gamma}(a)) = h(\vec{x})$ and $\phi(b) = h(\vec{\gamma}(b)) = h(\vec{y})$. $\square$

Proposition 3.1.4

Let D be a connected set and f, g potentials of the vector field $\vec{F} \in \mathsf{C}(D)$. Then $f - g \equiv$constant.

PROOF. Assume $\nabla f = \vec{F} = \nabla g$. Then, if $h := f - g$, $\nabla h = \vec{0}$, hence, from the Lemma, $h \equiv \mathit{constant}$. □

Example 3.1.5: $(*)$

Q. *Determine all the potentials for the field*

$$\vec{F}(x, y) = (y, x), \quad (x, y) \in \mathbb{R}^2 =: D.$$

A. We have to find f such that

$$\begin{cases} \partial_x f(x, y) = y, \\ \\ \partial_y f(x, y) = x, \end{cases} \quad (x, y) \in \mathbb{R}^2.$$

Consider the first equation, $\partial_x f(x, y) = y$. Looking at y as a parameter,

$$f(x, y) = \int y \, dx = yx + c,$$

where c is a constant w.r.t. x. Then we may imagine $c = c(y)$, that is

$$f(x, y) = xy + c(y).$$

To find c we use the second equation, $\partial_y f(x, y) = x$. Indeed

$$\partial_y f(x, y) = x, \iff x + c'(y) = x, \iff c'(y) = 0, \iff c(y) \equiv c,$$

and we deduce $f(x, y) = xy + c$, $c \in \mathbb{R}$.

3.2. Irrotational fields

In dimension 1 the problem to find a primitive of a function F has always an answer provided F is continuous. As the following example shows, in dimension $d \geqslant 2$ the story is quite different

Example 3.2.1: $(*)$

Q. *Show that the field*

$$\vec{F}(x, y) = (y, -x), \quad (x, y) \in \mathbb{R}^2 =: D,$$

has not any potential.

A. We have to find f such that

$$\begin{cases} \partial_x f(x, y) = y, \\ \\ \partial_y f(x, y) = -x, \end{cases} \quad (x, y) \in \mathbb{R}^2.$$

The first equation leads to

$$f(x, y) = \int y \, dx = xy + c(y).$$

Now, imposing the second equation we get:

$$\partial_y f(x, y) = -x, \iff x + c'(y) = -x, \iff c'(y) = -2x.$$

This is impossible because c must be constant in x! We deduce that it is impossible that f exists.

What is surprising in the previous example is that, while $\vec{F} = (y, x)$ is conservative, $\vec{F} = (y, -x)$ it is not. What is the explanation of this fact? The answer comes from a consequence of Schwarz second order crossed partial derivatives:

Proposition 3.2.2

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $\vec{F} = (f_1, \dots, f_d)$ be a conservative C^1 vector field (that is, $\partial_i f_j \in \mathsf{C}(D)$ for every $i, j = 1, \dots, d$). Then,

$$\partial_i f_j \equiv \partial_j f_i, \quad \text{on } D, \ \forall i, j = 1, \dots, n. \tag{3.2.1}$$

Proof. Since $\vec{F} = \nabla f$ for some f, we have $f_i = \partial_i f$. By assumption $\partial_j f_i = \partial_j(\partial_i f) \in \mathsf{C}(D)$, so according to Schwarz's theorem

$$\partial_j f_i = \partial_j \partial_i f \overset{Schwarz}{=} \partial_i \partial_j f = \partial_i f_j. \quad \square$$

Definition 3.2.3

A vector field $\vec{F}$ fulfilling (3.2.1) is called **irrotational vector field**.

Remark 3.2.4

It is easy to check that in the Example 3.2, the field is not irrotational. Indeed,

$$(y, -x) \text{ is irrotational} \iff \partial_y(y) = \partial_x(-x), \iff 1 = -1,$$

which is false. $\square$

Therefore, **conservative** $\Longrightarrow$ **irrotational**. The vice-versa is false, as the following example shows.

Example 3.2.5: $(**)$

Q. *Show that the field*

$$\vec{F}(x, y) := \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right), \quad (x, y) \in \mathbb{R}^2 \backslash \{0_2\}$$

is irrotational but not conservative on $\mathbb{R}^2 \backslash \{0_2\}$.

A. Let us check first that $\vec{F}$ is irrotational. $\vec{F}$ is irrotational iff

$$\partial_y \left(-\frac{y}{x^2+y^2}\right) \equiv \partial_x \left(\frac{x}{x^2+y^2}\right)$$

We have

$$\partial_y \left(-\frac{y}{x^2+y^2}\right) = -\frac{x^2+y^2-y2y}{(x^2+y^2)^2} = -\frac{x^2-y^2}{x^2+y^2}, \quad \partial_x \left(\frac{x}{x^2+y^2}\right) = -\frac{x^2-y^2}{x^2+y^2}.$$

Therefore $\vec{F}$ is irrotational. Let us assume that a potential f exists. Then

$$\partial_x f(x, y) = -\frac{y}{x^2+y^2}, \iff f(x, y) = \int -\frac{y}{x^2+y^2}\, dx + c(y) = -\frac{1}{y} \int \frac{1}{1+\left(\frac{x}{y}\right)^2}\, dx + c(y) = -\arctan \frac{x}{y} + c(y).$$

This if $y \neq 0$. If $y = 0$,
$$\partial_x f(x,0) = 0, \iff f(x,0) = c(0),$$
Therefore the candidate is
$$f(x,y) = \begin{cases} -\arctan\frac{x}{y} + c(y), & y \neq 0, \\ \\ c(0), & y = 0. \end{cases}$$
On the other hand, if $y \neq 0$,
$$\partial_y f(x,y) = \frac{x}{x^2+y^2}, \iff \partial_y\left(-\arctan\frac{x}{y} + c(y)\right) = \frac{x}{x^2+y^2}, \iff c'(y) \equiv 0,$$
so $c \equiv C$. We derive then that
$$f(x,y) = \begin{cases} -\arctan\frac{x}{y} + C, & y \neq 0, \\ \\ C, & y = 0. \end{cases}$$
We are done apparently. But... looking carefully to f we see that the f we found is not even continuous! To see this consider a point $(x,0)$ with $x > 0$. If $(x,y) \longrightarrow (x,0)$ with $y \longrightarrow 0+$ then
$$f(x,y) = -\arctan\frac{x}{y} + C \longrightarrow -\arctan(+\infty) + C = -\frac{\pi}{2} + C.$$
But if $(x,y) \longrightarrow (x,0)$ with $y \longrightarrow 0-$ we have
$$f(x,y) = -\arctan\frac{x}{y} + C \longrightarrow -\arctan(-\infty) + C = +\frac{\pi}{2} + C.$$
The conclusion is that $\lim_{(x,y)\longrightarrow(x,0)} f(x,y)$ doesn't exists, for any $x > 0$. Thus, f cannot be differentiable on $\mathbb{R}^2\backslash\{\vec{0}\}$, whence it cannot be a potential for $\vec{F}$. We got a contradiction.

3.3. Line Integral

An important concept of Physics is the *work done by a force along a path*. This quantifies the energy spent by a force to move a mass from a point to another along a certain path. This concept is made formally precise through the following definition.

Definition 3.3.1

Let $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a continuous vector field on D, $\vec{\gamma} \in \mathsf{C}^1([a,b];D)$ (that is $\vec{\gamma}, \vec{\gamma}' \in \mathsf{C}([a,b])$ and $\vec{\gamma} \subset D$) a **regular curve**. We call **line integral of $\vec{F}$ along $\vec{\gamma}$**
$$\int_{\vec{\gamma}} \vec{F} := \int_a^b \vec{F}(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t)\, dt.$$
If $\vec{\gamma}$ is a **circuit**, that is $\vec{\gamma}(a) = \vec{\gamma}(b)$, we call **circulation** of $\vec{F}$ along $\vec{\gamma}$ the integral
$$\oint_{\vec{\gamma}} \vec{F} := \int_{\vec{\gamma}} \vec{F}.$$

For convenience, we introduce some notations that will be useful working with line integrals. Given a regular path $\vec{\gamma} \in \mathsf{C}^1([a,b];D)$, we define
$$-\vec{\gamma} : [a,b] \longrightarrow \mathbb{R}^d, \ -\vec{\gamma}(t) := \vec{\gamma}(b+a-t).$$
In practice, $-\vec{\gamma}$ is $\vec{\gamma}$ traversed in the opposite direction. Then
$$\int_{-\vec{\gamma}} \vec{F} = \int_a^b \vec{F}(\vec{\gamma}(b+a-t)) \cdot (-\vec{\gamma}'(b+a-t))\, dt \overset{s:=b+a-t}{=} -\int_b^a \vec{F}(\vec{\gamma}(s)) \cdot \vec{\gamma}'(s)(-ds) = -\int_{\vec{\gamma}} \vec{F}.$$

Another important operation is the *path concatenation*. Suppose that $\vec{\gamma}_1 \in \mathsf{C}^1([a,b];D)$ and $\vec{\gamma}_2 \in \mathsf{C}^1([b,c];D)$ are two paths with the enpoint of $\vec{\gamma}_1$ coinciding with the initial point of $\vec{\gamma}_2$, that is $\vec{\gamma}_1(b) = \vec{\gamma}_2(b)$. We could merge the two paths into a unique one

$$\vec{\gamma}_1 + \vec{\gamma}_2 : [a,c] \longrightarrow \mathbb{R}^d, \; (\vec{\gamma}_1 + \vec{\gamma}_2)(t) := \begin{cases} \vec{\gamma}_1(t), & t \in [a,b], \\ \vec{\gamma}_2(t), & t \in [b,c]. \end{cases}$$

Then

$$\int_{\vec{\gamma}_1+\vec{\gamma}_2} \vec{F} = \int_a^b \vec{F}(\vec{\gamma}_1(t)) \cdot \vec{\gamma}_1'(t)\, dt + \int_b^c \vec{F}(\vec{\gamma}_2(t)) \cdot \vec{\gamma}_2'(t)\, dt = \int_{\vec{\gamma}_1} \vec{F} + \int_{\vec{\gamma}_2} \vec{F}.$$

Combining these two properties you can easily check that

$$\int_{\vec{\gamma}_1 - \vec{\gamma}_2} \vec{F} = \int_{\vec{\gamma}_1} \vec{F} - \int_{\vec{\gamma}_2} \vec{F}.$$

Warning 3.3.2

$-\vec{\gamma}$ **is not** the path $-\vec{\gamma}(t)$ (that would make even sense, but it is not what we mean here by $-\vec{\gamma}$), and, similarly, $\vec{\gamma}_1 + \vec{\gamma}_2$ **is not** the algebraic sum $\vec{\gamma}_1(t) + \vec{\gamma}_2(t)$.

The following result is the analogous of Fundamental Theorem of Integral Calculus for line integrals:

Proposition 3.3.3

Let $\vec{F}$ be a conservative vector field with potential f. Then, if $\vec{\gamma} \in \mathsf{C}^1([a,b];D)$

(3.3.1)
$$\int_{\vec{\gamma}} \vec{F} = f(\vec{\gamma}(b)) - f(\vec{\gamma}(a)).$$

In particular, if $\vec{\gamma}$ is a **circuit** in D,

(3.3.2)
$$\oint_{\vec{\gamma}} \vec{F} = 0.$$

Proof. If $\vec{F} = \nabla f$ then, by the fundamental formula of integral calculus,

$$\int_{\vec{\gamma}} \vec{F} = \int_a^b \nabla f(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \frac{d}{dt} f(\vec{\gamma}(t))\, dt = f(\vec{\gamma}(b)) - f(\vec{\gamma}(a)).$$

The (3.3.2) follows immediately because for a circuit $\vec{\gamma}$ we have $\vec{\gamma}(b) = \vec{\gamma}(a)$. □

Example 3.3.4: $(*)$

Q. *Compute* $\oint_{x^2+y^2=1} \vec{F}$ *for the* $\vec{F}$ *of Example* 3.2.5.

A. Let $\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$. We may parametrize the unit circle as $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\oint_{\gamma} \vec{F} = \int_0^{2\pi} \frac{-\sin t}{1}(-\sin t) + \frac{\cos t}{1} \cos t\, dt = \int_0^{2\pi} dt = 2\pi. \quad \square$$

So, property (3.3.2) of vanishing circulations is a new necessary condition for $\vec{F}$ to be conservative. It turns out that, if D is connected, then condition (3.3.2) is also sufficient

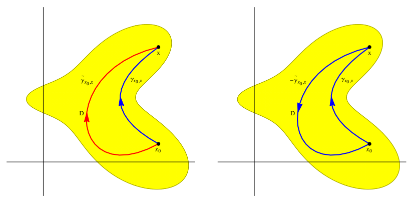
Theorem 3.3.5: fundamental theorem of calculus for fields

Let $\vec{F} \in \mathsf{C}(D)$ be a vector field on $D \subset \mathbb{R}^d$ open and connected and such that all circulations vanish (that is, (3.3.2) holds). Then $\vec{F}$ is conservative and all the possible potentials are

$$f(\vec{x}) = \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} + c,\ \vec{x} \in D,\ c \in \mathbb{R}, \tag{3.3.3}$$

where $\gamma_{\vec{x}_0,\vec{x}}$ is any regular path contained in D that joins $\vec{x}_0$ to $\vec{x}$.

Proof. We will prove that i) formula (3.3.3) is well posed, that is, the line integral is independent of the specific path connecting $\vec{x}_0$ to $\vec{x}$; ii) $\nabla f = \vec{F}$.



i) Let $\widetilde{\gamma}_{\vec{x}_0,\vec{x}}$ be a second path connecting $\vec{x}_0$ to $\vec{x}$. We want to prove that

$$\int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} = \int_{\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F}, \iff 0 = \int_{\widetilde{\gamma}_{\vec{x}_0,\vec{x}}} \vec{F} - \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} = \int_{\widetilde{\gamma}_{\vec{x}_0,\vec{x}} - \gamma_{\vec{x}_0,\vec{x}}} \vec{F}$$

But this follows from the assumption of null circulations being $\widetilde{\gamma}_{\vec{x}_0,\vec{x}} - \gamma_{\vec{x}_0,\vec{x}}$ a circuit. This proves the well posedness of definition (3.3.3).

ii) The second step consists in proving that $\nabla f = \vec{F}$. We have to prove that

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \vec{F}(\vec{x}) \cdot \vec{h} + o(\vec{h}).$$

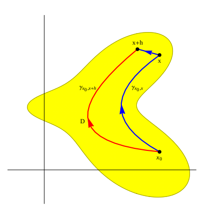
We have,

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \int_{\gamma_{\vec{x}_0,\vec{x}+\vec{h}}} \vec{F} - \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F}.$$

Since the integral does not depend on the particular path connecting $\vec{x}_0$ to $\vec{x}+\vec{h}$, we can write

$$\int_{\gamma_{\vec{x}_0,\vec{x}+\vec{h}}} \vec{F} = \int_{\gamma_{\vec{x}_0,\vec{x}}+[\vec{x},\vec{x}+\vec{h}]} \vec{F} = \int_{\gamma_{\vec{x}_0,\vec{x}}} \vec{F} + \int_{[\vec{x},\vec{x}+\vec{h}]} \vec{F},$$

where $[\vec{x}, \vec{x}+\vec{h}]$ stands for the linear segment joining $\vec{x}$ to $\vec{x}+\vec{h}$.



Therefore

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \int_{[\vec{x},\vec{x}+\vec{h}]} \vec{F}.$$

Now, the natural parametrization of the segment $[\vec{x}, \vec{x}+\vec{h}]$ is $\gamma(t) = \vec{x} + t\vec{h}$, $t \in [0,1]$ so $\gamma'(t) = \vec{h}$. Hence

$$\begin{aligned}\int_{[\vec{x},\vec{x}+\vec{h}]} \vec{F} &= \int_0^1 \vec{F}(\vec{x}+t\vec{h}) \cdot \vec{h}\, dt = \int_0^1 \left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x}) + \vec{F}(\vec{x})\right) \cdot \vec{h}\, dt \\ &= \vec{F}(\vec{x}) \cdot \vec{h} + \underbrace{\int_0^1 \left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right) \cdot \vec{h}\, dt.}_{\varepsilon(\vec{h})}.\end{aligned}$$

It remains to prove that $\varepsilon(\vec{h}) = o(\vec{h})$. Notice first that Now,

$$|\varepsilon(\vec{h})| = \left|\int_0^1 \left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right) \cdot \vec{h}\, dt\right| \overset{\triangle}{\leqslant} \int_0^1 \left\|\left(\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right) \cdot \vec{h}\right\|\, dt \overset{CS}{\leqslant} \left(\int_0^1 \left\|\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right\|\, dt\right) \|\vec{h}\|,$$

thus

$$\frac{|\varepsilon(\vec{h})|}{\|\vec{h}\|} \leqslant \int_0^1 \left\|\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\right\|\, dt.$$

Since $\vec{F}$ is continuous, $\|\vec{F}(\vec{x}+t\vec{h}) - \vec{F}(\vec{x})\| \longrightarrow 0$ when $\vec{h} \longrightarrow \vec{0}$ and by this (with some work to be done) the conclusion follows. □

In general, to check null circulations condition (3.3.2) can be a prohibitive task. Fortunately, under good conditions on the domain, a limited number of checks is sufficient. For illustrative purposes, here is an example of such tests.

Proposition 3.3.6

Let $\vec{F} : \mathbb{R}^2 \backslash \{\vec{p}_1, \dots, \vec{p}_N\} \longrightarrow \mathbb{R}^2$ an **irrotational** vector field and

$$\oint_{\vec{\gamma}_j} \vec{F} = 0,$$

where $\vec{\gamma}_j$ are circuits aroud $\vec{p}_j$ $(j = 1, \dots, N)$ not containing any other $\vec{p}_i$ $(i \neq j)$. Then, $\vec{F}$ is conservative.

The findings of this section are illustrated in the following example.

Example 3.3.7: $(**)$

Q. *Show that the field*

$$\vec{F}(x,y) = \left(\frac{y^2+2xy-x^2}{(x^2+y^2)^2}, -\frac{x^2+2xy-y^2}{(x^2+y^2)^2}\right), \quad (x,y) \in \mathbb{R}^2 \backslash \{0_2\} =: D,$$

is conservative and determine all its potentials.

A. The field $\vec{F}$ is defined on $\mathbb{R}^2 \backslash \{\vec{0}\}$. Easily, $\vec{F}$ is irrotational, that is

$$\partial_y \frac{y^2+2xy-x^2}{(x^2+y^2)^2} = \partial_x \left(-\frac{x^2+2xy-y^2}{(x^2+y^2)^2}\right).$$

(check it). To apply the Proposition 3.3.6, it is sufficient to che that just the circulation around the unique singularity $\vec{0}$ vanishes. Precisely, we will show that $\oint_{\partial B(0,1]} \vec{F} = 0$, where $\partial B(0,1]$ is parametrized in the standard way $\vec{\gamma}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$. We have

$$\oint_{\partial B(0,1]} \vec{F} = \int_0^{2\pi} \vec{F}(\vec{\gamma}(t)) \cdot \gamma'(t)\, dt = \int_0^{2\pi} f_1(\cos t, \sin t)(\cos t)' + f_2(\cos t, \sin t)(\sin t)'\, dt,$$

where $\vec{F} = (f_1, f_2)$. We have

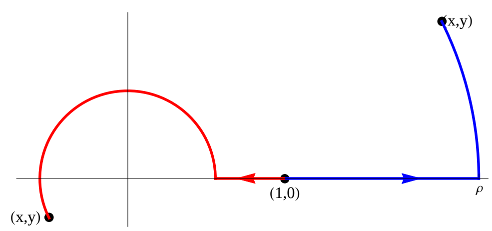
$$f_1(\cos t, \sin t) = (\sin t)^2 + 2\sin t\cos t - (\cos t)^2, \quad f_2(\cos t, \sin t) = -\left((\cos t)^2 + 2\cos t\sin t - (\sin t)^2\right).$$

Writing, for shortness, $C = \cos t$, $S = \sin t$, we have

$$\oint_\gamma \vec{F} = \int_0^{2\pi} -(S^2 + 2SC - C^2)S - (C^2 + 2SC - S^2)C\ dt = -\int_0^{2\pi} S^3 + C^3 - 2S + C^2S + S^2C\ dt.$$

It is easy to check that this integral equals 0. Therefore, by the Proposition 3.3.6, $\vec{F}$ is conservative on $\mathbb{R}^2\backslash\{\vec{0}\}$.

To compute the potentials, in this example we will use formula (3.3.3). We start fixing a reference point in the domain, for instance point $(1,0)$. If $(x,y) \in \mathbb{R}^2\backslash\{\vec{0}\}$ we connect $(1,0)$ to (x,y) with a path done first by a segment $[(1,0),(\rho,0)]$ where $\rho := \sqrt{x^2+y^2}$ followed by an arch of circumference of radius ρ up to (x,y).



Let $\arg(x,y) \in [0, 2\pi[$ the angle corresponding to the point (x,y). The path is then

$$\vec{\gamma}_{(1,0),(x,y)} = \vec{\gamma}_1 + \vec{\gamma}_2,$$

where (assuming for illustrative purposes that $\rho \geqslant 1$),

$$\vec{\gamma}_1(t) := (t, 0),\ t \in [1,\rho], \quad \vec{\gamma}_2(t) = (\rho\cos\theta, \rho\sin\theta),\ \theta \in [0, \arg(x,y)].$$

We have $\vec{\gamma}_1'(t) = (1,0)$ so

$$\int_{\vec{\gamma}_1} \vec{F} = \int_1^\rho f_1(t,0)\ dt = \int_1^\rho -\frac{t^2}{t^4}\ dt = -\int_1^\rho \frac{1}{t^2}\ dt = \frac{1}{t}\Big|_{t=1}^{t=\rho} = \frac{1}{\rho} - 1.$$

Similarly, $\vec{\gamma}_2'(\theta) = (-\rho\sin\theta, \rho\cos\theta) = (-\rho S, \rho C)$, so

$$\begin{aligned}\int_{\gamma_2} \vec{F} &= \int_0^{\arg(x,y)} \frac{\rho^2S^2 + 2\rho^2CS - \rho^2C^2}{(\rho^2C^2 + \rho^2S^2)^2}(-\rho S) - \frac{\rho^2C^2 + 2\rho^2CS - \rho^2S^2}{(\rho^2C^2+\rho^2S^2)^2}(\rho C)\ d\theta \\ &= \frac{1}{\rho}\int_0^{\arg(x,y)} \left(-S^3 - 2CS^2 + C^2S - C^3 - 2C^2S + S^2C\right)\ d\theta \\ &= \frac{1}{\rho}\int_0^{\arg(x,y)} \left(-S^3 - C^3 - S^2C - C^2S\right)\ d\theta.\end{aligned}$$

Let's compute the integrals:

$$\int_0^{\arg(x,y)} S^2C\ d\theta = \int_0^{\arg(x,y)} (\sin\theta)^2\cos\theta\ d\theta = \frac{1}{3}(\sin\theta)^3\Big|_{\theta=0}^{\theta=\arg(x,y)} = \frac{1}{3}(\sin\arg(x,y))^3.$$

Notice that $\sin \arg(x,y) = \frac{y}{\sqrt{x^2+y^2}}$ et, similarly, $\cos \arg(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, hence

$$\int_0^{\arg(x,y)} S^2 C \, d\theta = \frac{1}{3}\frac{y^3}{(x^2+y^2)^{3/2}}.$$

Similarly

$$\int_0^{\arg(x,y)} C^2 S \, d\theta = -\frac{1}{3}(\cos\theta)^3\Big|_{\theta=0}^{\theta=\arg(x,y)} = -\frac{1}{3}\frac{x^3}{(x^2+y^2)^{3/2}} + \frac{1}{3}.$$

Integrating by parts,

$$\begin{aligned}\int_0^{\arg(x,y)} S^3 \, d\theta &= \int_0^{\arg(x,y)} (-C)' S^2 \, d\theta = -CS^2\Big|_{\theta=0}^{\theta=\arg(x,y)} + 2\int_0^{\arg(x,y)} C^2 S \, d\theta \\ &= -\frac{xy^2}{(x^2+y^2)^{3/2}} - \frac{2}{3}\frac{x^3}{(x^2+y^2)^{3/2}} + \frac{2}{3}.\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^{\arg(x,y)} C^3 \, d\theta &= \int_0^{\arg(x,y)} S' C^2 \, d\theta = SC^2\Big|_{\theta=0}^{\theta=\arg(x,y)} + 2\int_0^{\arg(x,y)} S^2 C \, d\theta \\ &= \frac{x^2 y}{(x^2+y^2)^{3/2}} + \frac{2}{3}\frac{y^3}{(x^2+y^2)^{3/2}}.\end{aligned}$$

Therefore,

$$\int_{\gamma_2} \vec{F} = \frac{xy^2}{(x^2+y^2)^2} + \frac{x^3}{(x^2+y^2)^2} - \frac{x^2 y}{(x^2+y^2)^2} - \frac{y^3}{(x^2+y^2)^2} + \frac{1}{(x^2+y^2)^{1/2}},$$

and finally

$$f(x,y) = \frac{x^3 - y^3 + xy^2 - yx^2}{(x^2+y^2)^2} - 1.$$

To finish, being the domain connected, all the potentials are $f + c$, where $c \in \mathbb{R}$.

We can now summarize the investigations of this Chapter into the following scheme.

Check List 3.3.8: Discussing conservative vector fields

In order to establish whether a vector field $\vec{F} : D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is conservative and to determine its potentials:

(1) check whether $\vec{F}$ is irrotational or not:
$$\partial_j F_i \overset{D}{\equiv} \partial_i F_j \ ?$$
If not $\Longrightarrow \vec{F}$ is not conservative.

(2) (special test for $D = \mathbb{R}^2 \backslash \{\vec{p}_1, \dots, \vec{p}_N\}$), check circulations around each single singularity $\vec{p}_j$: if $\oint_{\gamma_j} \vec{F} = 0$ for every $j \Longrightarrow \vec{F}$ is conservative.

(3) to determine a potential, solve $\nabla f = \vec{F}$ either i) by a component-wise integration as
$$f(\vec{x}) = \int F_1(x_1, x_2, \dots, x_d) \, dx_1 + c(x_2, \dots, x_d),$$
and then use the other components to determine c; or ii) use the path integral formula (3.3.3).

(4) if D is connected, then all the potentials differ by an additive constant.

3.4. Exercises

Exercise 3.4.1 (*)**.** For each of the following vector fields on the given domains, check if they are irrotational, conservative and, in this case, find a potential:

(1) $\vec{F}(x,y) := (x, y-1)$, $(x,y) \in \mathbb{R}^2$;
(2) $\vec{F}(x,y) := (y,x)$, $(x,y) \in \mathbb{R}^2$;
(3) $\vec{F}(x,y) := (x,-y)$, $(x,y) \in \mathbb{R}^2$;
(4) $\vec{F}(x,y,z) := (y+z, x+z, x+y)$, $(x,y,z) \in \mathbb{R}^3$;

Exercise 3.4.2 (*)**.** Find al possible values for $a,b,c \in \mathbb{R}$ such that the field

$$\vec{F}(x,y) := \left(ax^3 + by + 3x^2y^2, cx^4 + 2x^3y + 1\right),$$

be irrotational on $\mathbb{R}^2$. In the case say if it is also conservative and find all the potentials.

Exercise 3.4.3 (*)**.** Consider the vector field

$$\vec{F}(x,y) := \left(\sin\frac{x}{y} + \frac{x}{y}\cos\frac{x}{y}, -\frac{x^2}{y^2}\cos\frac{x}{y} + 3\right), \text{ on } D = \mathbb{R}\times]0,+\infty[.$$

Check that $\vec{F}$ is irrotational and say if it is also conservative. In this case compute a potential f such that $f(\pi,1) = 3$.

Exercise 3.4.4 (**)**.** Consider the vector field

$$\vec{F}(x,y) := \left(\frac{y}{x^2}\cos\frac{y}{x}, -\frac{a}{x}\cos\frac{y}{x}\right), \text{ on } D =]0,+\infty[\times\mathbb{R}.$$

Find values of $a \in \mathbb{R}$ such that $\vec{F}$ is irrotational. For such value say if $\vec{F}$ is also conservative and, in the case, find the potentials.

Exercise 3.4.5 (**)**.** Consider the vector field

$$\vec{F}(x,y) := \left(-\frac{axy}{(x^2+y^2)^2}, \frac{bx^2-y^2}{(x^2+y^2)^2}\right), \text{ on } D = \mathbb{R}^2\backslash\{0_2\}.$$

Find values of $a,b \in \mathbb{R}$ such that $\vec{F}$ be irrotational. For such value say if $\vec{F}$ is also conservative and, in the case, find the potentials.

Exercise 3.4.6 (**)**.** Consider the vector field

$$\vec{F}(x,y,z) := \left(a(x,y,z), x^2 + 2yz, y^2 - z^2\right), \ (x,y,z) \in \mathbb{R}^3,$$

where a is a C^1 function. Find all the possible a in such a way that $\vec{F}$ be irrotational. Show that there is a unique a null as $y = z = 0$. In that case find all the potentials of $\vec{F}$.

Exercise 3.4.7 (**)**.** Find $a,b,c,d \in \mathbb{R}$ in such a way that the vector field

$$\vec{F}(x,y) := \left(\frac{ax+by}{x^2+y^2}, \frac{cx+dy}{x^2+y^2}\right), \ (x,y) \in \mathbb{R}^2\backslash\{0_2\}$$

be irrotational. For such values, find those such that $\vec{F}$ is conservative and find also its potentials.

Exercise 3.4.8 (**)**.** Let $\vec{F}$ be the vector field defined by

$$\vec{F}(x,y) := \left(\frac{ax^2+by^2}{(x^2+y^2)^2}, \frac{cxy}{(x^2+y^2)^2}\right), \ (x,y) \in D := \mathbb{R}^2\backslash\{0_2\}, \ (a,b,c \in \mathbb{R}).$$

i) Find $a,b,c \in \mathbb{R}$ such that $\vec{F}$ is irrotational. ii) Find $a,b,c \in \mathbb{R}$ such that $\vec{F}$ is conservative: for such a,b,c find the potentials of $\vec{F}$ (hint: . start with $\partial_y f = f_2(x,y)\dots$)

Exercise 3.4.9 (**)**.** Let $\vec{F}$ be the vector field defined as

$$\vec{F}(x,y) := \left(\frac{axy^2}{(x^2+y^2)^{1/2}}, \frac{bx^2y+cy^3}{(x^2+y^2)^{1/2}}\right), \quad (x,y) \in \mathbb{R}^2\backslash\{0_2\} =: D.$$

i) Find $a, b, c \in \mathbb{R}$ such that $\vec{F}$ be irrotational. ii) For the values found at i), say if $\vec{F}$ is conservative on $\mathbb{R}^2\backslash\{(0,y) \,:\, y \geqslant 0\}$ and on D. iii) For the values a, b, c such that $\vec{F}$ is conservative on D find the potentials of $\vec{F}$.

Exercise 3.4.10 (*)**.** Compute $\int_\gamma \vec{F}$ in the following cases:

(1) $\vec{F}(x,y) := (y^3 + x, -\sqrt{x})$ on $D = [0, +\infty[\times\mathbb{R}$, γ of equation $x = y^2$ connecting $(0,0)$ to $(1,1)$.
(2) $\vec{F}(x,y) := (y^2, 2xy + 1)$ on $D = \mathbb{R}^2$, γ of equation $y = \sqrt{|x-1|}$, $x \in [0,2]$.
(3) $\vec{F}(x,y) := (\sqrt{y}, x^3 + y)$ on $D = \mathbb{R} \times [0, +\infty[$ along $y = x^2$ connecting $(1,1)$ to $(2,4)$.
(4) $\vec{F}(x,y) := \left(\frac{x+1}{y-1}, \frac{y+1}{x-1}\right)$ on $D = \{(x,y) \in \mathbb{R}^2 \,:\, y \neq 1,\ x \neq 1\}$, along the segment connecting $(0,0)$ to $(1/2, 1/2)$.
(5) $\vec{F}(x,y) := (\log(1+y), \log(1+x))$, on $D =]-1, +\infty[^2$, along the segment connecting $(1,0)$ to $(0,1)$.
(6) $\vec{F}(x,y,z) := (y+z, x+z, x+y)$ on $D = \mathbb{R}^3$ along the helix $\gamma(t) = (r\cos t, r\sin t, kt)$, $t \in [0, 2\pi]$.

Exercise 3.4.11 (*)**.** Consider the vector field

$$\vec{F}(x,y) := \left(\frac{1}{1+y^2}, -\frac{2xy}{(1+y^2)^2}\right), \ (x,y) \in \mathbb{R}^2.$$

Is $\vec{F}$ irrotational? Conservative? Compute, the path integral $\int_\gamma \vec{F}$ where $\gamma(t) = \left(e^{\sin t}, \frac{2\cos t}{1+(\cos t)^2}\right)$, $t \in [0, \pi]$.

Exercise 3.4.12 (**)**.** Let $a, b, \alpha, \beta \neq 0$ and $\vec{F} \in C^1(D)$ be the vector field

$$\vec{F}(x,y) := \left(\frac{ax}{(x^2+y^2)^\alpha}, \frac{by}{(x^2+y^2)^\beta}\right), \quad (x,y) \in \mathbb{R}^2\backslash\{0_2\} =: D.$$

i) Find $a, b, \alpha, \beta \in \mathbb{R}\backslash\{0\}$ such that $\vec{F}$ be irrotational on D. ii) For the values found in i) compute $\int_\gamma \vec{F}$ where γ is the poligonal connecting $(2,0)$, $(0,1)$ and $(-2,0)$. iii) Find the values a, b, α, β such that $\vec{F}$ be conservative on D and compute the eventual potentials.

Exercise 3.4.13 (**)**.** Consider the vector field

$$\vec{F}(x,y) := \left(\frac{x}{\sqrt{x+y}}, \frac{ax+b}{\sqrt{x+y}}\right), \quad (x,y) \in D := \left\{(x,y) \in \mathbb{R}^2 \,:\, x+y > 0\right\}.$$

i) Find values $a, b \in \mathbb{R}$ such that $\vec{F}$ be irrotational. For such values may you say, without computing the potential, if $\vec{F}$ is also conservative? ii) For values $a, b \in \mathbb{R}$ such that $\vec{F}$ be conservative, find all its potentials.

Exercise 3.4.14 (**)**.** Let

$$\vec{F}(x,y,z) := \left(\frac{1}{x} + \frac{y^\alpha}{1+x^2y^2}, \frac{1}{y} + \frac{x}{1+x^2y^2}, \frac{1}{z}\right), \ (x,y,z) \in]0, +\infty[^3.$$

i) Find all the possible $\alpha > 0$ such that $\vec{F}$ be irrotational. ii) For the values α found in i), say if $\vec{F}$ is also conservative and compute all the potentials.

Exercise 3.4.15 (**)**.** Let $\alpha \in \mathbb{R}$ and consider the vector field

$$\vec{F}(x,y,z) := \left(\frac{x}{\sqrt{1+x^2+y^2}}, \frac{x+\alpha z}{\sqrt{1+x^2+y^2}} + e^{y+z}, e^{y+z}\right), \ (x,y,z) \in \mathbb{R}^3.$$

i) Find all the possible $\alpha > 0$ such that $\vec{F}$ be irrotational. ii) For the values α found in i), say if $\vec{F}$ is also conservative and compute all the potentials.

Exercise 3.4.16 (∗∗∗)**.** Consider the vector field $\vec{F}(x, y) := \left(\frac{x}{x^2+y^2}, u(x, y)\right)$ on $D = \mathbb{R}^2 \backslash \{0_2\}$, where $u \in \mathsf{C}^1(D)$. Find all the possible u in order that $\vec{F}$ be conservative.

Exercise 3.4.17 (∗∗∗)**.** Find all the possible functions $u = u(x, y)$ belonging to $\mathsf{C}^1(\mathbb{R}^2)$ such that the vector field $\vec{F}(x, y, z) := (2xz, yz, u(x, y))$ be conservative in $D = \mathbb{R}^3$.

Exercise 3.4.18 (∗∗∗)**.** Let $\vec{F}(x, y, z) := (y, x, z)$. Show that g is conservative and find all its potentials on $\mathbb{R}^3$. Now, let

$$\vec{H}(x, y, z) = u\left(x^2 + y^2\right) \vec{F}(x, y, z),\ (x, y, z) \in \mathbb{R}^3 \backslash \{(0, 0, z)\ :\ z \in \mathbb{R}\},$$

where $u \in \mathscr{C}^1(]0, +\infty[)$.

i) Compute $\oint_\gamma \vec{H}$ where $\gamma(t) := (r\cos t, r\sin t, k)$, $t \in [0, 2\pi]$, $r > 0$ and $k \in \mathbb{R}$ fixed parameters.
ii) Find all the possible u such that $\vec{H}$ be irrotational.
iii) Find all the possible u such that $\vec{H}$ be conservative.

CHAPTER 4

Differential Equations

In the first course of Calculus, basic Ordinary Differential Equations (ODEs) were introduced. The focus has been on methods to explicitly solve equations. However, as for algebraic equations, there is a limited class of solvable equations. This motivates new method and tools for general equations. Two main questions arise:

- *How to ensure existence of solutions when they cannot be explicitly computed?*
- What information can be drawn for solutions that cannot be explicitly computed?

A third important question, which is the focus of Numerical Methods: even if not explicitly solvable, *are there numerical ways to build solutions?* The scope of this chapter is to give an introduction to these questions and to give an idea on possible methods. ODEs is a wide and important topic of Mathematical Analysis that we are far from being explorable within the limitations of this course.

Chapter requirements: Basic Differential Equations, multi-dimensional Differential Calculus.

Learning objectives:

- (basic $*$) understanding of what is a solution of a differential equation and of a Cauchy problem, what is existence and uniqueness, drawing simple qualitative information as the domain of the equation, stationary solutions, regions of the domain where solutions are increasing/decreasing or searching for a first integral for separable systems.
- (intermediate $**$) how to check local/global existence and uniqueness, fine properties of the solutions, being able to plot a qualitative graph/phase portrait of solutions, searching for specific exact solutions.
- (advanced $***$) whatever is non standard.

4.1. First order scalar equations

We start considering a *first order scalar equation in normal form*, that is an equation of type

$$y'(t) = f(t, y(t)). \tag{4.1.1}$$

A solution is any function for which (4.1.1) makes sense and it is verified.

Definition 4.1.1: solution

Let $f = f(t, y) : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$. A function $y = y(t) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, I interval, is a **solution on** I of the equation (4.1.1) if

i) y is differentiable on I,
ii) $(t, y(t)) \in D$, $\forall t \in I$,
iii) $y'(t) = f(t, y(t))$, $\forall t \in I$.

Notice that, in particular, ii) implies that point $(t, y(t))$ belongs to D, the domain of f. Since points $(t, y(t))$ describe the graph of y, this is equivalent to say that *the graph of a solution must be contained into the domain, in the plane ty, of f*. For this reason, D is also called **domain of the equation**.

Warning 4.1.2: notations!

In the everyday use of differential equations, letter t is suppressed in the argument of y, so the equation (4.1.1) is usually written in the form
$$y' = f(t, y).$$
This could create some confusion about the notation $f(t, y)$:

- when we talk about the equation, $f(t, y)$ means $f(t, y(t))$, so y is actually $y(t)$, because y is a function solving the equation.
- when we talk about the function f, $f(t, y)$ means f evaluated at point $(t, y) \in \mathbb{R}^2$, so in this case y is a number.

We already know that, in general, an ODE has infinitely many solutions. In applications, it is important to find a solution that verifies some extra condition. Among these, the most important is the *passage (or initial) condition*, that is a solution whose value at some known t_0 is known $= y_0$:
$$CP(t_0, y_0) \ : \ \begin{cases} y' = f(t, y), \quad t \in I, \\ \\ y(t_0) = y_0. \end{cases}$$
This problem is called **Cauchy problem**. Existence of solutions to the Cauchy problem is ensured under very mild assumptions on f:

Theorem 4.1.3: Peano

Let $f = f(t, y) : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be **continuous** on D. Then, for every passage condition $(t_0, y_0) \in D$, there exists a solution of $CP(t_0, y_0)$.

Peano's theorem is a very weak result, it does not ensure uniqueness of the solution. Uniqueness is a very important property, because in many applications ODEs are used to make predictions. For example, known the state of a system at some time t_0 (that is $y(t_0) = y_0$), do a prediction on some future state ($y(t)$ for $t > t_0$) or on some past state ($y(t)$ for $t < t_0$). Knowing that the CP has a unique solution means that there is a unique possible future and a unique possible past. Conversely, if there are different solutions for the same CP, with different future/past values, makes the prediction impossible.

Example 4.1.4: $(**)$

Q. *Show that the Cauchy problem*
$$\begin{cases} y' = \frac{3}{2} y^{1/3}, \\ y(0) = 0. \end{cases}$$
has not a unique solution.

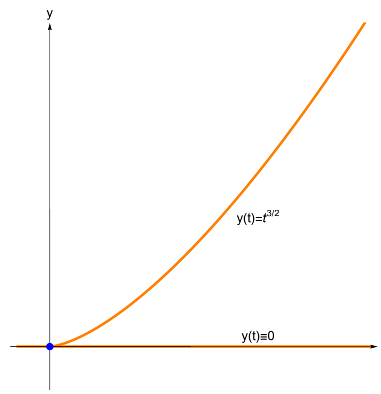
A. Here $f(t, y) := \frac{3}{2} y^{1/3}$, which is defined $\forall (t, y) \in \mathbb{R}^2$, so $D = \mathbb{R}^2$. Clearly $f \in \mathsf{C}(D)$, so Peano's theorem applies. A trivial solution of the CP is $y(t) \equiv 0$ (indeed, $y'(t) \equiv 0$ and $y(t)^{1/3} \equiv 0$, so the equation $y'(t) = \frac{3}{2} y(t)^{1/3}$ is verified for every t, and moreover $y(0) = 0$).

We look now for non zero solutions. We may notice that $y \neq 0$ is a solution of the equation iff
$$y'(t) = \frac{3}{2} y(t)^{1/3}, \iff \frac{2}{3} y(t)^{-1/3} y'(t) = 1, \iff \left(y(t)^{2/3} \right)' = 1, \iff y(t)^{2/3} = t + c,$$
where c is an arbitrary constant. This yields to
$$y(t) = (t + c)^{3/2}.$$

Here we may notice that, taking $c = 0$ yields $y(t) = t^{3/2}$, defined for $t \in [0, +\infty[$, which is a solution of the equation even at $t = 0$ (where y vanishes) because

$$y'(t) = \frac{3}{2}t^{1/2}, \;\; y(t)^{1/3} = t^{1/2}, \implies y'(t) = \frac{3}{2}y(t)^{1/3}, \; \forall t \geqslant 0,$$

and, moreover, $y(0) = 0$, so y solves the CP. So we have (at least) two solutions of the CP for $t \geqslant 0$: $y(t) \equiv 0$ and $y(t) = t^{3/2}$. As you can see, for $t > 0$, the two solutions are always different!



4.2. Global and local existence and uniqueness

In this section we present two important results concerning the existence and uniqueness of solutions of a CP. The major difference between them is that in one case we know in advance how long the solution y is defined (that is we know the entire *life span* of the solution), in the other we don't have this information (so we don't know exactly for how long the solution is defined). Moreover, while in the first case the life span of the solution is the same for all the solutions, in the second case the life span depends on the passage condition. We refer to the first result as **global existence and uniqueness** and to the second as **local existence and uniqueness** theorems.

4.2.1. Global existence and uniqueness. Since the local version, which is more general by the way, has a more complex statement, we prefer to start from the global version.

Theorem 4.2.1: Global Cauchy–Lipschitz

Let $f : D = [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathsf{C}([a, b] \times \mathbb{R})$;

ii) $\partial_y f$ be bounded on $[a, b] \times \mathbb{R}$, that is

$$\exists L > 0 \; : \; |\partial_y f(t, y)| \leqslant L, \; \forall (t, y) \in [a, b] \times \mathbb{R}.$$

Then, for every $(t_0, y_0) \in [a, b] \times \mathbb{R}$ there exists a unique solution to $CP(t_0, y_0)$.

Proof. (a sketch) The proof of this theorem is very complex and far beyond the scopes of this course. However, since it is a constructive proof on which also numerical methods can be based, it is worth of a short presentation of the main ideas behind it.

First step: transforming the Cauchy problem into a fixed point integral equation. We may notice that, if we integrate the equation on $[t_0, t]$, we get

$$\int_{t_0}^{t} y'(s)\, ds = \int_{t_0}^{t} f(s, y(s))\, ds, \iff y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))\, ds,$$

and since $y(t_0) = y_0$ (known) we end into the equation

$$\boxed{y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.} \tag{4.2.1}$$

It can be proved that the vice versa also holds: if y is a solution of (4.2.1), then y is a solution of $CP(t_0, y_0)$. So, solving (4.2.1) is 100% equivalent than solving the $CP(t_0, y_0)$. The equation (4.2.1) is a *fixed point equation*: y must be such that, if you plug it into the r.h.s. of (4.2.1), then you get back exactly your y. But, *why (4.2.1) should*

be better than our original Cauchy problem? This because there is a very simple and natural method to solve a fixed point equation. This is our next step.
Second step: solving fixed point equation. It is convenient, for a moment, to look at the equation (4.2.1) under a slightly more abstract way. At r.h.s. we have a "black box" that takes $y = y(t)$ and gives you back a new function of t:

$$Ty(t) := y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Now,

$$y \text{ solution of (4.2.1)} \iff Ty = y.$$

We may observe that $y(t) \equiv y_0$ is, for example, a function verifying the passage condition, but

$$Ty(t) = y_0 + \int_{t_0}^{t} f(s, y_0)\, ds \not\equiv y_0,$$

in general. Set $y_1 := Ty_0$, and try again: probably

$$\begin{aligned} &y_2 := Ty_1 \not\equiv y_1, \\ &y_3 := Ty_2 \not\equiv y_2, \\ &\vdots \\ &y_{n+1} := Ty_n \not\equiv y_n, \\ &\vdots \end{aligned}$$

but there is a reasonable hope that this recurrence sequence of functions will somehow converge to a y that will verify the equation $Ty = y$. Let us see why this belief is reasonable on an extremely simple example where we know the solution in advance. Consider the Cauchy problem

$$\begin{cases} y' = y, \\ \\ y(0) = 1. \end{cases} \implies y(t) = e^t.$$

Here, $f(t, y) = y$, so the (4.2.1) becomes,

$$y(t) = 1 + \int_0^t y(s)\, ds.$$

Now, applying the previous scheme we have:

$$\begin{aligned} &y_1(t) = y_0 + \textstyle\int_{t_0}^t y_0\, ds = 1 + \int_0^t 1\, ds = 1 + t, \\ &y_2(t) = y_0 + \textstyle\int_{t_0}^t y_1(s)\, ds = 1 + \int_0^t 1 + s\, ds = 1 + t + \frac{t^2}{2}, \\ &y_2(t) = y_0 + \textstyle\int_{t_0}^t y_2(s)\, ds = 1 + \int_0^t 1 + s + \frac{s^2}{2}\, ds = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!}, \\ &\vdots \\ &y_n(t) = 1 + t + \tfrac{t^2}{2} + \cdots + \tfrac{t^{n+1}}{(n+1)!}. \end{aligned}$$

Now,

$$\lim_n y_n(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t,$$

which is the solution of the Cauchy problem. In general, defining

$$\boxed{y_0(t) \equiv y_0, \quad y_{n+1}(t) := Ty_n(t) = y_0 + \int_{t_0}^t f(s, y_n(s))\, ds,} \tag{4.2.2}$$

it is possible to prove that, under the hypotheses of the theorem, $\lim_n y_n(t)$ exists and it defines a function $y(t)$ that solves the equation (4.2.1), which is equivalent to say that y solves the $CP(t_0, y_0)$. The (4.2.2) provides a numerical algorithm for approximate calculus of solutions: each y_n is computed recursively from the previous one. □

Example 4.2.2: (∗∗)

Q. *Check the global Cauchy–Lipschitz's theorem for the equation*

$$y' = \frac{\sin(ty)}{1+y^2}.$$

on a strip $[a,b]\times\mathbb{R}$.

A. Here $f(t,y) = \frac{\sin(ty)}{1+y^2}$ is clearly defined on $(t,y)\in\mathbb{R}\times\mathbb{R}$ and therein continuous. Moreover,

$$\partial_y f(t,y) = \frac{(1+y^2)t\cos(ty) - 2y\sin(ty)}{(1+y^2)^2}.$$

Therefore

$$|\partial_y f(t,y)| = \left|\frac{(1+y^2)t\cos(ty) - 2y\sin(ty)}{(1+y^2)^2}\right| \leqslant \frac{(1+y^2)|t| + 2|y|}{(1+y^2)^2} = \frac{|t|}{1+y^2} + \frac{2|y|}{(1+y^2)^2}.$$

Clearly $\frac{1}{1+y^2} \leqslant 1$ and because $2ab \leqslant a^2+b^2$, $\frac{2|y|}{(1+y^2)^2} \leqslant \frac{1+y^2}{(1+y^2)^2} = \frac{1}{1+y^2} \leqslant 1$, we have

$$|\partial_y f(t,y)| \leqslant |t| + 1 \leqslant \max\{|a|,|b|\} + 1 =: L,\ \forall (t,y)\in[a,b]\times\mathbb{R}. \qquad \square$$

4.2.2. Local existence and uniqueness. The global Cauchy–Lipschitz's theorem has two important limitations:

- *the domain that must be a strip* $[a,b]\times\mathbb{R}$
- *the* $\partial_y f$ *that must be bounded.*

If one or both these conditions are not fulfilled by f, the conclusions of the global theorem, can be false.

Example 4.2.3: (∗∗)

Q. *Solve the Cauchy problem*

$$\begin{cases} y' = y^2, \\ y(0) = y_0. \end{cases}$$

A. Let $f(t,y) = y^2$. The domain of the equation is $D = \mathbb{R}\times\mathbb{R}$ and $f \in \mathsf{C}(D)$. However, $\partial_y f(t,y) = 2y$ is unbounded on every strip $[a,b]\times\mathbb{R}$,

$$\sup_{(t,y)\in[a,b]\times\mathbb{R}} |\partial_y f(t,y)| = \sup_{(t,y)\in[a,b]\times\mathbb{R}} 2|y| = +\infty,$$

so the global Cauchy-Lipschitz's theorem does not apply.
The differential equation is a separable variables equation, and solutions can be easily found by separation of variables. We first notice that $y \equiv 0$ is a trivial solution, defined for $t \in]-\infty,+\infty[$. Then, if $y \neq 0$ is a solution,

$$y'(t) = y(t)^2, \iff \frac{y'(t)}{y(t)^2} = 1, \iff \left(-\frac{1}{y(t)}\right)' = 1, \iff -\frac{1}{y(t)} = t + c, \iff y(t) = -\frac{1}{t+c}.$$

We notice that this solution is never $= 0$ therefore, we have the following alternative

- either a solution is $y \equiv 0$,
- or, if $y \neq 0$ at some point, it is always $\neq 0$ and $y(t) = -\frac{1}{t+c}$ for some c.

This is the general integral of the equation. Imposing the passage condition we distinguish two cases:

- case $y_0 = 0$: since $y(0) = 0$, there is a unique solution of the equation verifying this, it is $y(t) \equiv 0$.
- case $y_0 \neq 0$: since $y(0) = y_0 \neq 0$, the solution must be always $\neq 0$ and $y(t) = -\frac{1}{t+c}$ for some c. By imposing the passage condition

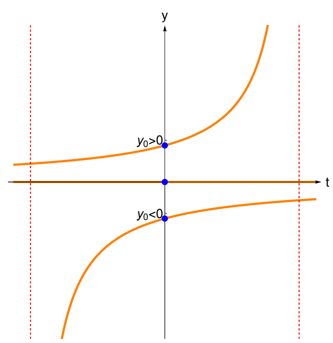
$$y(0) = y_0, \iff -\frac{1}{c} = y_0, \iff c = -\frac{1}{y_0},$$

from which

$$y(t) = -\frac{1}{t - \frac{1}{y_0}} = \frac{1}{\frac{1}{y_0} - t}.$$

As a function, this y is defined for $t \neq \frac{1}{y_0}$, that is on $]-\infty, \frac{1}{y_0}[\cup]\frac{1}{y_0}, +\infty[$. However, **as a solution**, this y must be defined on an interval containing $t = 0$. The biggest possible interval is either $]-\infty, \frac{1}{y_0}[$ or $]\frac{1}{y_0}, +\infty[$, and the choice is the interval that contains $t = 0$, so

$$y(t) = \frac{1}{\frac{1}{y_0} - t},\ t \in]-\infty, \tfrac{1}{y_0}[, \quad \text{if } y_0 > 0,$$
$$y(t) = \frac{1}{\frac{1}{y_0} - t},\ t \in]\tfrac{1}{y_0}, +\infty[, \quad \text{if } y_0 < 0.$$



This example shows that, in general, the life-span of a solution varies with the initial condition. This is why we cannot way in advance what this will be. In the example we found the life-span by explicitly solving the Cauchy problem. But what if we cannot do this?

Theorem 4.2.4: Local Cauchy–Lipschitz

Let $f : D \subset \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that

i) $f \in C(D)$;

ii) $\partial_y f \in C(D)$.

Then, for every $(t_0, y_0) \in \mathrm{Int}(D)$ there exists a unique solution $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ of $CP(t_0, y_0)$ defined on the largest possible interval $]\alpha, \beta[$. In particular, if $u : J \longrightarrow \mathbb{R}$ is any solution of $CP(t_0, y_0)$ then $J \subset]\alpha, \beta[$ and $u = y$ on J.

The solution y is called **maximal solution**.

Example 4.2.5: $(**)$

Q. *Show that the equation*

$$y' = \frac{e^{ty}}{1+y^2}$$

fulfills the local Cauchy–Lipschitz condition on $\mathbb{R}\times\mathbb{R}$ *but* **not** *the global condition on any strip* $[a,b]\times\mathbb{R}$.

A. Here $f(t,y) = \frac{e^{ty}}{1+y^2}$ is defined on $D = \mathbb{R}\times\mathbb{R}$. Clearly f is continuous and

$$\partial_y f(t,y) = \frac{te^{ty}(1+y^2) - 2ye^{ty}}{(1+y^2)^2} = e^{ty}\frac{t(1+y^2)-2y}{(1+y^2)^2} \in \mathsf{C}(\mathbb{R}^2).$$

This shows that the local Cauchy–Lipschitz condition holds. However, $\partial_y f(t,y)$ is clearly unbounded on any strip $[a,b]\times\mathbb{R}$. Indeed: if $t \neq 0$ is fixed, $|\partial_y f(t,y)| \longrightarrow +\infty$ as $y \longrightarrow +\infty$.

In general, the explicit determination of the life-span for a maximal solution is difficult if not impossible. Nonetheless, we can say something on the behavior of the solution at endpoints of the life-span. The idea is that when the solution "expires" (in the past, when $t \longrightarrow \alpha+$, in the future, when $t \longrightarrow \beta-$) something "dramatic" should happens. For example: *the solution cannot "die" in the interior of D*. Indeed, let us imagine that the "final" limit point $(\beta, y(\beta-)) \in \mathrm{Int}(D)$. Applying local existence and uniqueness, the $CP(\beta, y(\beta-))$ would have a solution $\widehat{y} :]\widehat{\alpha}, \widehat{\beta}[\longrightarrow \mathbb{R}$, with $\beta \in]\widehat{\alpha}, \widehat{\beta}[$, so in particular $\widehat{\beta} > \beta$. But this seems impossible: defining

$$\widetilde{y}(t) := \begin{cases} y(t), & t < \beta, \\ \\ \widehat{y}(t), & \beta \leqslant t < \widehat{\beta}, \end{cases}$$

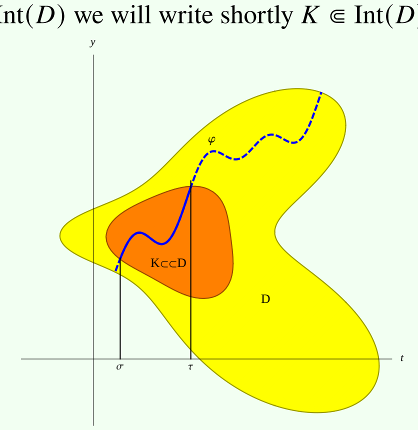
we would have a solution of the differential equation on $]\alpha, \widehat{\beta}[$ such that $\widetilde{y}(t_0) = y(t_0) = y_0$, that is $\widetilde{y}$ would solve $CP(t_0, y_0)$. Therefore $]\alpha, \widehat{\beta}[\subset]\alpha, \beta[$ and, in particular, $\widehat{\beta} < \beta$, in contradiction with $\widehat{\beta} > \beta$. This is, actually, a special case of general result we mention here, because, as we will see, it plays an important role while doing a qualitative study of solutions of a differential equation.

Theorem 4.2.6: exit from compact sets

Let $f : D \subset \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be fulfilling the hypotheses of local Cauchy-Lipschitz's theorem. Then, every maximal solution y must leave every compact set $K \subset \mathrm{Int}(D)$. Precisely:

$$\forall K \subset \mathrm{Int}(D),\ K \text{ compact} \ : \ \exists \sigma, \tau \ : \ (t, y(t)) \notin K,\ \forall t < \sigma,\ \forall t > \tau.$$

To write K compact included in $\mathrm{Int}(D)$ we will write shortly $K \Subset \mathrm{Int}(D)$.



4.3. Qualitative Study of Scalar Equations

The goal of this section is to illustrate, through examples, the problem of a *qualitative study* of the solution of a Cauchy problem. The main question we will answer is: *is it possible to plot a graph of the solution of a CP without computing the solution itself?* Apparently, this looks to be impossible. However, the differential equation yields a relation expressing y' in function of y. The idea is to exploit this to draw conclusions on the behavior of the solution. In general, each problem is different from any other, and even if some arguments are standard, their combination is not.

Example 4.3.1: $(***)$

Q. *Consider the Cauchy problem*

$$\begin{cases} y' = \dfrac{\tan y}{1+y^2}, \\ \\ y(0) = y_0, \end{cases}$$

i) Show that local existence and uniqueness hold. ii) Find constant solutions and regions of D where the solutions are increasing/decreasing. iii) Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ *the maximal solution with* $y_0 \in]0, \frac{\pi}{2}[$. *Show that y is monotone and deduce that* $\alpha = -\infty$, *computing also* $y(-\infty)$. *iv) Show that* $\beta < +\infty$ *and compute* $y(\beta-)$. *v) Show that* $y \in \mathsf{C}^2$ *and find the concavity of y. Use this to show again that* $\beta < +\infty$ *and to deduce an estimate of* β. *vi) Plot a qualitative graph of y.*

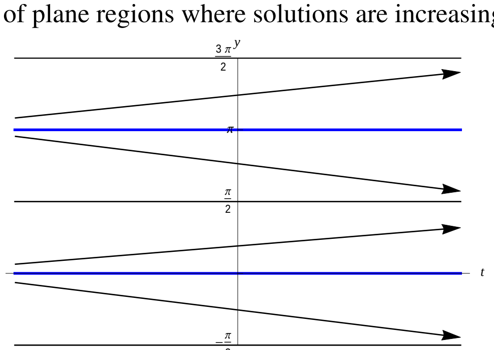
A. i) Let $f(t,y) = \frac{\tan y}{1+y^2}$, defined on $D :=]-\infty, +\infty[\times \left(\mathbb{R} \backslash \left\{ \frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z} \right\}\right)$. On D, $f, \partial_y f$ are clearly continuous: therefore, local existence and uniqueness are fulfilled.
ii) $y \equiv c$ is a constant solution iff

$$0 = y' = \frac{\tan y}{1+y^2} = \frac{\tan c}{1+c^2}, \iff \tan c = 0, \iff c = k\pi, \ k \in \mathbb{Z}.$$

Let y be a generic solution. Then

$$y \nearrow, \text{ on } I \iff y' = \frac{\tan y}{1+y^2} \geqslant 0, \text{ on } I, \iff k\pi \leqslant y < k\pi + \frac{\pi}{2}, \ k \in \mathbb{Z}.$$

By this we deduce the picture of plane regions where solutions are increasing/decreasing.



iii) Let $y :]a, b[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(0, y_0)$ with $y_0 \in]0, \frac{\pi}{2}[$. We prove that

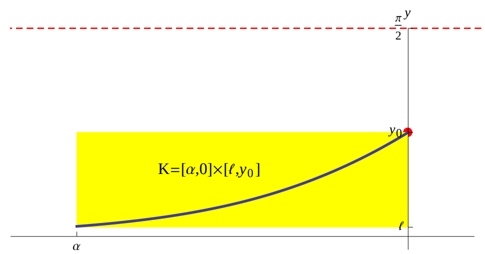
$$0 < y(t) < \frac{\pi}{2}, \ \ \forall t \in]a, b[,$$

and by ii) this implies that $y \nearrow$. Suppose, by contradiction, that there exists $\exists t_1 \in]a, b[$ such that $y(t_1) \leqslant 0$. Then, by intermediate values theorem, $y(\widehat{t}) = 0$ for some $\widehat{t}$. But then, we have 0 which is a solution of $CP(\widehat{t}, 0)$ and, by previous argument, also y is a solution of $CP(\widehat{t}, 0)$. By uniqueness, $y \equiv 0$, so in particular

$y_0 = y(0) = 0$, which is in contradiction with $y_0 > 0$. Similarly, it is impossible that there exists t_1 such that $y(t_1) \geqslant \frac{\pi}{2}$ otherwise there should be $\widehat{t}$ such that $y(\widehat{t}) = \frac{\pi}{2}$. In this case $(\widehat{t}, y(\widehat{t})) \notin D$, and this is impossible for a solution.
So, we now know that $y \nearrow$. Therefore, by properties of monotone functions, $\exists \lim_{t \to a+} y(t) = \ell$. Moreover, because $0 < y(t) < \frac{\pi}{2}$, we have $0 \leqslant \ell \leqslant \frac{\pi}{2}$ (actually $< \frac{\pi}{2}$). Since, by moving to the left, the solution decreases but it can never cross 0 and, at same time, it cannot die in $\mathrm{Int}(D)$, we guess that $\alpha = -\infty$. Indeed, if false, that is $\alpha > -\infty$, we would have

$$(t, y(t)) \in [\alpha, 0] \times [\ell, y_0] =: K \Subset \mathrm{Int}(D), \ \forall t < 0,$$

So, the solution would not get out (in the past) by the compact K, and this contradicts the exit from compact sets.



Thus $\alpha = -\infty$. About ℓ we have two alternatives: either $\ell = 0$ or $\ell > 0$. The second is impossible: taking the equation and passing to the limit as $t \longrightarrow -\infty$ we would have

$$y' \longrightarrow \frac{\tan \ell}{1 + \ell^2}.$$

But being $y \longrightarrow \ell$ as $t \longrightarrow -\infty$ **it cannot be** $\frac{\tan \ell}{1+\ell^2} \neq 0$. This follows from this general fact:

Proposition 4.3.2

Let $y \in \mathsf{C}^1$ be such that

$$\lim_{t \to \pm\infty} y(t) = \ell, \quad \lim_{t \to \pm\infty} y'(t) = \ell', \quad \text{with } \ell, \ell' \in \mathbb{R}, \quad \Longrightarrow \quad \ell' = 0.$$

Proof. It follows from the Hopital's rule:

$$0 = \lim_{t \to \pm\infty} \frac{y(t)}{t} \overset{(H)}{=} \lim_{t \to \pm\infty} y'(t) = \ell'. \quad \square$$

Therefore $\frac{\tan \ell}{1+\ell^2} = 0$, iff $\tan \ell = 0$, iff $\ell = k\pi$, and being $0 \leqslant \ell < y_0 < \frac{\pi}{2}$ necessarily $\ell = 0$. This proves that $\lim_{t \to -\infty} y(t) = 0$.
iv) We know that $y \nearrow$ so $\exists \lim_{t \to \beta-} y(t) =: \ell$. Since $0 < y(t) < \frac{\pi}{2}$, then $0 \leqslant \ell \leqslant \frac{\pi}{2}$ (actually $\ell \geqslant y_0$). We now deduce that $\beta < +\infty$. Indeed, if $\beta = +\infty$ we would have

$$y'(t) \longrightarrow \frac{\tan \ell}{1 + \ell^2} =: \ell'.$$

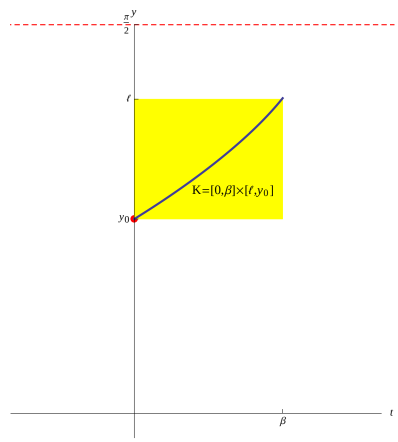
Now: either $\ell < \frac{\pi}{2}$ or $\ell = \frac{\pi}{2}$.

- Case $\ell < \frac{\pi}{2}$. In this case, $\ell' \in \mathbb{R}$. Then, by last proposition, $\ell' = 0$, that is $\frac{\tan \ell}{1+\ell^2} = 0$. This implies $\ell = k\pi$, and since $y_0 < \ell \leqslant \frac{\pi}{2}$, this is impossible.
- Case $\ell = \frac{\pi}{2}$, hence $\ell' = +\infty$. But also this is impossible: if $\ell' = +\infty$ then $y' \geqslant 1$ for t large, thus $y(t) \longrightarrow +\infty$, but this is incompatible with $y(t) \longrightarrow \frac{\pi}{2}$.

Since by assuming $\beta = +\infty$ we got a contradiction, it means that $\beta < +\infty$.
Let's see that $\ell = \frac{\pi}{2}$. If $\ell < \frac{\pi}{2}$ we would have that

$$\forall t > 0,\ (t, y(t)) \in [0, \beta] \times [y_0, \ell] \Subset \mathrm{Int}(D)$$

contradicting the exit by compact sets.



v) Concavity. By the equation, $y' = \frac{\tan y}{1+y^2}$, and computing the derivative w.r.t. t (do not forget that $y = y(t)$) we get

$$\begin{aligned} y'' &= \frac{(\tan y)'(1+y^2) - (\tan y)(1+y^2)'}{(1+y^2)^2} = \frac{(1+(\tan y)^2)y'(1+y^2) - (\tan y)2yy'}{(1+y^2)^2} \\ &= y'\frac{1+(\tan y)^2 + y^2 + y^2(\tan y)^2 - 2y\tan y}{(1+y^2)^2} = y'\frac{1+y^2(\tan y)^2 + (y-\tan y)^2}{(1+y^2)^2}. \end{aligned}$$

From this it is evident that $y'' \geqslant 0$ iff $y' \geqslant 0$, and because this is true by previous discussion, y is convex. Taking the tangent at $t = 0$ of equation

$$y = y_0 + \varphi'(0)t = y_0 + \frac{\tan y_0}{1+y_0^2}t,$$

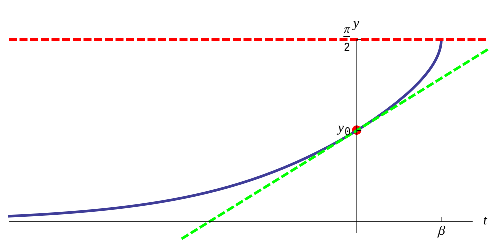
by properties of convex functions it follows that

$$y(t) \geqslant y_0 + \frac{\tan y_0}{1+y_0^2}t.$$

And because

$$y_0 + \frac{\tan y_0}{1+y_0^2}t \leqslant \frac{\pi}{2}, \quad \Longleftrightarrow \quad t \leqslant \left(\frac{\pi}{2} - y_0\right)\frac{1+y_0^2}{\tan y_0}$$

we deduce necessarily that $\beta < \left(\frac{\pi}{2} - y_0\right)\frac{1+y_0^2}{\tan y_0}$.
vi) Plot of y:



Example 4.3.3: (∗∗∗)

Q. *Consider the differential equation*

$$y' = \frac{1}{t^2 + y^2 - 1}.$$

i) Determine the domain of local existence and uniqueness. ii) Let now y be the solution of the $CP(0,0)$. iii) Show that y is odd decreasing and iv) find its concavity. v) Provide an estimate for the life-span of y and plot its graph.

A. i) Let $f(t,y) = \frac{1}{t^2+y^2-1}$. Clearly $f \in \mathsf{C}(D)$ where $D = \{(t,y) \in \mathbb{R}^2 \; : \; t^2 + y^2 \neq 1\}$ (the plane minus the circle centered in $(0,0)$ with radius 1). Moreover

$$\partial_y f = -\frac{2y}{(t^2 + y^2 - 1)^2} \in \mathsf{C}(D),$$

so hypotheses of local existence and uniqueness theorem are verified.
ii) Let now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(0,0)$. To show that y is odd we need to prove that

$$y(-t) = -y(t), \; \forall t \iff y(t) = -y(-t), \; \forall t$$

We will use the following standard argument: let z be defined as $z(t) := -y(-t)$. Notice that z is still a solution of the same Cauchy problem solved by y. Indeed: $z(0) = -y(-0) = -y(0) = -0 = 0$ and

$$z'(t) = (-y(-t))' = y'(-t) = \frac{1}{(-t)^2 + (y(-t))^2 - 1} = \frac{1}{t^2 + z(t)^2 - 1},$$

By uniqueness $z = y$, that is $-y(-t) = y(t)$ for every t.
iii) Monotonicity. We have

$$y \searrow, \iff y' \leqslant 0, \; \forall t, \iff t^2 + y^2 < 1, \; \forall t.$$

Now, as $t = 0$ this is true (because $y(0) = 0$). If for some t_1 it would be $t_1^2 + y(t_1)^2 \geqslant 1$ then, either $t_1^2 + y(t_1)^2 = 1$ (but this means $(t_1, y(t_1)) \notin D$, impossible) or $t_1^2 + y(t_1)^2 > 1$. Since $g(t) := t^2 + y(t)^2$ is a continuous function, $g(0) = 0$ and $g(t_1) > 1$, then for some t_2 we would have $g(t_2) = 1$. So, $t_2^2 + y(t_2)^2 = 1$, and again we get a contradiction ($(t_2, y(t_2)) \notin D$). Therefore $t^2 + y(t)^2 < 1$ for every t, from which $y \searrow$.
iv) Concavity. We compute y'':

$$y'' = -\frac{\left(t^2 + y^2 - 1\right)'}{\left(t^2 + y^2 - 1\right)^2} = -\frac{2t + 2yy'}{(t^2 + y^2 - 1)^2}$$

Therefore

$$y'' \geqslant 0, \iff t + yy' \leqslant 0.$$

We know that $y' < 0$ always. Being $y(0) = 0$ we deduce that

$$y \geqslant 0, \iff t \leqslant 0.$$

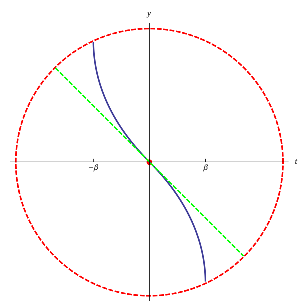
Therefore:

- $t \leqslant 0 \Longrightarrow y \geqslant 0$, and since $y' < 0$ we have $t + yy' \leqslant 0$.
- $t \geqslant 0 \Longrightarrow y \leqslant 0$, and since $y' < 0$ we have $t + yy' \geqslant 0$.

In conclusion: y is convex for $t \leqslant 0$, concave for $t \geqslant 0$.
v) In particular, for $t \geqslant 0$, y lies below of its tangent at $t = 0$, that is

$$y(t) \leqslant y(0) + y'(0)t = -t,$$

and since the straight line $y = -t$ crosses $t^2 + y^2 = 1$ at $t = \frac{\sqrt{2}}{2}$, it follows that $\beta < \frac{\sqrt{2}}{2}$. Plot of y follows.



These few examples provides some insight into the algorithm for draw a qualitative plot of a solution of a Cauchy problem.

Check List 4.3.4: plotting solutions of ODEs

To plot the solution of a $CP(t_0, y_0)$,

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0 \end{cases}$$

(1) Determine the domain D of the equation (where f is defined), check if local or global existence and uniqueness apply (is $f \in \mathsf{C}(D)$? is $\partial_y f \in \mathsf{C}(D)$? is $\partial_y f$ bounded on D strip?)
(2) Determine, if possible, where $f \geqslant 0$ (you get regions of plane ty where solutions are increasing)
(3) Determine, if any, stationary/constant solutions (they cannot be crossed by non constant solutions if existence and uniqueness holds)
(4) Determine monotonicity: is y staying all time in a region where f has constant sign?
(5) Behavior at endpoint of life-span $]\alpha, \beta[$: is $\alpha > -\infty$? what about $\lim_{t \to \alpha} y(t)$? Same discussion for $t \to \beta$ (it might be useful to discuss also limits of y': pass to the limit in the differential equation)
(6) Determine concavity: compute y'' by differentiating the equation, try to discuss $y'' \geqslant 0$.

Always remind that uniqueness is an important atout: you can use to prove symmetries (when present), as for example proving that y is even ($y(-t) \equiv y(t)$).

4.4. Systems of Differential Equations

Many real world systems are described by several quantities function of a parameter (e.g. *time*) through differential equations where each quantities enters in the equation for the others. The best way to understand this is through a couple of (important) example.

Example 4.4.1: prey-predator system

The prey-predator system was originally introduced by Volterra-Lotka in 1925/26. The goal is to describe the interaction between a prey population and a predator population. The main assumptions are:

i) in absence of predators, preys grow at constant rate $\alpha > 0$, in absence of preys, predators die at constant rate $\gamma > 0$.
ii) predators affect preys growth rate proportionally to their size (coefficient $\beta > 0$).
iii) the preys consumption determines an increase of predators growth rate proportional to the size of preys (coefficient $\delta > 0$).
iv) No other factor is considered.

Describe the dynamics of the system.

A. Let $x(t)$ =*size of preys* and $y(t)$ =*size of predators* at time t. The instantaneous growth rate of preys is

$$\frac{x(t+h)-x(t)}{h\cdot x(t)} \approx \frac{x'(t)}{x(t)}.$$

By i) and ii) we have

$$\frac{x'(t)}{x(t)} = \underbrace{+\alpha}_{preys\ growth} \quad \underbrace{-\beta y(t)}_{predation\ rate}, \implies x'(t) = \alpha x(t) - \beta x(t) y(t).$$

Similarly, from i) and iii)

$$\frac{y'(t)}{y(t)} = -\gamma + \delta x(t), \implies y'(t) = -\gamma y(t) + \delta x(t) y(t).$$

Thus, we end up with the following system of differential equations:

$$\begin{cases} x' = \alpha x - \beta x y, \\ \\ y' = -\gamma y + \delta x y. \end{cases}$$

Example 4.4.2: SIR pandemic

The SIR model is a simple compartmental framework for modeling the spread of an infectious disease through a fixed population. We assume that each individual can be in only one of the states: **susceptible** *(not yet infected but susceptible to infection),* **infected** *and* **removed** *(that is, either died or immunized, thus no more susceptible for further infection). Furthermore, we assume that*

i) susceptibles becomes infected at rate proportional to the number of infected;
ii) infected grow because of new infected susceptibles and decrease because of recovering or deceases.
iii) No other factor is considered.

Describe the dynamics of the system.

A. Let $S(t)$ =*number of susceptibles at time t*, and similarly $I(t)$ be the *number of infected*. According to i) and ii),

$$\frac{S'(t)}{S(t)} = -\alpha I(t), \quad \frac{I'(t)}{I(t)} = +\alpha S(t) - \beta, \implies \begin{cases} S' = -\alpha S I, \\ \\ I' = +\alpha S I - \beta I. \end{cases}$$

In general, we are led to consider a 2×2 system of differential equations

$$\begin{cases} x' = f(t,x,y), \\ \\ y' = g(t,x,y), \end{cases} \tag{4.4.1}$$

or, in a vector form,

$$\vec{x} := \begin{pmatrix} x \\ y \end{pmatrix}, \quad \vec{F}(t,\vec{x}) := \begin{pmatrix} f(t,x,y) \\ g(t,x,y) \end{pmatrix} \quad \vec{x}'(t) = \vec{F}(t,\vec{x}(t)). \tag{4.4.2}$$

With this formalism, the arguments we will see in this section can be extended to a system of n equations in n unknowns. For ease of presentation, we will remain on the case of 2×2 systems.

We assume that

$$f, g : D \subset \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}.$$

The set D is the **domain** of the system. A solution of system (4.4.1) is a pair $(x, y) : I \subset \mathbb{R} \longrightarrow \mathbb{R}$, I interval, such that x, y are differentiable on I, $(t, x(t), y(t)) \in D$ for every $t \in I$ (thus the r.h.s. of (4.4.1) makes sense) and they verify the system. The **Cauchy problem** consists in searching for a solution of the system verifying a passage condition, precisely

$$CP(t_0; x_0, y_0) \ : \ \begin{cases} x' = f(t, x, y), \\ y' = g(t, x, y), \\ x(t_0) = x_0, \\ y(t_0) = y_0. \end{cases} \tag{4.4.3}$$

In general, the existence of a solution for a Cauchy problem is ensured under very mild assumptions, that is for $f, g \in \mathsf{C}(D)$ (Peano's theorem). This is, however, too weak to have also uniqueness as we know. As for the case of scalar equations, with some further requirement on f, g we have also uniqueness.

Theorem 4.4.3: Global Cauchy–Lipschitz

Let $f, g : [a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathsf{C}([a, b] \times \mathbb{R}^2)$;

ii) $\partial_x f, \ \partial_y f, \ \partial_x g, \ \partial_y g$ are bounded in $[a, b] \times \mathbb{R}^2$, that is

$$\exists L > 0 \ : \ |\partial_x f(t, x, y)|, \ |\partial_y f(t, x, y)|, \ |\partial_x g(t, x, y)|, \ |\partial_y g(t, x, y)| \leqslant L, \ \forall (t, x, y) \in [a, b] \times \mathbb{R}^2.$$

Then, for every $(t_0, x_0, y_0) \in [a, b] \times \mathbb{R}^2$ there exists a unique solution to $CP(t_0; x_0, y_0)$.

A sensible weakening is the local existence and uniqueness theorem:

Theorem 4.4.4: Local Cauchy–Lipschitz

Let $f, g : D \subset \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathsf{C}(D)$;

ii) $\partial_x f, \ \partial_y f, \ \partial_x g, \ \partial_y g \in \mathsf{C}(D)$.

Then, for every $(t_0, x_0, y_0) \in \mathrm{Int}(D)$ there exists a unique solution $x, y :]\alpha, \beta[\longrightarrow \mathbb{R}$ of $CP(t_0; x_0, y_0)$ defined on the largest possible interval $]\alpha, \beta[$. In particular, if $u, v : J \longrightarrow \mathbb{R}$ is any solution of $CP(t_0; x_0, y_0)$ then $J \subset]\alpha, \beta[$ and $u \equiv x$, $v \equiv y$ on J.

Dealing with systems is far more complicate respect to scalar equations. The naive approach of solving first an equation (for instance, the equation for x), plug in the solution into the remaining equation and solve it does not work. Each equation depends, in general, on the "other" unknown. Even at level of qualitative study, it seems to be difficult to plot a graph of one or both components of a solution.

4.5. Autonomous systems

An *autonomous system* is a 2×2 system of differential equations (4.4.1) with f, g that do not depend explicitly on t, that is a system of the form

$$\begin{cases} x' = f(x, y), \\ \\ y' = g(x, y), \end{cases} \tag{4.5.1}$$

For example, both prey-predator and SIR pandemic are systems of this type.

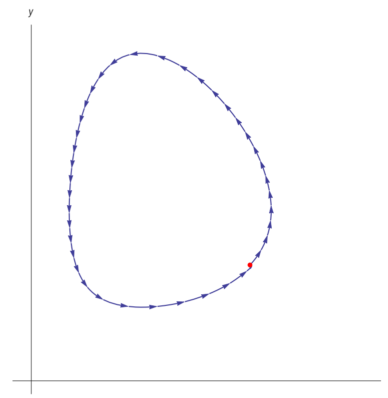
Notice that we are still using notations f, g we used for the general system (4.4.1). Here, f, g do not depend explicitly on t. So, if $f, g : E \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ are the functions involved into the system (4.5.1), the domain of the equation is

$$D = \mathbb{R} \times E \subset R \times \mathbb{R}^2.$$

We assume local existence and uniqueness hypotheses verified here, that is i) $f, g \in \mathsf{C}(E)$ and $\partial_x f, \partial_y f, \partial_x g, \partial_y g \in \mathsf{C}(E)$.

For 2×2 autonomous systems there is an interesting way to have a qualitative picture of the behavior of the system. The idea is to think to the solution $(x(t), y(t))$ as a time variable state in the *state plane* xy.

The map
$$t \longmapsto (x(t), y(t)),$$
is a curve in the xy plane. The trace left from the curve in the plane xy, that is the set
$$\{(x(t), y(t)) \ : \ t \in]\alpha, \beta[\} \subset E.$$
is called **orbit**. Like planet orbits, an orbit is just a geometrical line in the plane xy, thus it does not provide any information about the "motion", that is how the solution $(x(t), y(t))$ is moving along the orbit. To represent this, we introduce the **oriented orbit**, which is an orbit equipped with an orientation provided by the direction of the tangent vector $(x'(t), y'(t))$. Actually, since the solution moves along the orbit, the behavior of one of the two coordinates provides the orientation.



So, for example, the solution $(x(t), y(t))$ is moving from left to right (with respect to the $x-$axis) iff
$$x \nearrow, \iff x'(t) \geqslant 0, \iff f(x(t), y(t)) \geqslant 0, \iff \text{orbit} \subset \{(x, y) \ : \ f(x, y) \geqslant 0\}.$$
Similarly, the solution is moving upward iff
$$y \nearrow, \iff y'(t) \geqslant 0, \iff g(x(t), y(t)) \geqslant 0, \iff \text{orbit} \subset \{(x, y) \ : \ g(x, y) \geqslant 0\}.$$
Special orbits are those corresponding to stationary/constant solutions:
$$(x(t), y(t)) \equiv (x_0, y_0), \implies \{(x(t), y(t)) \ : \ t \in]\alpha, \beta[\} = \{(x_0, y_0)\},$$
that is the orbit is made of just one single point. These points are also called **equilibrium points** for the system.

Remark 4.5.1

$$(x_0, y_0) \text{ equilibrium} \iff \begin{cases} f(x_0, y_0) = 0, \\ \\ g(x_0, y_0) = 0. \end{cases}$$

Other particular orbits are those of *periodic solutions*. A periodic solution is a solution $(x(t), y(t))$ for which there exists $T > 0$ such that
$$(x(t + T), y(t + T)) = (x(t), y(t)), \ \forall t \in]\alpha, \beta[.$$
The smallest T for which this happens is called *period* of the solution. It is clear that the orbit of a periodic solution is a **circuit**. The vice-versa is non trivial:

Theorem 4.5.2

Assume that the system (4.7.1) fulfills local existence and uniqueness hypotheses. Then, a solution is periodic if and only if its orbit is a circuit.

A major problem when studying the qualitative behavior of a 2×2 system is to draw a plot of its oriented orbits. Actually, we do not need to plot all possible orbits (that would be impossible) but, rather, to plot the typical ones.

The figure obtained is called **phase portrait** of the system. In next sections we will explore how, in certain cases, this task can be performed.

4.6. Autonomous Linear Systems

An *autonomous linear system* is a system of the form

$$\left\{ \begin{array}{l} x' = ax + by, \\ \\ y' = cx + dy, \end{array} \right. \qquad a, b, c, d \in \mathbb{R} \text{ (constants)}. \tag{4.6.1}$$

Here $(f, g) := (ax + by, cx + dy)$, so both f, g fulfill the hypotheses of the global existence and uniqueness theorem being

$$\partial_x f \equiv a, \;\; \partial_y f \equiv b, \quad \partial_x g \equiv c, \;\; \partial_y g \equiv d.$$

We expect that some explicit solution formula should hold. We start noticing that

$$(x_0, y_0) \text{ equilibrium}, \iff \left\{ \begin{array}{l} ax_0 + by_0 = 0, \\ \\ cx_0 + dy_0 = 0. \end{array} \right.$$

Of course $(x_0, y_0) = 0_2$ is a solution. This is the unique equilibrium if and only if

$$\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0.$$

Throughout this Section we will assume $\det A \neq 0$ (we leave the case $\det A = 0$ in the exercises). We will also assume that at least one between b, c is $\neq 0$. This because if both $b = c = 0$, the system is actually a decoupled system of two first order equations

$$\left\{ \begin{array}{l} x' = ax, \\ \\ y' = dy, \end{array} \right. \qquad \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{at} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{dt} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For this reason we will assume $b \neq 0$. To recap, along this section we assume

$$\boxed{\det A = ad - bc \neq 0, \;\; b \neq 0.} \tag{4.6.2}$$

To determine non constant solutions, notice that by deriving side by side the first equation we obtain

$$x'' = ax' + by' = ax' + b(cx + dy) = ax' + bcx + d(x' - ax) = (a + d)x' - (ad - bc)x,$$

that is x fulfills a second order linear equation with constant coefficients. The characteristic polynomial for this equation is

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

Incidentally, notice that this is the same of

$$0 = \det(\lambda \mathbb{I} - A) = \det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix}.$$

In other words, λ are the eigenvalues of A. As expected, we will have three main situations according $\Delta = (a + d)^2 - 4(ad - bc) \gtreqless 0$.

Case $\Delta > 0$. The characteristic polynomial has two real roots $\lambda_2 > \lambda_1$. Therefore,

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \;\; c_1, c_2 \in \mathbb{R}.$$

Now, by $x' = ax + by$ it follows

$$by = x' - ax = c_1(\lambda_1 - a)e^{\lambda_1 t} + c_2(\lambda_2 - a)e^{\lambda_2 t}.$$

Since $b \neq 0$,

(4.6.3) $$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} =: c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2.$$

As easily checked, *vectors $\vec{v}_{1,2}$ are linearly independent eigenvectors of A for, resp., eigenvalues $\lambda_{1,2}$.* So, $(\xi, \eta) := (c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t})$ are the coordinates of (x, y) in $\vec{v}_1, \vec{v}_2$ basis. Since λ_1, λ_2 cannot be both $= 0$, assuming for instance $\lambda_1 \neq 0$, we have

$$\eta = c_1 e^{\lambda_2 t} = c_1 \left(e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} = k \xi^{\lambda_2/\lambda_1},$$

that is, orbits are described by lines $\eta = k\xi^\gamma$ with $\gamma = \frac{\lambda_2}{\lambda_1}$ in $\vec{v}_1, \vec{v}_2$ basis.

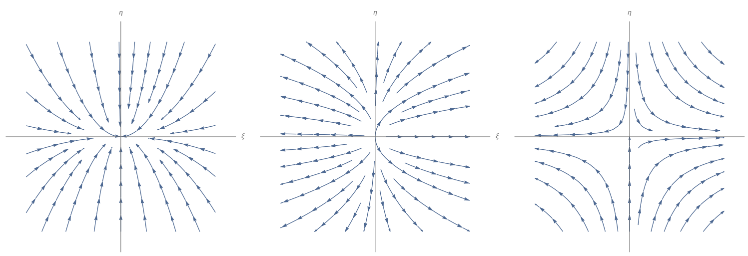


FIGURE 1. From the left to the right: $\lambda_2 > \lambda_1 > 0$, $0 > \lambda_2 > \lambda_1$, $\lambda_2 > 0 > \lambda_1$.

The orientation depends on the sign of λ_1, λ_2:

- If $\lambda_2 > \lambda_1 > 0$, the solution "escapes" to infinity in the future: in this case we say that 0_2 is an **unstable knot**;
- if $0 > \lambda_2 > \lambda_1$, the solution is "attracted" by 0_2 in the future: in this case we way that 0_2 is a stable a **stable knot**;
- if $\lambda_2 > 0 > \lambda_1$, the solution "escapes" in the first coordinate while it is attracted in the second: we say that 0_2 is a **saddle**.

Finally, to have the real phase portrait we have to remind that (ξ, η) are the coordinates of the solution (x, y) in the base $\vec{v}, \vec{w}$.

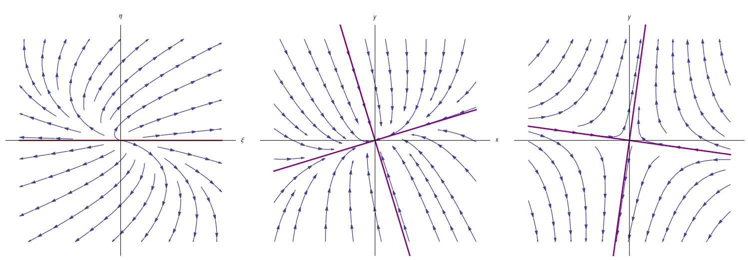


FIGURE 2. From the left: *unstable knot*, *stable knot*, *saddle*.

Example 4.6.1: $(*)$

Q. *Find the general solution and phase portrait of the system*

$$\begin{cases} x' = -2x + y, \\ y' = -2y + x. \end{cases}$$

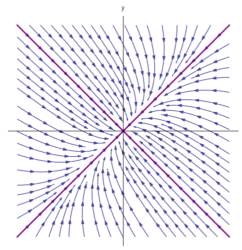
A. The characteristic equation is

$$p(\lambda) = \det \begin{bmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{bmatrix} = (\lambda+2)^2 - 1 = 0, \iff \lambda + 2 = \pm 1, \iff \lambda = -1, -3.$$

Therefore, according to (4.6.3),

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Because $\lambda_1, \lambda_2 < 0$ we have that $(0,0)$ is a stable knot.



Case $\Delta = 0$. The characteristic polynomial has a unique root $\lambda_1 \neq 0$ and

$$x(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}, \quad c_1, c_2 \in \mathbb{R}.$$

Again, by $x' = ax + by$

$$by = x' - ax = (c_1(\lambda_1 - a) + c_2)\, e^{\lambda_1 t} + c_2(\lambda_2 - a) e^{\lambda_2 t},$$

and since $b \neq 0$, we get

(4.6.4)
$$\begin{pmatrix} x \\ y \end{pmatrix} = (c_1 + c_2 t) e^{\lambda_1 t} \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} + c_2 e^{\lambda_1 t} \begin{pmatrix} 0 \\ \frac{1}{b} \end{pmatrix} \equiv (c_1 + c_2 t) e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_1 t} \vec{v}_2.$$

As before, denoting by $(\xi, \eta) := ((c_1 + c_2 t)e^{\lambda_1 t}, c_2 e^{\lambda_1 t})$, we have

$$\xi = k\eta + h\eta \log \eta.$$

Also in this case the orientation can be easily deduced by the sign of λ_1.

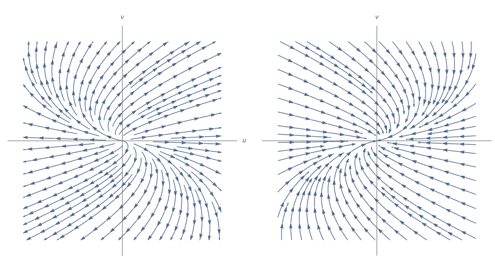


FIGURE 3. From left $\lambda_1 > 0$ (**unstable improper knot**), $\lambda_1 < 0$ (**stable improper knot**).

Example 4.6.2: $(*)$

Q. *Find the phase portrait of*

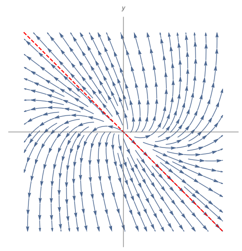
$$\begin{cases} x' = x - y, \\ y' = x + 3y. \end{cases}$$

A. The characteristic polynomial is

$$\det(\lambda\mathbb{I} - A) = \det\begin{bmatrix} \lambda-1 & 1 \\ -1 & \lambda-3 \end{bmatrix} = (\lambda-1)(\lambda-3)+1 = \lambda^2-4\lambda+4 = (\lambda-2)^2,$$

hence the unique eigenvalue is $\lambda = 2$. According to (4.6.4),

$$\begin{pmatrix} x \\ y \end{pmatrix} = (c_1 + c_2 t)e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$



Case $\Delta < 0$**.** Let $\lambda_{1,2} = \alpha \pm i\omega$, $\omega \neq 0$. Then

$$x(t) = c_1 e^{\alpha t}\cos(\omega t) + c_2 e^{\alpha t}\sin(\omega t).$$

Setting $\phi = e^{-\alpha t}x$, $\psi = e^{-\alpha t}y$, we have

$$\phi' = -c_1\omega\sin(\omega t) + c_2\omega\cos(\omega t),$$

and because $by = x' - ax$,

$$b\psi = be^{-\alpha t}y = be^{-\alpha t}x' - ae^{-\alpha t}x = be^{-\alpha t}x' - a\phi.$$

Now,

$$\phi' = \left(e^{-\alpha t}x\right)' = -\alpha e^{-\alpha t}x + e^{-\alpha t}x' = -\alpha\phi + e^{-\alpha t}x',$$

we obtain finally,

$$\psi = (\phi' + \alpha\phi) - \frac{a}{b}\phi = \phi' + (\alpha - \frac{a}{b})\phi.$$

Therefore

$$\begin{aligned}
\begin{pmatrix} x \\ y \end{pmatrix} &= e^{\alpha t}\begin{pmatrix} \phi \\ \psi \end{pmatrix} = e^{\alpha t}\begin{pmatrix} c_1\cos(\omega t) + c_2\sin(\omega t) \\ -c_1\omega\sin(\omega t) + c_2\omega\cos(\omega t) + (\alpha - \frac{a}{b})\left(c_1\cos(\omega t) + c_2\sin(\omega t)\right) \end{pmatrix} \\
&= e^{\alpha t}\left((c_1\cos(\omega t) + c_2\sin(\omega t))\begin{pmatrix} 1 \\ \alpha - \frac{a}{b} \end{pmatrix} + (-c_1\sin(\omega t) + c_2\cos(\omega t))\begin{pmatrix} 0 \\ -\omega \end{pmatrix}\right) \\
&= e^{\alpha t}\left(\xi\vec{v} + \eta\vec{w}\right), \quad \text{where } \vec{v} = \begin{pmatrix} 1 \\ \alpha - \frac{a}{b} \end{pmatrix}, \ \vec{w} = \begin{pmatrix} 0 \\ -\omega \end{pmatrix}.
\end{aligned}$$

Clearly, $\vec{v}$ and $\vec{w}$ are linearly independent. Furthermore, $\xi^2 + \eta^2 = c_1^2 + c_2^2$ constant, therefore

$$\xi\vec{v} + \eta\vec{w}$$

are points of an ellipse with axes $\vec{v}$ and $\vec{w}$. The factor $e^{\alpha t}$ leads to three sub cases:

- if $\alpha > 0$, orbits are elliptic spirals and solutions escape to infinity as $t \longrightarrow +\infty$: we say that 0_2 is an **unstable focus**;
- if $\alpha = 0$, orbits are ellipses: we say that 0_2 is a **center**;
- if $\alpha < 0$, orbits are elliptic spirals and solutions go to 0_2 as $t \longrightarrow +\infty$: we say that 0_2 is a **stable focus**.

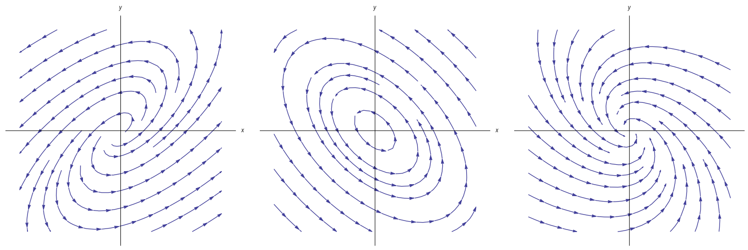


FIGURE 4. From left to right: *unstable focus*, *center*, *stable focus*.

Example 4.6.3: $(*)$

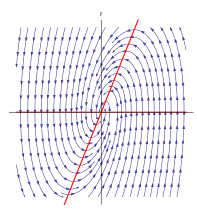
Q. *Find the phase portrait of the system*

$$\begin{cases} x' = -y, \\ y' = 5x - 2y. \end{cases}$$

A. We have

$$A = \begin{bmatrix} 0 & -1 \\ 5 & -2 \end{bmatrix}, \quad \det(\lambda \mathbb{I} - A) = \det \begin{bmatrix} \lambda & 1 \\ -5 & \lambda + 2 \end{bmatrix} = \lambda(\lambda + 2) + 5 = \lambda^2 + 2\lambda + 5 = 0,$$

by which $\lambda_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$, that is $\alpha = -1$, $\omega = 2$. The point $(0,0)$ is a stable focus then.



4.7. Non Linear systems: First Integrals

The plot of the phase portrait for a general 2×2 system

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases} \tag{4.7.1}$$

can be very difficult. Even in the case of linear systems, where solutons can be explicitly determined, the discussion is non trivial. In this Section we will see a method to get a picture of the phase portrait that works in some important cases.

The key idea is based on a physical principle known as the *conservation of energy*. This means that, in certain physical systems there is a quantity, named *energy*, defined on states of the system (here $H = H(x,y)$) which is conserved along trajectories of motion (that is $H(x(t), y(t))$ is constant in t). There are always functions E that verify this: just take any constant function. We are interested into non trivial functions verifying this property, independently of the physical nature of the system. This lead to the following definition.

Definition 4.7.1: first integral

Given a 2×2 system (4.7.1), a **first integral** is any non constant function $H = H(x, y)$ such that
$$H(x(t), y(t)) \equiv H(x(t_0), y(t_0)),\ \forall t \in I,\ (\forall (x(t), y(t)) \text{ solution of } (4.7.1).$$

In particular, if H is a non trivial first integral, *the orbits of the system are contained into the level sets of* H, that is
$$\{(x, y)\ :\ H(x, y) = H_0\}.$$

We are not saying that orbits coincide with level sets of a non constant first integral, as we will see the story is more complicate than this. Nonetheless, first integrals may help a lot in identifying orbits. This motivates the search for some condition helping to determine such type of functions.

Proposition 4.7.2

H is a first integral for the system (4.7.1) if and only if
$$\nabla H \cdot (f, g) \equiv 0. \tag{4.7.2}$$
In particular,
$$\exists \lambda = \lambda(x, y)\ :\ \nabla H = \lambda(g, -f),\ \forall (x, y) \in E \backslash S, \tag{4.7.3}$$
where S is the set of stationary points of the system.

Proof. Let $(x_0, y_0) \in E$ and $(x(t), y(t))$ be a solution of the Cauchy problem $x(t_0) = x_0$, $y(t_0) = y_0$. Since $H(x(t), y(y)) \equiv H(x_0, y_0)$, deriving this w.r.t. t, we have
$$\begin{aligned} 0 = \tfrac{d}{dt} H(x(t), y(t)) &= \partial_x H(x(t), y(t)) x'(t) + \partial_y H(x(t), y(t)) y'(t) \\ &= \partial_x H(x(t), y(t)) f(x(t), y(t)) + \partial_y H(x(t), y(t)) g(x(t), y(t)) \end{aligned}$$
At $t = t_0$ we get
$$\nabla H(x_0, y_0) \cdot (f(x_0, y_0), g(x_0, y_0) = 0,$$
and since this holds for every $(x_0, y_0) \in E$ we have the conclusion.

Now, if (x_0, y_0) is not a stationary point (for which we already know the orbits), then $(f, g)(x_0, y_0) \neq \vec{0}$, so
$$\nabla H \cdot (f, g) = 0, \iff \nabla H \perp (f, g), \iff \nabla H \parallel (g, -f), \iff \exists \lambda\ :\ \nabla H = \lambda(g, -f).$$
Since at every point (x_0, y_0) the multiplier λ could change, we have $\lambda = \lambda(x, y)$. □

The condition (4.7.3) says that H is a *potential* for the vector field $\lambda(g, -f)$. Function λ is called **integrating factor** and what (4.7.3) says is that, modulo an *integrating factor*, the field $(g, -f)$ must be conservative. Of course, if $(g, -f)$ itself is conservative, any potential is a first integral. We remind that, if f, g are regular (differentiable with partial derivatives continuous) a necessary (but not sufficient) condition to ensure $(g, -f)$ be conservative is that
$$(g, -f) \text{ irrotational} \iff \partial_y g \equiv \partial_x(-f), \iff \partial_y g = -\partial_x f.$$

If useful, since H is constant along solutions iff $aH + b$ is constant, if useful we can add/subtract and divide/multiply by constants without loosing the nature of first integral.

Example 4.7.3: $(*)$

Q. *Consider the system*

$$\begin{cases} x' = -y, \\ y' = x. \end{cases}$$

Determine a non trivial first integral and describe the orbits of the system.

A. Here $f = -y$, $g = x$ so $(g, -f) = (x, y) = (x, y)$. Clearly, (x, y) is irrotational ($\partial_y x \equiv 0 \equiv \partial_x y$) and

$$(x, y) = \nabla H, \iff \begin{cases} \partial_x H = x, \\ \partial_y H = y, \end{cases} \iff H(x, y) = \frac{1}{2}(x^2 + y^2) + k.$$

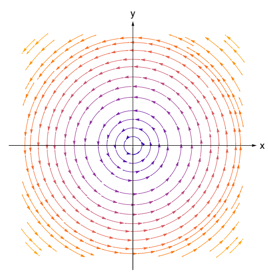
Thus, $H(x, y) = x^2 + y^2$ is a non trivial first integral for the system. The level sets of H are

$$\{H = H_0\} = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = H_0\}.$$

For $H_0 < 0$ the set is empty, for $H_0 = 0$ the level set is $\{(0, 0)\}$ (which is the unique stationary point of the system by the way). For $H_0 > 0$, the level set is non empty and, precisely, it is a circle centered in the origin with radius $\sqrt{H_0}$. So, orbits of non constant solutions are contained into circles centered at the origin. We can determine the orientation:

$$x \nearrow, \iff x' \geqslant 0, \iff -y \geqslant 0, \iff y \leqslant 0,$$

so, a solution moves from left to right iff (x, y) is in the lower half plane.



The previous example suggests that orbits are circles centered at the origin. In general, as we will see, level sets of first integrals can contain several orbits. This happens when, on the level set $\{H = H_0\}$ are contained stationary points (corresponding to stationary solutions) of the system. However, it is possible to prove the following

Proposition 4.7.4

Let H be a non trivial first integral for an autonomous system (4.7.1) fulfilling local existence and uniqueness. Let S be the set of stationary points for the system. Then, each connected component of $\{H = H_0\}\backslash S$ is an orbit of the system.

Remark 4.7.5

Returning to the last example, since each level set $\{x^2 + y^2 = H_0\}$, with $H_0 > 0$, does not contain stationary points (the unique of them is $(0, 0) \notin \{H = H_0\}$ when $H_0 > 0$), we have $\{H = H_0\}\backslash S = \{H = H_0\}$, and since this is a connected set it has a unique connected component, namely the circle centered at the origin with radius $\sqrt{H_0}$. This circle is then an orbit. Notice that, in particular, since the orbit is a circuit, the solution is periodic.

Warning 4.7.6

Different systems may have a same first integral. For example,

$$\begin{cases} x' = -y, \\ y' = x. \end{cases} \quad \begin{cases} x' = y, \\ y' = -x. \end{cases}$$

have the same first integral $H(x, y) = x^2 + y^2$.

Example 4.7.7: $(*)$

Q. *Find stationary points, a non trivial first integral and the phase portrait of the system*

$$\begin{cases} x' = y(x - y), \\ y' = -x(x - y), \end{cases}$$

A. (a, b) is stationary point iff

$$\begin{cases} 0 = b(a - b), \\ 0 = -a(a - b). \end{cases} \iff (a, b) = (0, 0), \text{ or } a = b.$$

Therefore, all points (infinitely many) of the straight line $y = x$ are stationary points. Each of them is then an orbit of a constant solution.
Let's discuss for a non trivial first integral. Here,

$$(g, -f) = (x(y - x), y(y - x)).$$

Since

$$\partial_y g = x, \quad \partial_x(-f) = -y, \quad \partial_y g \neq \partial_x(-f),$$

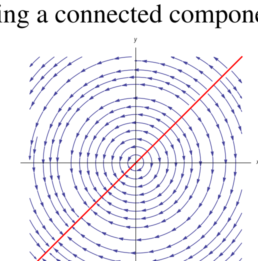
so $(g, -f)$ is not irrotational. However,

$$\underbrace{\frac{1}{y - x}}_{=:\lambda(x,y)}(g, -f) = (x, y) = \nabla \frac{1}{2}(x^2 + y^2)$$

we see that λ is an integrating factor, thus H is a first integral for the system. We can also consider $H(x, y) := x^2 + y^2$. The level sets of H are $\{x^2 + y^2 = H_0\}$. For $H_0 > 0$, these are circles centered at $(0, 0)$ with radius $\sqrt{H_0}$. On each circle, points $P_\pm := \pm(\sqrt{H_0}, \sqrt{H_0})$ correspond to stationary solutions. The set

$$\{H = H_0\} \backslash S$$

is made of two half circles, each one being a connected component, so an orbit of the system.



In this case, then, for $H_0 > 0$ the level set $\{H = H_0\}$ contains four different orbits: the two constant solutions $\{P_\pm\}$ and the two non constant solutions (the half circles).
To finish, we determine the orientation. We have

$$x \nearrow, \iff x' = y(x-y) \geqslant 0, \iff (y \geqslant 0, \text{ and } x - y \geqslant 0) \text{ or } (y \leqslant 0, \text{ and } x - y \leqslant 0).$$

So: the solution moves from left to right if (x, y) fulfils $0 \leqslant y \leqslant x$ or $x \leqslant y \leqslant 0$).

The last example is a particular case of the following proposition.

Proposition 4.7.8

Consider an autonomous system (4.7.1) verifying local existence and uniqueness. Suppose that

$$\frac{g(x,y)}{f(x,y)} = \frac{a(x)}{b(y)} \tag{4.7.4}$$

Then,

$$H(x,y) := \int a(x)\, dx - \int b(y)\, dy, \tag{4.7.5}$$

is a first integral for the system. When the condition (4.7.4) we say that the system (4.7.1) is **separable**.

Proof. By (4.7.4), we have

$$(g, -f) = f\left(\frac{g}{f}, -1\right) = f\left(\frac{a(x)}{b(y)}, -1\right) = \frac{f(x,y)}{b(y)}\,(a(x), -b(y))\,,$$

so, setting $\lambda(x,y) := \frac{b(y)}{f(x,y)}$, we have

$$\lambda(g, -f) = (a(x), -b(y)) = \nabla H, \iff \begin{cases} \partial_x H = a(x), \\ \partial_y H = -b(y), \end{cases}$$

that is

$$H(x,y) = \int a(x)\, dx - \int b(y)\, dy + c. \quad \square$$

Example 4.7.9: (∗∗) prey–predator system

Q. *Find the phase portrait of positive solutions (that is $x, y > 0$) of the prey-predator system*

$$\begin{cases} x' = \alpha x - \beta xy, \\ \\ y' = -\gamma y + \delta xy. \end{cases}$$

A. **Existence and uniqueness.** Here

$$f(x,y) = \alpha x - \beta xy, \quad g(x,y) = -\gamma y + \delta xy,$$

and clearly local existence and uniqueness hold.
Equilibriums. We have (a, b) is an equilibrium iff

$$\begin{cases} \alpha a - \beta ab = 0, \\ -\gamma b + \delta ab = 0, \end{cases} \iff \begin{cases} a(\alpha - \beta b) = 0, \\ b(-\gamma + \delta a) = 0, \end{cases} \iff (a,b) = (0,0), \ \vee\ (a,b) = \left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right).$$

The first solution is a trivial equilibrium: no prey, no predators, no life! The second one, is non trivial equilibrium: it represents a configuration where the two species stay stationary in time.
First integral. Since

$$\partial_y g = -\gamma + \delta x, \quad \partial_x(-f) = -\alpha + \beta y, \quad \partial_x g \neq \partial_y(-f),$$

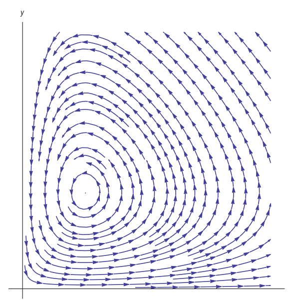
so, definitely $(g, -f)$ is not conservative. However, we may notice that

$$\frac{g(x,y)}{f(x,y)} = \frac{y(-\gamma+\delta x)}{x(\alpha-\beta y)} = \frac{(-\gamma+\delta x)/x}{(\alpha-\beta y)/y} =: \frac{a(x)}{b(y)},$$

so we have a separable system. A first integral is provided by

$$H(x,y) = \int \frac{-\gamma+\delta x}{x}\,dx - \int \frac{\alpha-\beta y}{y}\,dy = -\gamma \log x + \delta x - \alpha \log y + \beta y,$$

being $x, y > 0$. It is not easy to do an elementary plot of lines $H = H_0$, this can be easily accomplished by a computer.



About the orientation, we notice that

$$x \nearrow, \iff x' = x(\alpha - \beta y) \geqslant 0, \overset{x,y>0}{\iff} \alpha - \beta y \geqslant 0, \iff y \leqslant \frac{\alpha}{\beta}.$$

The phase portrait clearly shows a periodic behavior of the system.

A remarkable feature of first integrals is that they allow to do a "dimensional reduction" of the original system. Indeed, known that

$$H(x(t), y(t)) \equiv H(x(t_0), y(t_0)) =: H_0,$$

we could use this identity to explicit one of $x(t)$ or $y(t)$ as a function of the other. If, for example, we can draw $y(t)$ out of this relation,

$$y(t) = \varphi(x(t), H_0). \tag{4.7.6}$$

then, plugging this into the equation for x we get

$$x'(t) = f(x(t), \varphi(x(t), H_0)),$$

which is now a closed form differential equation that we could solve for x. Once this has been found, by (4.7.6) we automatically have also y.

Example 4.7.10: $(***)$

Q. *Consider the system*

$$\begin{cases} x' = xy, \\ \\ y' = -x^2 + 2x^4. \end{cases}$$

i) Find the equilibriums. ii) Determine a non trivial first integral. iii) Plot the phase portrait of the system: are there periodic solutions? iv) Find the x solution of the Cauchy problem $x(0) = 2,\ y(0) = 2\sqrt{3}$.

A. i) A point (a, b) is an equilibrium iff

$$\begin{cases} ab = 0, \\ \\ 2a^4 - a^2 = 0, \end{cases} \iff a = 0,\ \forall b,\ \vee\ b = 0,\ a^2(2a^2 - 1) = 0, \iff a = 0,\ \forall b,\ \vee\ b = 0,\ a = \pm\frac{1}{\sqrt{2}}.$$

All points $(0, b)$ ($b \in \mathbb{R}$) and $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ are stationary points.

ii) We have $f(x, y) = xy$, $g(x, y) = -x^2 + 2x^4$, so $\partial_y g \equiv 0$ while $\partial_x(-f) = -y$, so $(g, -f)$ is definitely not conservative. However,

$$\frac{g(x, y)}{f(x, y)} = \frac{-x^2 + 2x^4}{xy} = \frac{-x + 2x^3}{y} \equiv \frac{a(x)}{b(y)},$$

so

$$H(x, y) = \int 2x^3 - x\, dx - \int y\, dy = \frac{x^4}{2} - \frac{x^2}{2} - \frac{y^2}{2},$$

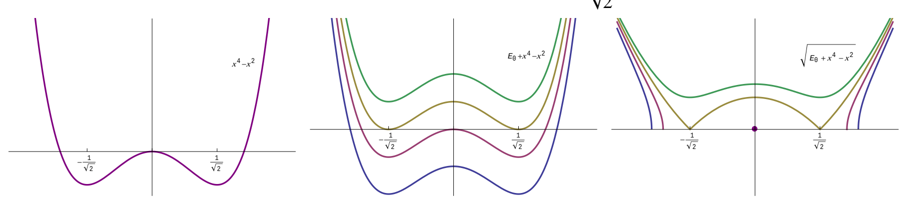
is a non constant first integral. For computational ease we will consider $H(x, y) = y^2 + x^2 - x^4$.

iii) We have

$$H = H_0, \iff y^2 + x^2 - x^4 \equiv H_0, \iff y = \pm\sqrt{x^4 - x^2 + H_0}.$$

To plot these lines for different values of H_0 we can proceed as follows:

(1) plot $x^4 - x^2$: it is even, $(x^4 - x^2)' = 4x^3 - 2x = 2x(2x^2 - 1) = 0$ as $x = 0, \pm\frac{1}{\sqrt{2}}$. Easily we deduce the monotonicity and we see that $x = 0$ is a local max, $x = \pm\frac{1}{\sqrt{2}}$ are global mins (fig. below left)



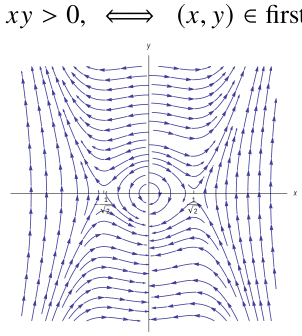
(2) plot $x^4 - x^2 + H_0$: just translate up and down the plots of $x^4 - x^2$ (fig. above center).

(3) plot $\sqrt{x^4 - x^2 + H_0}$: cut off the previous plot when negative, then $\sqrt{y}$ has the same monotonicity of y (fig above right).

(4) plot $\pm\sqrt{x^4 - x^2 + H_0}$: just reflect the previous plot w.r.t. x axis.

FInally, for the phase portrait, we determine the orientation of the orbits:

$$x \nearrow, \iff x' = xy > 0, \iff (x, y) \in \text{first and third quarter.}$$



We see that are periodic solutions for $0 < H_0 < \frac{1}{2}$ on the component of the level sets with $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.
iv) The sought solution fulfills $H(x,y) \equiv H(x(0),y(0)) = H(2,2\sqrt{3}) = (2\sqrt{3})^2+2^2-2^4 = 12+4-16 = 0$. Then,

$$y^2 = x^4 - x^2, \iff y = \pm\sqrt{x^4-x^2} = \pm|x|\sqrt{x^2-1} = \pm x\sqrt{x^2-1}.$$

Being $x(0) = 2$ and $y(0) = 2\sqrt{3}$, we deduce that

$$y = x\sqrt{x^2-1}.$$

Plugging this into the first equation we get

$$x' = xy = x\cdot x\sqrt{x^2-1} = x^2\sqrt{x^2-1}, \iff \frac{x'}{x^2\sqrt{x^2-1}} = 1.$$

This is a separable variable equation: integrating and setting $u = x(t)$

$$t + C = \int \frac{1}{u^2\sqrt{u^2-1}}\,du = \int \frac{1}{u^2\sqrt{u^2-1}}\,du \overset{u=\cosh v,\ du=\sinh v\ dv}{=} \int \frac{1}{(\cosh v)^2\sqrt{(\cosh v)^2-1}}\sinh v\,dv.$$

Now $\sqrt{(\cosh v)^2-1} = \sqrt{(\sinh v)^2} = |\sinh v| = \sinh v$ if $v \geqslant 0$, hence

$$t + C = \int \frac{1}{(\cosh v)^2}\,dv = \tanh v, \iff v = \tanh^{-1}(t+C), \iff x(t) = \cosh^{-1}\left(\tanh^{-1}(t+C)\right).$$

By imposing the initial condition we find C.

We can now summarize the principal techniques presented in this section.

Check List 4.7.11: discussing 2×2 systems

To plot a phase portrait of the system

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y), \end{cases}$$

(1) check if local/global existence and uniqueness apply ($f, g \in C$? $\partial_x f, \partial_y f, \partial_x g, \partial_y g \in C$? also bounded?)
(2) determine constant/stationary solutions $x \equiv a$, $y \equiv b$ (plug into the system and determine all possible pairs (a,b))
(3) determine a non constant first integral (if any), either
  (a) checking if the system is separable

$$\frac{f(x,y)}{g(x,y)} \equiv \frac{a(x)}{b(y)} \implies H(x,y) = \int a(x)\,dx - \int b(y)\,dy.$$

  (b) or searching for an integrating factor $\lambda = \lambda(x,y)$ such that

$$\lambda(g,-f) = \nabla H.$$

(4) plot typical level sets of H ($\{H(x,y) \equiv H_0\}$
(5) determine orbit orientation (for instance, $x \nearrow$ iff $f(x,y) \geqslant 0$)
(6) use H to explicitly solve Cauchy problems: use $H(x,y) \equiv H(x(0),y(0)) =: H_0$ to explicit $y = y(x)$ (or $x = x(y)$), then plug into the equations to get a closed form equation for x (or for y).

4.8. Differential Equations of order n

So far, we discussed differential equations of first order, that is when the derivative of the unknowns appear up to the first derivative. There are important differential equations of order higher than one. Perhaps, the most important example is the case of *Newton's equations*. Newton's second law of Classical Physics says that a (constant) mass m subject to a force $\vec{F}$ gets an acceleration $\vec{a}$ through the formula

$$m\vec{a} = \vec{F}.$$

Assuming, for simplicity, that the mass moves along a one dimensional straight guide, and denoting by $y(t)$ the position on the guide, then $a = y''(t)$. In Classical Physics, forces can depend on time t, position $y(t)$ and velocity $y'(t)$. Thus, Newton's second law boils down to a *second order differential equation*

$$my''(t) = F(t, y(t), y'(t)).$$

In general, we call *$n-$th order differential equation in normal form* an equation of type

$$y^{(n)}(t) = f(t, y(t), y'(t), \dots, y^{(n-1)}(t)).$$

For illustrative purposes, here we will focus on second order equations

$$y''(t) = f(t, y(t), y'(t)), \tag{4.8.1}$$

but what we say here extends to the more general case of $n-$th order equations.

The first remarkable fact is that *there is a standard equivalence between an $n-$th order equation and a first order system of differential equations*. We illustrate this on the case of the second order equation.

Proposition 4.8.1

$$y \text{ solution on } I \text{ of } y'' = f(t, y, y'), \iff (y, v) := (y, y') \text{ solution on } I \text{ of } \begin{cases} y' = v, \\ v' = f(t, y, v). \end{cases} \tag{4.8.2}$$

Proof. The proof is straightforward: if y solves (4.8.1) on I then, setting $v := y'$ we have that

$$y' = v, \quad v' = (y')' = y'' = f(t, y, y') = f(t, y, v),$$

so (y, v) is a solution of the system. Vice versa, if (y, v) solves the system, being $v = y'$ it will be $v' = *y')' = y''$, thus from the second equation we deduce

$$y'' = v' = f(t, y, v) = f(t, y, y'),$$

so y solves (4.8.1). □

The equivalence (4.8.2) between the second order equation and a suitable first order 2×2 system naturally yields a definition of Cauchy problem. Indeed, for the equivalent system (4.8.2), the Cauchy problem is

$$\begin{cases} y' = v, \\ v' = f(t, y, v), \\ y(t_0) = y_0, \\ v(t_0) = v_0. \end{cases}$$

Since $v = y'$ we get the following

Definition 4.8.2: Cauchy problem

The Cauchy problem for the second order equation (4.8.1) is

$$CP(t_0; y_0, v_0) \ : \ \begin{cases} y'' = f(t, y, y'), \\ y(t_0) = y_0, \\ y'(t_0) = v_0. \end{cases}$$

Thinking to a Newton's equation, this is the classical problem of determining the motion of a mass given its initial position and velocity. Global and local existence and uniqueness for the Cauchy problem can now be easily obtained from global/local existence and uniqueness theorems 4.4.3 and 4.4.4 for 2×2 systems. We have:

Theorem 4.8.3: Global Cauchy–Lipschitz

Let $f = f(t, y, v) : [a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathsf{C}([a, b] \times \mathbb{R}^2)$;

ii) $\partial_y f,\ \partial_v f$ are bounded in $[a, b] \times \mathbb{R}^2$, that is

$$\exists L > 0 \ :\ |\partial_y f(t, x, y)|,\ |\partial_v f(t, x, y)| \leqslant L,\ \forall (t, x, y) \in [a, b] \times \mathbb{R}^2.$$

Then, for every $(t_0, y_0, v_0) \in [a, b] \times \mathbb{R}^2$ there exists a unique solution to $CP(t_0; y_0, v_0)$.

Theorem 4.8.4: Local Cauchy–Lipschitz

Let $f = f(t, y, v) : D \subset \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that

i) $f \in \mathsf{C}(D)$;

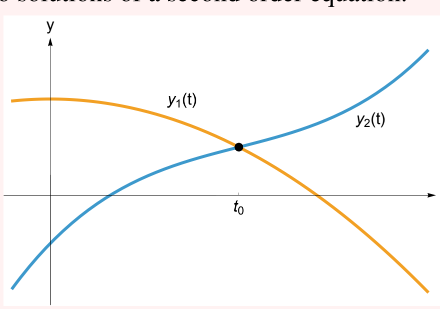
ii) $\partial_y f,\ \partial_v f \in \mathsf{C}(D)$.

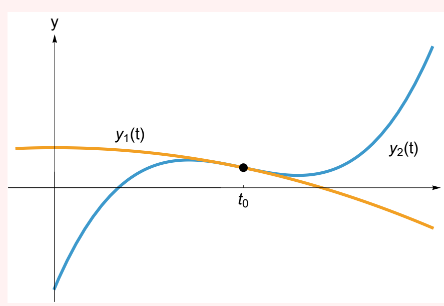
Then, for every $(t_0; y_0, v_0) \in \mathrm{Int}(D)$ there exists a unique solution $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ of $CP(t_0; y_0, v_0)$ defined on the largest possible interval $]\alpha, \beta[$. In particular, if $u : J \longrightarrow \mathbb{R}$ is any solution of $CP(t_0; y_0, v_0)$ then $J \subset]\alpha, \beta[$ and $u \equiv y$ on J.

The proof of these theorems is left as an exercise (see Exercise 4.10.13).

Warning 4.8.5

For a second order equation it may well happen that two solutions intersect at some point t_0, that is $y_1(t_0) = y_2(t_0)$ with $y_1 \not\equiv y_2$. This is not in contradiction with uniqueness! In fact, uniqueness requires that also $y_1'(t_0) = y_2'(t_0)$. In this case, under uniqueness, $y_1 \equiv y_2$. In other words, **if uniqueness holds**: the left figure below is **impossible** for any two solutions of a **first order equation**, while **it is well possible** for two solutions of a **second order equation**. If, however, the two solutions intersect at some point t_0 **with same tangent** (that is $y_1'(t_0) = y_2'(t_0)$) then, necessarily, $y_1 \equiv y_2$. So the right figure below is **impossible** for any two solutions of a second order equation.





4.9. Conservative Newton's equations

Conservative Newton's equations are second order equations where $f = f(t, y, y')$ actually is independent of t and $y'(t)$ and it depends only on $y(t)$, that is, it is a positional force (like gravitation or Coulomb electric force). Furthermore, it is supposed that the force field f has a potential V, that is $f(y) = \partial_y V(y)$. Therefore, Newton's equation takes the form

$$y'' = \partial_y V(y). \tag{4.9.1}$$

The reason why these equations are said to be *conservative* is because of the

Proposition 4.9.1

If y is a solution of (4.9.1) then the *mechanical energy*

$$E(y,v) := \frac{1}{2}v^2 - V(y),$$

is constant along the solution, that is $E(y(t), y'(t))$ is constant.

Proof. We have

$$\frac{d}{dt}E(y,y') = \left(\frac{1}{2}y'^2 - V(y)\right)' = \frac{1}{2}2y'y'' - \partial_y V(y)y' = y'\left(y'' - \partial_y V(y)\right) \equiv 0.$$

With a suitable transformation, we can see this as a particular case of first integral. Indeed, we start noticing that setting $v := y'$, then Newton's equation (4.9.1) is equivalent to the 2×2 system

$$\begin{cases} y' = v & := h(y,v), \\ v' = y'' = \partial_y V(y) & := g(y,v). \end{cases} \tag{4.9.2}$$

This is an autonomous system (4.7.1) in variables (y, v). It is a separable system being

$$\frac{h(y,v)}{g(y,v)} = \frac{v}{\partial_y V(y)}, \implies E(y,v) := \int v\, dv - \int \partial_y V(y)\, dy = \frac{1}{2}v^2 - V(y),$$

is a first integral. This correspondence makes natural to adopt the language and the methods developed for systems:

- an orbit is a curve in the space (y, v) (space×velocity/momentum) which is called *phase space*;
- the orientation of orbits follows a simple rule: by the first equation of the system (4.9.2)

$$y \nearrow \iff y' = v \geqslant 0, \iff (y,v) \text{ belongs in the upper half plane of the phase space.}$$

Example 4.9.2: (∗∗∗) pendulum without friction

Q. *Plot the phase portrait for the pendulum without friction, described by the equation*

$$m\ell\theta''(t) = -mg\sin(\theta(t)).$$

A. First notice that the equation may be written as

$$\theta'' = -\frac{g}{\ell}\sin\theta = \frac{g}{\ell}\partial_\theta \cos\theta = \partial_\theta\left(\frac{g}{\ell}\cos\theta\right)$$

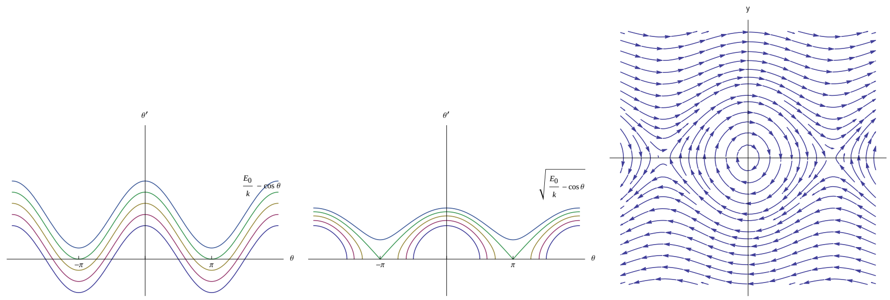
that is the equation is conservative. The mechanical energy is

$$E(\theta,\theta') = \frac{1}{2}\theta'^2 - \frac{g}{l}\cos\theta.$$

Now

$$E(\theta,\theta') = E_0, \iff \frac{\theta'^2}{2} - \frac{g}{l}\cos\theta = E_0, \iff \theta'^2 = 2\left(E_0 + \frac{g}{l}\cos\theta\right), \ \theta' = \pm\sqrt{2\left(E_0 + \frac{g}{l}\cos\theta\right)}.$$

It is not very difficult to plot the surfaces $E = E_0$. The orientation being standard we obtain the following picture:



The phase portrait shows that the classical periodic oscillatory motion (the closed cycles). The limit case when the cycle closes on an equilibrium correspond to a non periodic motion reaching in an infinite time in the future/past the equilibrium position $\theta = \pi$. Finally the not closed orbits in the upper and lower half plane corresponds to rotations: when the mass receive initially more than certain minimum energy the mass rotate infinitely many times. This is visible being θ an increasing/decreasing according to the direction anti/clockwise of the motion. $\square$

The conservation of the energy yields to a order reduction. Indeed, since

$$\frac{1}{2}y'^2 - V(y) \equiv E_0, \tag{4.9.3}$$

we get

$$y' = \pm\sqrt{2(E_0 - V(y))}.$$

This is a first order separable variables equation that can be solved (modulo integrations).

Example 4.9.3: (∗∗∗)

Q. *Consider the differential equation*

$$y'' = y^2 - y, \quad (\star).$$

i) Check local existence and uniqueness holds and determine any stationary solutions. ii) Show that if $y : I \longrightarrow \mathbb{R}$ is a solution of $(\star)$ then also $y(-t)$ is a solution. Deduce that the maximal solution of the Cauchy problem with initial conditions $\varphi(0) = a$, $\varphi'(0) = 0$ is even. iii) Determine the energy of the system and use this to solve the Cauchy problem $y(0) = 3$, $y'(0) = \sqrt{3}$.

A. i) We have a second order equation

$$y'' = f(t, y, y'), \text{ where } f(t, y, v) = y^2 - y.$$

Clearly $f \in \mathsf{C}(\mathbb{R} \times \mathbb{R}^2)$ and since $\partial_y f = 2y - 1 \in \mathsf{C}(\mathbb{R} \times \mathbb{R}^2)$ and $\partial_v f \equiv 0$, local existence and uniqueness holds on $\mathbb{R} \times \mathbb{R}^2$. Notice that global existence does not apply because $|\partial_y f| = |2y - 1|$ is unbounded in y. Stationary solutions: we have $y(t) \equiv C$ is a solution iff $0 = C^2 - C = C(C - 1)$, that is for $C = 0$, $C = 1$. So que have two stationary/constant solutions, $y \equiv 0$ and $y \equiv 1$.
ii) Let $y : I \longrightarrow \mathbb{R}$ be a solution and define $u : -I \longrightarrow \mathbb{R}$, as $u(t) := y(-t)$. We have $u'(t) = -y'(-t)$, $u''(t) = y''(-t)$, so

$$u''(t) = y''(-t) = y(-t)^2 - y(-t) = u(t)^2 - u(t),$$

that is, u is a solution of the equation $(\star)$. If now $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ is the (unique) solution of the Cauchy problem $y(0) = a$ and $y'(0) = 0$, we have that $u(t) := y(-t)$ solves the equation and, moreover, $u(0) = y(-0) = y(0) = a$ while $u'(0) = -y'(-0) = -y'(0) = 0$, so u solves the same Cauchy problem solved by y. By uniqueness, $y \equiv u$, from which $y(t) = y(-t)$, that is y is even.

iii) Since

$$y^2 - y = \partial_y \left(\frac{y^3}{3} - \frac{y^2}{2} \right),$$

the energy of the system is

$$E(y, v) := \frac{v^2}{2} - \left(\frac{y^3}{3} - \frac{y^2}{2} \right).$$

So, if y is the solution fulfilling $y(0) = 3$, $y'(0) = \sqrt{3}$, we have $E(3, \sqrt{3}) = \frac{9}{2} - \left(\frac{27}{3} - \frac{9}{2} \right) = 9 - 9 = 0$, so

$$E(y, y') \equiv 0, \iff \frac{y'^2}{2} - \frac{y^2}{2}\left(\frac{2}{3}y - 1 \right) \equiv 0, \iff y' = \pm |y| \sqrt{\frac{2}{3}y - 1}.$$

Now $y'(0) = 3$ and $y(0) = 3$, so

$$y' = y\sqrt{\frac{2}{3}y - 1}, \iff \frac{y'}{y\sqrt{\frac{2}{3}y - 1}} = 1, \iff \int \frac{y'}{y\sqrt{\frac{2}{3}y - 1}}\, dt = t + c.$$

Now

$$\int \frac{y'}{y\sqrt{\frac{2}{3}y-1}} \overset{u=y(t)}{=} \int \frac{1}{u\sqrt{\frac{2}{3}u - 1}}\, du \overset{v=\sqrt{\frac{2}{3}u-1},\, u=\frac{3}{2}(v^2+1),\ du=3\,dv}{=} \int \frac{1}{\frac{3}{2}(v^2+1)v} 3v\, dv = 2\arctan v$$

$$= 2\arctan \sqrt{\frac{2}{3}u - 1} = 2\arctan \sqrt{\frac{2}{3}y - 1},$$

hence

$$2\arctan \sqrt{\frac{2}{3}y(t) - 1} - \frac{\pi}{2} = t + c.$$

Setting $t = 0$ we find $2\arctan 1 = c$, that is $c = \frac{\pi}{2}$, so the solution is $y(t) = \frac{3}{2} + \frac{3}{3}\left(\tan \frac{2t+\pi}{4}\right)^2$. □

Check List 4.9.4: discussing Newton's conservative equations

To plot a phase portrait of

$$y'' = f(y),$$

(1) check if local/global existence and uniqueness apply ($f \in \mathsf{C}$? $\partial_y f \in \mathsf{C}$? also bounded?)
(2) determine constant/stationary solutions $y \equiv c$
(3) determine a non constant first integral

$$E(y, v) = \frac{1}{2}v^2 - V(y),\ V(y) = \int f(y)\, dy.$$

(4) plot typical level sets of E ($\{E(y, v) \equiv E_0\}$)
(5) easy orientation: $y \nearrow$ iff $v \geqslant 0$.
(6) use E to explicitly solve Cauchy problems: use $y' = \pm\sqrt{2(E_0 - V(y))}$, the $\pm$ must be decided by using the initial condition.

4.10. Exercises

Exercise 4.10.1 ($**$)**.** Consider the equation

$$y' = y\left(e^y - 1\right).$$

i) $(*)$ Check that local existence and uniqueness hypotheses are fulfilled in the domain of the equation.
ii) $(*)$ Find stationary solutions (if any), draw regions of plane ty where solutions are increasing/decreasing.
iii) $(**)$ Let y be the maximal solution with $y(0) = y_0 > 0$. Determine monotonicity and concavity of y. Find the life-span for the solution and the limits at the endpoints of it. Plot a graph of y.

Exercise 4.10.2 $(***)$**.** Consider the equation

$$y' = \frac{\log y}{1+y}.$$

i) $(*)$ Determine the domain of the equation, discuss local and global existence and uniqueness.
ii) $(*)$ Determine any stationary solutions and regions of the plane ty where solutions are increasing/decreasing.
iii) $(**)$ Since now let $y :]\alpha, \beta[\longrightarrow \mathbb{R}$ be the maximal solution with $y(0) = y_0 \in]0, 1[$. Show that y is monotone and determine its concavity.
iv) $(***)$ Determine if α, β are finite and compute limits of y when $t \longrightarrow \alpha, \beta$.
v) $(**)$ Plot a graph of the solution.

Exercise 4.10.3 $(***)$**.** Consider the equation

$$y' = t(y^2 - 1) \arctan y.$$

i) $(*)$ Determine the domain D of the equation and check if local/global existence and uniqueness holds.
ii) $(**)$ Determine any stationary/constant solution, and determine the regions of plane ty where solutions are increasing/decreasing.
iii) $(**)$ Let now $y :], \beta[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(0, y_0)$, with $y_0 \in]0, 1[$. Discuss whether y is odd or even.
iv) $(***)$ For the solution at iii), determine the monotonicity, discuss whether α, β are finite or not and compute the limits of the solution when $t \longrightarrow \alpha, \beta$.
v) $(*)$ Plot a qualitative graph of the solution at iii).

Exercise 4.10.4 $(***)$**.** Consider the equation

$$y' = \frac{1}{\log y + t}.$$

i) $(*)$ Determine the domain D of the equation and check if local/global existence and uniqueness holds.
ii) $(**)$ Determine any stationary/constant solution, and determine the regions of plane ty where solutions are increasing/decreasing.
iii) $(**)$ Let now $y :], \beta[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(1, 1)$ (that is, $y(1) = 1$). Determine the monotonicity and the concavity of y.
iv) $(***)$ What can be said on α and β? What about $\lim_{t\to\alpha} y'(t)$ and $\lim_{t\to\beta} y(t)$?
v) $(*)$ Plot a qualitative graph of the solution at iii).

Exercise 4.10.5 $(***)$**.** Consider the equation

$$y' = \frac{t}{y^2 - t^2}.$$

i) $(*)$ Determine the domain D of the equation and check if local/global existence and uniqueness holds.
ii) $(**)$ Determine any stationary/constant solution, and determine the regions of plane ty where solutions are increasing/decreasing.
iii) $(***)$ Let now $y :], \beta[\longrightarrow \mathbb{R}$ be the maximal solution of $CP(0, 1)$ (that is, $y(0) = 1$). Show that y is even and determine the monotonicity of y.
iv) $(***)$ What can be said on α and β? What about $\lim_{t\to\beta} y(t)$?
v) $(*)$ Plot a qualitative graph of the solution at iii).

Exercise 4.10.6 (∗)**.** For each of the following systems, determine the nature of the equilibrium $(0,0)$.

$$1.\ \begin{cases} x' = 3x + y, \\ \\ y = -x + y. \end{cases} \quad 2.\ \begin{cases} x' = 4x - y, \\ \\ y = -2x + 3y. \end{cases} \quad 3.\ \begin{cases} x' = -x - 2y, \\ \\ y = -3y. \end{cases} \quad 4.\ \begin{cases} x' = 2x - 2y, \\ \\ y = 3x - y. \end{cases}$$

Exercise 4.10.7 (∗)**.** For each of the following systems determine its solution and classify the equilibrium at $(0,0)$.

$$1.\ \begin{cases} x' = -2x + \frac{y}{2}, \\ y' = 2x - 2y. \end{cases} \quad 2.\ \begin{cases} x' = 5x + 4y, \\ y' = x + 2y. \end{cases} \quad 3.\ \begin{cases} x' = x - y, \\ y' = 5x - y. \end{cases} \quad 4.\ \begin{cases} x' = 9x - 4y, \\ y' = 12x - 5y. \end{cases}$$

Exercise 4.10.8 (∗∗)**.** For each of the following systems

i) Determine if local/global existence and uniqueness apply, and determine any stationary solution.
ii) Determine a non constant first integral for the system, plot the phase portrait of the system. Determine whether are there periodic solutions or not.
iii) Solve the assigned Cauchy problem.

$$1.\ \begin{cases} x' = y, \\ y' = -x + x^3. \\ \\ \begin{pmatrix} x(0) = 2 \\ y(0) = 2 \end{pmatrix} \end{cases} \quad 2.\ \begin{cases} x' = 2y(y - 2x), \\ y' = (1 - x)(y - 2x). \end{cases} \quad 3.\ \begin{cases} x' = x(y + 1), \\ y' = -y(x + 1). \\ \\ \begin{pmatrix} x(0) = -1 \\ y(0) = 1 \end{pmatrix} \end{cases} \quad 4.\ \begin{cases} x' = 2x^2 y, \\ y' = y^2 x + x. \\ \\ \begin{pmatrix} x(0) = 1 \\ y(0) = 0 \end{pmatrix} \end{cases}$$

Exercise 4.10.9 (∗∗)**.** For each of the following second order equations

i) Determine if local/global existence and uniqueness apply, the determine any stationary solution.
ii) Determine the energy of the system and use it to plot the phase portrait of the system. Are there periodic solutions?
iii) Determine the solution of the assigned Cauchy problem.

$$1.\ \begin{cases} y'' = y^3 - y, \\ \\ \begin{pmatrix} y(0) = 2 \\ y'(0) = 2 \end{pmatrix} \end{cases} \quad 2.\ \begin{cases} y'' = \sin(2y), \\ \\ \begin{pmatrix} y(0) = \frac{\pi}{4} \\ y'(0) = -1 \end{pmatrix} \end{cases} \quad 3.\ \begin{cases} y'' = (\cosh y)(\sinh y), \\ \\ \begin{pmatrix} y(0) = \log(1 + \sqrt{2}) \\ y'(0) = 1 \end{pmatrix} \end{cases}$$

For problems 2 and 3, determine under which conditions on a, b the solution of the Cauchy problem $y(0) = a$, $y'(0) = b$ is odd or even.

Exercise 4.10.10 (∗∗)**.** Consider the Cauchy problem

$$\begin{cases} y'' = -y^2, \\ y(0) = y_0, \\ y'(0) = y_0', \end{cases}$$

i) Show that local existence and uniqueness holds $\forall y_0, v_0 \in \mathbb{R}^2$. ii) Determine any stationary solution. iii) Determine the energy of the system and plot the phase portrait. iv) Determine explicitly the non constant solutions.

Exercise 4.10.11 ((∗∗∗) SIR pandemic)**.** Consider the SIR pandemic model,

$$\begin{cases} x' = -axy, \\ \\ y' = axy - by, \end{cases}$$

with $a, b > 0$. Here we are interested only at solutions $x, y \geqslant 0$. i) Discuss local/global existence and uniqueness and determine the equilibrium points. ii) Determine a non trivial first integral and plot the orbits, discussing the behavior of the system as much as you can.

Exercise 4.10.12 (($***$) prey-predator model with logistic growth)**.** Two species are in competition. In absence of interaction, each one of the two follows a logistic model $x' = ax(b-x)$, $y' = cy(d-y)$. The interaction works to reduce both proportionally to their size in such a way that a model could be described by the equations

$$\begin{cases} x' = ax(b-x) - \alpha xy, \\ \\ y' = cy(d-y) - \beta xy, \end{cases}$$

with $a, b, c, d, \alpha, \beta > 0$. Discuss the behavior of the system as better as you can.

Exercise 4.10.13 ($***$)**.** Use the equivalence (4.8.2) to deduce theorems 4.8.3 and 4.8.4, respectively from theorems 4.4.3 and 4.4.4.

CHAPTER 5

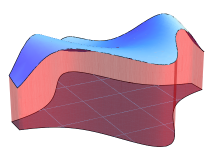
Multiple Integrals

In the first course of Mathematical Analysis, the concept of *integral* for a function depending on one real variable has been introduced. Integration is of paramount relevance in Analysis and applications since it gives a method to solve geometrical problems (calculus of areas of figures), probability problems (modern Probability is based on calculus of integral, to compute probabilities of events to expected values of random variables), as well as to Physics problems, Engineering etc. We recall that if $f = f(x) : [a, b] \longrightarrow [0, +\infty[$,

$$\int_{[a,b]} f(x)\, dx = \text{Area}\,(\text{Trap}(f)), \quad \text{where } \text{Trap}(f) := \{(x, y) \in \mathbb{R}^2 \;:\; x \in [a, b],\; 0 \leqslant y \leqslant f(x)\},$$

where AreaTrap(f) is defined through an *exhaustion method*. This Chapter extends this operation to the case of functions f of several variables. For instance, if $f = f(x, y) : E \subset \mathbb{R}^2 \longrightarrow [0, +\infty[$,

$$\int_E f(x, y)\, dxdy = \text{Volume}\,(\text{Trap}(f)), \text{ where } \text{Trap}(f) := \{(x, y, z) \in \mathbb{R}^3 \;:\; (x, y) \in E,\; 0 \leqslant z \leqslant f(x, y)\}.$$



More in general, given $f = f(\vec{x}) : E \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, we aim to define the integral

$$\int_E f(\vec{x})\, d\vec{x} = \text{measure}\left(\left\{(\vec{x}, y) \in \mathbb{R}^{m+1} \;:\; x \in E,\; 0 \leqslant y \leqslant f(\vec{x})\right\}\right).$$

The scope of this Chapter is then to give a precise definition to this operation. As for one variable integrals, almost never we compute and integral applying the definition: this is simply beyond of our possibilities. It is therefore important to have efficient techniques of calculus. In the case of multiple integrals, two main tools are

- the *reduction formula*, that allows to transform the calculus of an integral for a function of m variables into the calculus of m integrals each on a single variable (this basically reduces calculus to the well known one dimensional integral)
- the *change of variables formula*, a well known technique with integrals that allows to simplify calculations under special coordinate systems.

Along this Chapter we will provide precise definitions and statements but we will omit all the proofs. These are too technical and much beyond our scope here. Sometimes, we will provide informal justifications to the several results. With some technical work, these can be made true proofs.

Chapter requirements: differential calculus for functions of several variables, one variable integration and primitive calculus.

Learning objectives:

- (basic $*$) understanding of multiple integral, simple application of reduction formula for double integrals when the domain of integration is simple, integration in polar coordinates.
- (intermediate $**$) application of the reduction formula and the change of variable formula in complex cases with non evident standard parametrization of the domain, triple integrals and integration with spherical and cylindrical coordinates.
- (advanced $***$) general $m-$dimensional integration, abstract reasoning (proofs).

5.1. Measure of a trapezoid

In this section we define the operation of integral for a positive function. To prepare the ground, we introduce some useful definition. We call $m-$**dimensional interval** any set of type

$$I = [a_1, b_1] \times \cdots \times [a_m, b_m].$$

So, a 1-dim. interval is just an interval, a 2-dim. interval is a rectangle, a 3-dim. interval is a parallelepiped. For brevity, we will just call **interval** any multidimensional interval. The **measure** of an interval is, by definition, the number

$$|I| := (b_1 - a_1) \cdots (b_m - a_m).$$

Notice that

- in dimension $m = 1$, an *interval* is just an $[a, b]$, its measure is its *length* $|I| = b - a$;
- in dimension $m = 2$, $I = [a_1, b_1] \times [a_2, b_2]$ is a rectangle, its measure $|I| = (b_1 - a_1)(b_2 - a_2)$ is its *area*;
- in dimension $m = 3$, $I = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$, its measure $|I| = (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)$ is its *volume*.

Definition 5.1.1

Let $I \subset \mathbb{R}^m$ be an interval. A family $\pi := (I_j)_{j=1,\dots,n}$ of intervals is called **partition** of I if

i) $I = \bigcup_{k=1}^n I_k$;
ii) $|I_k \cap I_j| = 0$ for $j \neq k$.

We denote by $\Pi(I)$ the class made of all the partitions of I.

Condition i) means that I is splitted into rectangles I_k, like tiles on a floor. Condition ii) says that the "tiles" I_k can overlap but the "size" of the overlapping must be negligible. In other words, since the intersections $I_k \cap I_j$ of the "tiles" is itself a rectangle, its size $|I_k \cap I_j|$ must be 0. If you think to plane rectangles this basically means that rectangles can overlap only on their hedges.

Definition 5.1.2

Let $f : I \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, I **interval** and f **bounded**. Given a partition $\pi \in \Pi(I)$ we set

$$\underline{S}(\pi) := \sum_{k=1}^n m_k |I_k|, \quad \text{where } m_k := \inf_{\vec{x} \in I_k} f(\vec{x}),$$

$$\overline{S}(\pi) := \sum_{k=1}^n M_k |I_k|, \quad \text{where } M_k := \sup_{\vec{x} \in I_k} f(\vec{x}).$$

$\underline{S}(\pi)$ and $\overline{S}(\pi)$ are called, respectively, **inferior sum** (**superior sum**) of the partition π.

As for one dimensional integral, $\underline{S}(\pi)$ and $\overline{S}(\pi)$ are, respectively, an approximation by defect (excess) of the measure of $\mathrm{Trap}(f)$. We now introduce the best approximations by defect and excess:

Definition 5.1.3

Let $f : I \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, I **interval** and f **bounded**.

$$\underline{A}(f) := \sup_{\pi\in\Pi(I)} \underline{S}(\pi), \quad (\textbf{inferior measure of Trap}(f)),$$

$$\overline{A}(f) := \inf_{\pi\in\Pi(I)} \overline{S}(\pi), \quad (\textbf{superior measure of Trap}(f))$$

Easily, $\underline{A}(f) \leqslant \overline{A}(f)$. When they coincide, we say that

Definition 5.1.4: Measure of trapezoid

Let $f : I \subset \mathbb{R}^m \longrightarrow [0, +\infty[$, I **interval** and f **bounded**. If $\underline{A}(f) = \overline{A}(f)$ we define

$$\int_I f := \underline{A}(f) = \overline{A}(f) \in [0, +\infty[$$

and we call this **measure of the trapezoid** Trap(f).

Similarly to one dimensional definition, it may well happen that $A(f)f$ might not be defined:

Example 5.1.5: $(***)$ Dirichlet function

Q. *Let*

$$f = f(x, y) : [0, 1]^2 \longrightarrow [0, +\infty[, \quad f(x, y) := \begin{cases} 0, & (x, y) \in \mathbb{Q} \times \mathbb{Q}, \\ \\ 1, & (x, y) \notin \mathbb{Q} \times \mathbb{Q}. \end{cases}$$

Show that $\underline{A}(f) = 0 < 1 = \overline{A}(f)$.

A. The argument is similar to the one dimensional case. If $\pi = (I_k)$ is a partition of $I = [0, 1]^2$, then, because of the density of rational numbers and irrational numbers in $\mathbb{R}$, certainly in each I_k there are points of $\mathbb{Q} \times \mathbb{Q}$ as well as points of $(\mathbb{R}\backslash\mathbb{Q}) \times (\mathbb{R}\backslash\mathbb{Q})$. Then $m_k = \inf_{I_k} f(x, y) = 0$ while $M_k = \sup_{I_k} f(x, y) = 1$. Therefore

$$\underline{S}(\pi) = 0, \quad \overline{S}(\pi) = \sum_k |I_k| = |I| = 1.$$

As a consequence, $\underline{A}(f) = 0$, while $\overline{A}(f) = 1$.

Intervals are very special sets in $\mathbb{R}^m$ but, differently from $\mathbb{R}$, they are not particularly interesting because, easily, domains of functions $f = f(\vec{x})$ are not of this type. So, we need to extend the area of a trapezoid to the case of a generic base domain. We start with the case D bounded. In this case, there exists a multi-interval $I \supset D$. So, setting

$$f1_D(x) := \begin{cases} f(x), & x \in D, \\ 0, & x \in I\backslash D, \end{cases}$$

we have

Definition 5.1.6

Let $f : D \subset \mathbb{R}^m \longrightarrow [0, +\infty[$ bounded on D bounded. We pose

$$\int_D f := \int_I f 1_D,$$

provided this last makes sense.

Remark 5.1.7

It is possible to prove that previous definition does not depend on I (provided $I \supset D$).

As the intuition suggests, since $f \geqslant 0$, if $D_1 \subset D_2 \subset D$ then

$$\int_{D_1} f \leqslant \int_{D_2} f.$$

This leads to the idea to define $\int_D f$ when $D \subset \mathbb{R}^m$ is unbounded. Let

$$C_N := [-N, N]^m = [-N, N] \times \cdots \times [-N, N],$$

the hyper-cube centered at $\vec{0}$ with sides of length $2N$, and set

$$D_N := D \cap C_N.$$

Since $D_N \subset D_{N+1}$ we have

$$\int_{D_N} f \leqslant \int_{D_{N+1}} f.$$

Definition 5.1.8

Let $f = f(\vec{x}) : D \subset \mathbb{R}^m \longrightarrow [0, +\infty[$. We set

$$\int_D f := \lim_{N \to +\infty} \int_{D_N} f.$$

This way, $\int_D f$ is now defined for $\geqslant 0$. In particular, taking $f \equiv 1$ on D we have the

Definition 5.1.9

Let $D \subset \mathbb{R}^m$. We call (m dimensional) **measure of** D the number

$$\lambda_m(D) := \int_D 1, \tag{5.1.1}$$

provided this last is well defined (in this case we say that D is **measurable**).

The class of measurable sets is large enough to contain sets used in most of the applications:

Proposition 5.1.10

Every open set and every closed closed set is measurable.

However, not every set is measurable, as the following Dirichlet function based example shows:

Example 5.1.11: $(***)$

Q. *Let* $D := \{(x, y) \in [0,1]^2 \;:\; (x, y) \notin \mathbb{Q} \times \mathbb{Q}\}$*. Show that* $\lambda_2(D)$ *is not defined.*

A. Just notice that 1_D is the Dirichlet function, thus $\int_D 1 = \int_{[0,1]^2} 1_D$ which is not defined as shown in Example 5.1.5.

5.2. Integral

So far, we defined the integral of a positive function. Let's now consider $f = f(\vec{x}) : D \subset \mathbb{R}^m \longrightarrow \mathbb{R}$ and define

$$f_+(\vec{x}) := \begin{cases} f(\vec{x}), & \text{if } f(\vec{x}) \geqslant 0, \\ 0, & \text{if } f(\vec{x}) < 0, \end{cases} \qquad f_-(\vec{x}) := \begin{cases} -f(\vec{x}), & \text{if } f(\vec{x}) \leqslant 0, \\ 0, & \text{if } f(\vec{x}) > 0. \end{cases}$$

Functions $f_\pm$ are called, respectively, **positive part** and **negative part** of f. Both $f_\pm$ are positive and

$$f = f_+ - f_-, \quad |f| = f_+ + f_-.$$

We want to introduce the concept of *area with sign* or *integral of* f:

Definition 5.2.1

We say that f is **integrable** on D if $\int_D f_\pm < +\infty$. We pose

$$\int_D f := \int_D f_+ - \int_D f_-,$$

We write $f \in \mathsf{R}(D)$.

Notice that, since

$$\int_D f_+ + \int_D f_- = \int_D (f_+ + f_-) = \int_D |f|,$$

we have

$$\int_D f_\pm < +\infty, \iff \int_D |f| < +\infty.$$

In general, it is practically impossible to check if a given function is integrable by using the definition. This also happens with the one dimensional Riemann integral. In certain common cases, integrability can be easily drawn from good properties of function and integration domain:

Theorem 5.2.2

Let $f \in \mathsf{C}(D)$, D compact set. Then f is integrable on D.

In general, if D is not compact, in particular if D is closed and unbounded, continuity is not sufficient to ensure integrability (trivially, take $f \equiv 1$ on $D = \mathbb{R}^2$, then $\int_{\mathbb{R}^2} 1 = +\infty$). We have the following test

Proposition 5.2.3: absolute integrability

Let $f \in \mathsf{C}(D)$, D closed. Then, if

$$\int_D |f| < +\infty \tag{5.2.1}$$

f is integrable on D. When (5.2.1) holds we say that f is **absolutely integrable**.

For continuous function it is possible to prove that, if f is integrable on D, then for every $\varepsilon > 0$ there exists

- a family of multi-intervals $(I_k)_{k\in\mathbb{N}}$ such that $D \subset \bigcup_k I_k$;
- points $\vec{x}_k \in I_k \cap D$;

such that

$$\left| \int_D f - \sum_k f(\vec{x}_k)|I_k| \right| \leqslant \varepsilon. \tag{5.2.2}$$

This justifies the idea that, for good functions on good domains,

$$\int_D f \approx \sum_k f(\vec{x}_k) dx_1 \cdots dx_m.$$

Properties of the integral are very similar to those of one dimensional integral:

Proposition 5.2.4

The following properties hold true:

i) (linearity) if $f, g \in \mathsf{R}(D)$ then $\alpha f + \beta g \in \mathsf{R}(D)$ for any $\alpha, \beta \in \mathbb{R}$ and $\int_D (\alpha f + \beta g) = \alpha \int_D f + \beta \int_D g$;
ii) (isotonicity) if $f \leqslant g$ on D with $f, g \in \mathsf{R}(D)$ then $\int_D f \leqslant \int_D g$;
iii) (triangular inequality) if $f \in \mathsf{R}(D)$ then $\left|\int_D f\right| \leqslant \int_D |f|$;
iv) (decomposition) if $f \in \mathsf{R}(D_1), \mathsf{R}(D_2)$ with $D_1 \cap D_2 = \varnothing$, then $f \in \mathsf{R}(D_1 \cup D_2)$ and $\int_{D_1 \cup D_2} f = \int_{D_1} f + \int_{D_2} f$.
v) (null sets) if $f \in \mathsf{R}(D)$ and $E \subset D$ is such that $\lambda_m(D \backslash E) = 0$ (that is, the excess of D w.r.t. E has negligible measure), then
$$\int_D f = \int_E f.$$

The last property is very useful for practical purposes. It says that the integral is not affected by eliminating/adding a measure 0 set from/to a domain D. So we can always, if needed or convenient, add/subtract measure zero sets to the integration domain not changing the calculation of the integral.

With this, the definition of integral is completed. Of course, as for one dimensional integral, we need efficient tools to check integrability and to compute an integral. The two most important tools are *reduction formula* and *change of variables formula*, which we illustrate in next sections.

5.3. Reduction formula

In this Section we introduce the technique based on the *reduction formula* that allows to reduce the calculus of a multiple variables integral to iterated one variable integrals. For pedagogical reasons we present first the case of double integrals, then we will extend to the general case.

5.3.1. Double Integrals. To understand the idea, let's consider the problem of computing

$$\int_D f(x, y)\, dxdy.$$

Assuming f continuous and integrable,

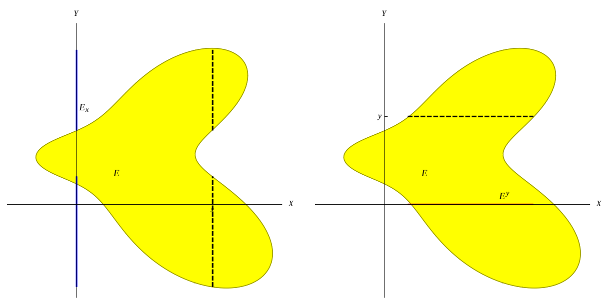
$$\int_D f(x, y)\, dxdy \approx \sum_{(x,y)\in D} f(x, y) dxdy.$$

Informally, associative and commutative properties lead to

$$\sum_{(x,y)\in D} f(x,y)dxdy = \sum_{x\in\mathbb{R}}\left(\sum_{y\,:\,(x,y)\in D} f(x,y)\;dy\right)\;dx.$$

Now,

$$\sum_{y\,:\,(x,y)\in D} f(x,y)\;dy \approx \int_{D_x} f(x,y)\;dy,\;\;\text{where } D_x := \{y\;:\;(x,y)\in D\}.$$



We call D_x the $x-$section of D. Notice that D_x is the set of ordinates of points of D with abscissas = x. Thus, denoting by

$$F(x) := \int_{D_x} f(x,y)\;dy,$$

(this is a function of x, y is "integrated" and it does not appear out of the integral), we would have

$$\int_D f(x,y)\;dxdy \approx \sum_{x\in\mathbb{R}} F(x)\;dx \approx \int_{\mathbb{R}} F(x)\;dx \equiv \int_{\mathbb{R}}\left(\int_{D_x} f(x,y)\;dy\right)\;dy.$$

Similarly, flipping the role of x and y we have a similar formula with exchanged order of the integrations. Of course, this is not a proof, but the conclusion is a true fact:

Proposition 5.3.1: reduction formula

Let $f\in \mathsf{C}(D)$ be absolutely integrable on D measurable. Then

$$\int_D f(x,y)\;dxdy = \int_{\mathbb{R}}\left(\int_{D_x} f(x,y)\;dy\right)\;dx = \int_{\mathbb{R}}\left(\int_{D^y} f(x,y)\;dx\right)\;dy. \tag{5.3.1}$$

where

- $D_x := \{y\in\mathbb{R}\;:\;(x,y)\in D\}$;
- $D^y := \{x\in\mathbb{R}\;:\;(x,y)\in D\}$.

Notice that D_x (D^y) may be empty for certain values of x (y). For such x (y), clearly $\int_{D_x} f = 0$ ($\int_{D^y} f = 0$). Therefore, formula (5.3.1) and be actually written as

$$\int_D f(x,y)\;dxdy = \int_{x\,:\,D_x\neq\varnothing}\left(\int_{D_x} f(x,y)\;dy\right)\;dx = \int_{y\,:\,D^y\neq\varnothing}\left(\int_{D^y} f(x,y)\;dx\right)\;dy.$$

However, for future use we prefer to keep a lighter notation as in (5.3.1).

Reduction formula (RF) says that we can reduce the calculation of a "double integral" $\int_D f(x,y)\;dxdy$ to two iterated one variable integrals:

- first, one computes integral $\int_{D_x} f(x,y)\;dy$: the output is a function $F(x)$ of x;
- second, one computes integrale $\int_{\mathbb{R}} F(x)\;dx$.

Example 5.3.2: (∗)

Q. *Compute*
$$\int_D \cos(x+y)\ dxdy, \quad \textit{where } D := \left\{(x,y) \in \mathbb{R}^2 \ :\ 0 \leqslant y \leqslant x \leqslant \pi\right\}.$$

A. First notice that $f(x,y) = \cos(x+y) \in \mathsf{C}(\mathbb{R}^2)$, thus $|f| \in \mathsf{C}(D)$, D is clearly closed and bounded hence compact. Therefore, f is absolutely integrable on D. To compute the integral we apply the RF. In this case, it is indifferent which one of the two forms, thus we write
$$\int_D \cos(x+y)\ dxdy = \int_{x\in\mathbb{R}} \left(\int_{y\in D_x} \cos(x+y)\ dy\right)\ dx.$$
Notice that
$$D_x = \{y \ :\ (x,y) \in D\} = \begin{cases} [0,x], & x \in [0,\pi], \\ \varnothing, & x \notin [0,\pi]. \end{cases}$$
Thus
$$\int_D \cos(x+y)\ dxdy = \int_0^\pi \left(\int_0^x \cos(x+y)\ dy\right)\ dx.$$
Now,
$$\int_0^x \cos(x+y)\ dy = [\sin(x+y)]_{y=0}^{y=x} = \sin(2x) - \sin x,$$
thus
$$\int_D \cos(x+y)\ dxdy = \int_0^\pi (\sin(2x) - \sin x)\ \ dx = \left[-\frac{\cos(2x)}{2} + \cos x\right]_{x=0}^{x=\pi} = \left(-\frac{1}{2} - 1\right) - \left(-\frac{1}{2} + 1\right) = -2.$$

The reduction formula (5.3.1) requires that f be absolutely integrable, that is
$$\int_D |f(x,y)|\ dxdy < +\infty.$$
To check this, in principle we should compute the double integral of $|f|$. Applying the reduction formula
$$\int_D |f(x,y)|\ dxdy = \int_{\mathbb{R}} \left(\int_{D_x} |f(x,y)|\ dy\right)\ dx = \int_{\mathbb{R}} \left(\int_{D^y} |f(x,y)|\ dx\right)\ dy,$$
so, if f is absolutely integrable, then
$$\int_{\mathbb{R}} \left(\int_{E_x} |f(x,y)|\ dy\right)\ dx,\ \int_{\mathbb{R}} \left(\int_{E^y} |f(x,y)|\ dx\right)\ dy\ < +\infty.$$
It turns out that also the vice versa holds true:

Proposition 5.3.3

Let $f \in \mathsf{C}(D)$, D measurable, be such that at least one of

(5.3.2) $$\int_{\mathbb{R}} \left(\int_{D_x} |f(x,y)|\ dy\right)\ dx,\ \int_{\mathbb{R}} \left(\int_{D^y} |f(x,y)|\ dx\right)\ dy$$

is finite. Then, f is absolutely integrable on D and reduction formula (5.3.1) holds.

Combining the previous Propositions, we get an algorithm to check integrability and compute double integrals.

Check List 5.3.4: computing double integrals

To compute $\int_D f(x,y)\,dxdy$

(1) is $f \in \mathsf{C}(D)$? Is D measurable (for instance, is D open or closed)?
(2) Check that at least one of the iterated integrals (5.3.2) is finite
(3) Apply RF (5.3.1) to compute the integral.

If f has constant sign, e.g. $f \geqslant 0$, then

$$\int_{\mathbb{R}}\int_{D_x} |f|\,dy\,dx = \int_{\mathbb{R}}\int_{D_x} f\,dy\,dx,$$

so, once step (2) is done we automatically have also the value of the integral according to RF without doing step (3).

Example 5.3.5: $(*)$

Q. *Discuss if* $f(x,y) := x^3 e^{-yx^2}$ *is integrable on* $D = [0,+\infty[\times[1,2]$ *and, in this case, compute its integral.*

A. Clearly $f \in \mathsf{C}(D)$ where $D = [0,+\infty[\times[1,2]$ is closed. Trivially,

$$D_x = \begin{cases} \varnothing, & x < 0, \\ [1,2], & x \geqslant 0. \end{cases}$$

Moreover, $f \geqslant 0$ on D, thus $|f| = f$ and

$$\begin{aligned}\int_{\mathbb{R}}\int_{D_x} |f|\,dy\,dx &= \int_0^{+\infty}\left(\int_1^2 x^3 e^{-yx^2}\,dy\right)dx = \int_0^{+\infty} x\left[-e^{-yx^2}\right]_{y=1}^{y=2} dx \\ &= \int_0^{+\infty} xe^{-x^2} - xe^{-2x^2}\,dx = \left[\frac{-e^{-x^2}}{2}\right]_{x=0}^{x=+\infty} - \left[\frac{-e^{-2x^2}}{4}\right]_{x=0}^{x=+\infty} = \frac{1}{4}.\end{aligned}$$

We deduce f is integrable and being $f \geqslant 0$, the previous calculation provides also $\int_{[0,+\infty[\times[1,2]} f = \frac{1}{4}$.

Example 5.3.6: $(*)$

Q. *Discuss if* $f(x,y) := e^{-x}$ *is integrable on* $D := \{(x,y) \in \mathbb{R}^2 \ :\ x \geqslant 0,\ 0 \leqslant y \leqslant x^2\}$*. In such case compute the integral of* f *on* D*.*

A. Clearly $f \in \mathsf{C}(D)$ and D is closed (defined by large inequalities on continuous functions). Applying (5.3.2), notice that

$$D_x = \begin{cases} \varnothing, & x < 0, \\ [0,x^2], & x \geqslant 0. \end{cases}$$

Therefore,

$$\int_{\mathbb{R}}\int_{D_x} |f|\,dy\,dx = \int_0^{+\infty}\left(\int_0^{x^2} e^{-x}\,dy\right)\,dx = \int_0^{+\infty} x^2e^{-x}\,dx = \int_0^{+\infty} x^2(-e^{-x})'\,dx$$
$$= \left[-x^2e^{-x}\right]_{x=0}^{x=+\infty} + \int_0^{+\infty} 2xe^{-x}\,dx = 2\int_0^{+\infty} x(-e^{-x})'\,dx$$
$$= 2\left[\left[-xe^{-x}\right]_{x=0}^{x=+\infty} + \int_0^{+\infty} e^{-x}\,dx\right] = 2\left[-e^{-x}\right]_{x=0}^{x=+\infty} = 2.$$

This says that f is integrable on D and, at same time, $\int_D f = 2$.

A particular case of RF (5.3.1) is obtained by taking $f \equiv 1$. Recalling that $\int_D 1 = \lambda_2(D)$ we obtain

$$\lambda_2(D) = \int_{\mathbb{R}}\left(\int_{D_x} 1\,dy\right)\,dx = \int_{\mathbb{R}} \lambda_1(D_x)\,dx = \int_{\mathbb{R}} \lambda_1(D^y)\,dy. \tag{5.3.3}$$

Formula (5.3.3) is called *slicing formula*.

Example 5.3.7: $(**)$

Q. *Compute the area of a disk of radius* r.

A. Let
$$D = \left\{(x,y)\in\mathbb{R}^2 \;:\; x^2+y^2 \leqslant r^2\right\}.$$
This set D is closed, hence measurable. According to the slicing formula,
$$\lambda_2(D) = \int_{\mathbb{R}} \lambda_1(D_x)\,dx.$$
Let's determine an $x-$section. We have
$$y\in D_x, \iff (x,y)\in D, \iff x^2+y^2\leqslant r^2, \iff y^2 \leqslant r^2-x^2, \iff \begin{cases} \varnothing, & |x|>r,\\ \left[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}\right], & |x|\leqslant r.\end{cases}$$
Therefore
$$\lambda_2(D) = \int_{\mathbb{R}} \lambda_1(D_x)\,dx = \int_{|x|\leqslant r} \lambda_1\left(\left[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}\right]\right)\,dx = \int_{|x|\leqslant r} 2\sqrt{r^2-x^2}\,dx.$$
Setting $x = r\sin\theta$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$, we have
$$\lambda_2(D) = 2r\int_{-\pi/2}^{\pi/2} \sqrt{1-(\sin\theta)^2}\,r\cos\theta\,d\theta = 2r^2\int_{-\pi/2}^{\pi/2} (\cos\theta)^2\,d\theta.$$
Now $\int(\cos\theta)^2 = \int\cos\theta(\sin\theta)' = \cos\theta\sin\theta + \int(\sin\theta)^2 = \frac{1}{2}\sin(2\theta) + \theta - \int(\cos\theta)^2$ hence
$$\lambda_2(D) = 4r^2\left[\frac{1}{4}\sin(2\theta) + \frac{\theta}{2}\right]_{\theta=0}^{\theta=\pi/2} = \pi r^2. \quad \square$$

Warning 5.3.8

It might happen that **both iterated integrals**
$$\int_{\mathbb{R}} \left(\int_{D_x} f(x,y)\, dy \right)\, dx, \quad \int_{\mathbb{R}} \left(\int_{D^y} f(x,y)\, dx \right)\, dy$$
make sense, but they are different! Of course, in this case, f cannot be absolutely integrable (otherwise the two iterated integrals should coincide). The next example shows a case of this situation.

Example 5.3.9: $(*)$

Q. *Let*
$$f(x,y) = \frac{x-y}{(x+y)^3}, \ (x,y) \in D := [0,1]^2.$$
Then $\int_{\mathbb{R}} \left(\int_{D_x} f\, dy \right)\, dx \neq \int_{\mathbb{R}} \left(\int_{D^y} f\, dx \right)\, dy$.

A. Notice first that
$$D_x = \{y \in \mathbb{R} \ : \ (x,y) \in [0,1]^2\} = \begin{cases} \emptyset, & x \notin [0,1], \\ \\ [0,1] & x \in [0,1] \end{cases}$$
and similarly for D^y. Therefore
$$\int_{D^y} f(x,y)\, dx = \begin{cases} 0, & y \notin [0,1], \\ \\ \displaystyle\int_0^1 \frac{x-y}{(x+y)^3}\, dx = \int_0^1 \frac{1}{(x+y)^2}\, dx - 2y \int_0^1 \frac{1}{(x+y)^3}\, dx. & y \in [0,1]. \end{cases}$$
Apart for $y = 0$, both integrals are finite equal to
$$\left[\frac{(x+y)^{-1}}{-1} \right]_{x=0}^{x=1} - 2y \left[\frac{(x+y)^{-2}}{-2} \right]_{x=0}^{x=1} = \frac{1}{y} - \frac{1}{y+1} + y \left(\frac{1}{(y+1)^2} - \frac{1}{y^2} \right) = -\frac{1}{(y+1)^2}.$$
Hence
$$\int_{\mathbb{R}} \left(\int_{D^y} f(x,y)\, dx \right)\, dy = \int_0^1 \left(-\frac{1}{(y+1)^2} \right)\, dy = \left[(y+1)^{-1} \right]_{y=0}^{y=1} = \frac{1}{2} - 1 = -\frac{1}{2}.$$
Exchanging x with y we obtain the same result except for the sign: $\int_{\mathbb{R}} \left(\int_{D_x} f(x,y) dy \right)\, dx = \frac{1}{2}$. □

5.3.2. General Multiple Integrals. The previous mechanism can be extended to functions f of m variables. Let $f = f(x_1, \dots, x_m)$, and imagine we group $(x_1, \dots, x_m)$ into two blocks, one of k variables and the remaining of $m-k$ variables, as
$$(x_1, \dots, x_m) = (\underbrace{x_1, \dots, x_k}_{\vec{x}}, \underbrace{x_{k+1}, \dots, x_m}_{\vec{y}}) \equiv (\vec{x}, \vec{y}), \ \vec{x} \in \mathbb{R}^k, \ \vec{y} \in \mathbb{R}^{m-k}.$$
An important point is that the two sub-arrays are not required to be the first k coordinates and the next $m-k$. So, for example, if $(x,y,z) \in \mathbb{R}^3$ we could group into two sub arrays of resp. 2 and 1 components as follows:
$$(x,y) \text{ and } (z), \quad (x,z) \text{ and } (y), \quad (y,z) \text{ and } (x).$$
With this notation we may write
$$f(x_1, \dots, x_m) = f(\vec{x}, \vec{y}).$$
We have the

Theorem 5.3.10

Let $f \in \mathsf{C}(D)$ be absolutely integrable on D measurable. Then

$$\int_D f = \int_{\mathbb{R}^k} \left(\int_{D_{\vec{x}}} f(\vec{x}, \vec{y})\, d\vec{y} \right) d\vec{x} = \int_{\mathbb{R}^{m-k}} \left(\int_{D^{\vec{y}}} f(\vec{x}, \vec{y})\, d\vec{x} \right) d\vec{y}. \tag{5.3.4}$$

Moreover, if one of integrals

$$\int_{\mathbb{R}^k} \left(\int_{D_{\vec{x}}} |f(\vec{x}, \vec{y})|\, d\vec{y} \right) d\vec{x}, \quad \int_{\mathbb{R}^{m-k}} \left(\int_{D^{\vec{y}}} |f(\vec{x}, \vec{y})|\, d\vec{x} \right) d\vec{y},$$

is finite, then f is absolutely integrable on D (and the reduction formula (5.3.4) holds). In particular, by taking $f = 1$ we have the **slicing formula**

$$\lambda_m(D) = \int_{\mathbb{R}^k} \lambda_{m-k}(D_{\vec{x}})\, d\vec{x} = \int_{\mathbb{R}^{m-k}} \lambda_m(D^{\vec{y}})\, d\vec{y}. \tag{5.3.5}$$

Remark 5.3.11

Consider a function of three variables $f = f(x, y, z) \in \mathsf{C}(D)$, $D \subset \mathbb{R}^3$ open/closed. In this case, there are six different possible applications of reduction formula:

$$\int_D f = \int_{\mathbb{R}} \left(\int_{(y,z)\in D_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left(\int_{x\in D_{(y,z)}} f\, dx \right) dydz,$$

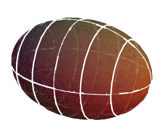
$$\int_D f = \int_{\mathbb{R}} \left(\int_{(x,z)\in D_y} f\, dxdz \right) dy = \int_{\mathbb{R}^2} \left(\int_{y\in D_{(x,z)}} f\, dy \right) dxdz,$$

$$\int_D f = \int_{\mathbb{R}} \left(\int_{(y,z)\in E_x} f\, dydz \right) dx = \int_{\mathbb{R}^2} \left(\int_{x\in D_{(y,z)}} f\, dx \right) dydz,$$

Which choice is the best one depends by the complexity of calculus.

Example 5.3.12: ($*$)

Q. *Compute the volume of a rugby ball* $D := \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1 \right\}$, $(a, b > 0)$.



A. Clearly D is closed and bounded in $\mathbb{R}^3$, hence measurable. Slicing D along the z-axis,

$$\lambda_3(D) = \int_{\mathbb{R}} \lambda_2(D_z)\, dz.$$

Now,

$$(x,y,z)\in D, \iff \frac{x^2+y^2}{a^2}\leqslant 1-\frac{z^2}{b^2}, \iff \begin{cases} \varnothing, & |z|>b, \\ B\left(0_2, \sqrt{1-\frac{z^2}{b^2}}\right], & |z|\leqslant b. \end{cases}$$

Thus

$$\begin{aligned} \lambda_3(E) &= \int_{|z|\leqslant b} \lambda_2\left(B\left(0_2, a\sqrt{1-\frac{z^2}{b^2}}\right]\right)\,dz = \int_{|z|\leqslant b} \pi a^2\left(1-\frac{z^2}{b^2}\right)\,dz = \int_{-b}^{b} \pi a^2\left(1-\frac{z^2}{b^2}\right)\,dz \\ &= \pi a^2\left([z]_{-b}^{b} - \left[\frac{z^3}{3b^2}\right]_{-b}^{b}\right) = \pi a^2\left(2b-\frac{2}{3}b\right) = \pi\frac{4}{3}a^2 b. \end{aligned}$$

Taking $a=b=r$ we obtain the volume of a sphere of radius r, the well known $\frac{4}{3}\pi r^3$.

5.4. Change of variable

Change of variable is an important technique of calculus for integrals. To compute

$$\int_a^b f(x)\,dx,$$

introducing the "new" variable $y := \phi(x)$, or $x=\phi^{-1}(y)$ (the change of variable must be invertible) then, provided $\phi,\phi^{-1}\in \mathsf{C}$ in their respective domains, we have

$$\int_a^b f(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy = \begin{cases} \int_{\phi(a)}^{\phi(b)} f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy, & \text{if } \phi \nearrow, \\ \int_{\phi(b)}^{\phi(a)} f(\phi^{-1}(y))(\phi^{-1})'(y)\,dy, & \text{if } \phi \searrow. \end{cases}$$

Denoting with $\phi([a,b])$ the image of $[a,b]$ through ϕ, we have, in a unique formula

$$\int_a^b f(x)\,dx = \int_{\phi([a,b])} f(\phi^{-1}(y))|(\phi^{-1})'(y)|\,dy.$$

This formula extends to the case of functions of vector variable as follows:

Theorem 5.4.1

Let $f\in \mathsf{C}(D)$, D closed or open domain of $\mathbb{R}^d$. Suppose that $\Phi : D \longrightarrow \Phi(D)$ is a **diffeomorphism on** D that is

- Φ is a bijection: $\exists \Phi^{-1} : \Phi(D) \longrightarrow D$;
- Φ, Φ^{-1} are *regular*, that is $\Phi,\Phi^{-1}\in \mathsf{C}^1$ on their domains.

Then

$$\boxed{\int_D f(\vec{x})\,d\vec{x} \overset{\vec{y}=\Phi(\vec{x}),\ \vec{x}=\Phi^{-1}(\vec{y})}{=} \int_{\Phi(D)} f(\Phi^{-1}(\vec{y}))|\det(\Phi^{-1})'(\vec{y})|\,d\vec{y}.} \tag{5.4.1}$$

Remark 5.4.2

The determinant $\det(\Phi^{-1})'(\vec{y})$ is the determinant of the jacobian matrix of Φ^{-1}. An interesting fact to know is that

$$\boxed{\det(\Phi^{-1})'(\vec{y}) = \frac{1}{\det \Phi'(\Phi^{-1}(\vec{y}))}.} \tag{5.4.2}$$

To check this identity we notice that, according to the chain rule,

$$\Phi(\Phi^{-1}(\vec{y})) \equiv \vec{y}, \implies \Phi'(\Phi^{-1}(\vec{y}))(\Phi^{-1})'(\vec{y}) = \mathbb{I},$$

therefore, by taking both side the determinant and recalling that, if A,B are square matrices, $\det(AB) = \det A \det B$, we get

$$\det \Phi'(\Phi^{-1}(\vec{y})) \det(\Phi^{-1})'(\vec{y}) = 1,$$

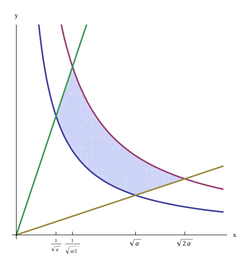
from which the conclusion follows.

Example 5.4.3: $(**)$

Q. *By using the change of variables $u = xy$, $v = \frac{y}{x}$, compute*

$$\int_{1 \leqslant xy \leqslant 2,\ 0 < ax \leqslant y \leqslant \frac{x}{a}} \frac{y^4 \arctan(xy)}{(x^2+y^2)^2}\, dxdy, \quad 0 < a < 1.$$

A. The integration domain D_a is closed and bounded in $\mathbb{R}^2$ and $f \in \mathsf{C}(D)$, thus it is integrable.



Define now the map

$$(u, v) = \Phi(x, y) = \left(xy, \frac{y}{x}\right).$$

We check that Φ is a diffeomorphism on D_a. It is clear that $\Phi \in \mathsf{C}^1(D_a)$. We check that Φ is invertible. Let $(x, y) \in D_a$. Then

$$(u, v) \in \Phi(x, y), \iff \begin{cases} u = xy, \\ v = \frac{y}{x}, \end{cases} \iff \begin{cases} u = vx^2, \\ y = vx, \end{cases}$$

Notice that, since $(x, y) \in D_a$, both $x, y > 0$ then, necessarily, also $u, v > 0$. So, the previous system yields,

$$\begin{cases} x = \sqrt{\frac{u}{v}}, \\ y = \sqrt{uv}, \end{cases} \iff (x, y) = \Phi^{-1}(u, v) = \left(\sqrt{\frac{u}{v}}, \sqrt{uv}\right).$$

Moreover:

$$(x,y)\in D_a, \iff \begin{cases} 1\leqslant xy\leqslant 2, \\ 0<ax\leqslant y\leqslant \frac{x}{a}, \end{cases} \iff \begin{cases} 1\leqslant u\leqslant 2, \\ a\leqslant v\leqslant \frac{1}{a}. \end{cases} \iff (u,v)\in \underbrace{[1,2]\times\left[a,\frac{1}{a}\right]}_{\Phi(D_a)}.$$

It is clear that both Φ,Φ^{-1} are C^1 on their domains D_a and $\Phi(D_a)$. We can apply the change of variable formula. To this aim, we need to calculate $\det(\Phi^1)'(u,v)$. We can do this **directly** by computing the jacobian matrix of Φ^{-1}, or **indirectly** using the formula (5.4.2). Since

$$\Phi'(x,y)=\begin{bmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x}\end{bmatrix}, \implies \det\Phi'(x,y)=\frac{y}{x}+x\frac{y}{x^2}=2\frac{y}{x}=2v.$$

by (5.4.2) we have

$$\det(\Phi^{-1})'(u,v)=\frac{1}{2v}.$$

Therefore,

$$I_a:=\int_{1\leqslant xy\leqslant 2,\ 0<ax\leqslant y\leqslant\frac{x}{a}}\frac{y^4\arctan(xy)}{(x^2+y^2)^2}\,dxdy \ =\int_{1\leqslant u\leqslant 2,\ a\leqslant v\leqslant\frac{1}{a}}\frac{v^4}{(1+v^2)^2}\arctan u\left|\frac{1}{2v}\right|\,dudv$$

$$\overset{RF}{=}\frac{1}{2}\left(\int_1^2\arctan u\,du\right)\left(\int_a^{\frac{1}{a}}\frac{v^3}{(1+v^2)^2}\,d\eta\right).$$

Now

$$\int_1^2\arctan u\,du=[\xi\arctan u]_1^2-\int_1^2\frac{u}{1+u^2}\,du=2\arctan 2-\frac{\pi}{4}-\frac{1}{2}\log\frac{5}{2},$$

while

$$\int_a^{\frac{1}{a}}\frac{v^3}{(1+v^2)^2}\,d\eta=\int_a^{\frac{1}{a}}\frac{v}{1+v^2}\,dv-\int_a^{\frac{1}{a}}\frac{v}{(1+v^2)^2}\,dv=-\log a+\frac{1}{2}\frac{1-a^2}{1+a^2}.$$

From this we get

$$I_a=\left(2\arctan 2-\frac{\pi}{4}-\frac{1}{2}\log\frac{5}{2}\right)\left(\frac{1}{2}\frac{1-a^2}{1+a^2}-\log a\right).$$

In the next subsections we illustrate the change of variable formula with particular change of coordinates in the case of double and triple integrals.

5.4.1. Integration in polar coordinates. A very important change of variable in plane integration is

$$\begin{cases} x=\rho\cos\theta, \\ y=\rho\sin\theta, \end{cases}\qquad (\rho,\theta)\in[0,+\infty[\times[0,2\pi].$$

Here we may notice that change of variable is defined in the form $(x,y)=\Psi(\rho,\theta)$. This means that, referring to notations of (5.4.1), present Ψ is just Φ^{-1}. Thus

$$\det(\Phi^{-1})'=\det\Psi'=\det\begin{bmatrix}\cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta\end{bmatrix}=\rho(\cos^2\theta+\sin^2\theta)=\rho,$$

and (5.4.1) becomes

(5.4.3) $$\int_D f(x,y)\,dxdy=\int_{\Phi_{pol}(D)} f(\rho\cos\theta,\rho\sin\theta)\rho\,d\rho d\theta.$$

Here, $\Phi_{pol}(D)$ is nothing but D represented in polar coordinates and notice also that we don't really need to have Φ explicitly.

Remark 5.4.4: $(***)$

The scope of this remark is to show that, **even if the map** $(\rho, \theta) = \Phi(x, y)$**,** $(x, y) = \Phi^{-1}(\rho, \theta)$ **is not a bijection between the cartesian plane** $\mathbb{R}^2$ **and the strip** $[0, +\infty[\times[0, 2\pi]$**, formula (5.4.3) if 100% correct!**. Let us see why (the discussion below is a bit technical and it can be skipped, unless you are curious!).

First fact: **Φ is not a bijection**. This because, for example, $(x, y) = (0, 0)$ corresponds to $(\rho, \theta) = (0, \theta)$ for any $\theta \in [0, 2\pi]$. To make Φ a true bijection, we have to restrict a bit the domains taking

$$\Phi : \mathbb{R}^2\backslash\{\vec{0}\} \longrightarrow]0, +\infty[\times[0, 2\pi[.$$

It is now possible to prove that Φ is a bijection with inverse Φ^{-1}.

Second fact: **even if Φ is now a bijection, it is not continuous!** Indeed, take a point $(x^*, 0)$ with $x^* > 0$ (positive half of $x-$axis). Now, if $(x, y) \longrightarrow (x^*, 0)$ coming from the upper half plane (that is $y > 0$), then $\theta(x, y) \longrightarrow 0$ while, if $(x, y) \longrightarrow (x^*, 0)$ coming from the lower half plane (that is $y < 0$), then $\theta(x, y) \longrightarrow 2\pi$. So, if $\Phi(x, y) = (\rho(x, y), \theta(x, y))$, the second component has not a limit for $(x, y) \longrightarrow (x^*, 0)$. Therefore, it cannot be continuous. To fix this problem, we need a further restriction, considering

$$\Phi : \mathbb{R}^2\backslash\mathbb{R}_+ \longrightarrow]0, +\infty[\times]0, 2\pi[, \quad \text{where } \mathbb{R}_+ := \{(x, 0) \ : \ x \geqslant 0\}.$$

It is now possible to prove that Φ is a diffeomorphism between these domains.

Third fact: **integration formula (5.4.3) holds!** The key point is that that the difference between $\mathbb{R}^2$ and $\mathbb{R}^2\backslash\mathbb{R}_+$ is the half axis $\mathbb{R}_+$, for which, as you might expect, $\lambda_2(\mathbb{R}_+) = 0$. Thus, by null invariance of integrals,

$$\int_D f = \int_{D\cap(\mathbb{R}^2\backslash\mathbb{R}_+)} f \overset{CV}{=} \int_{\Phi(D\cap(\mathbb{R}^2\backslash\mathbb{R}_+))} f(\rho\cos\theta, \rho\sin\theta)\rho \ d\rho d\theta.$$

Now, $\Phi(D\cap(\mathbb{R}^2\backslash\mathbb{R}_+))$ are polar coordinates of points of D which are not on $\mathbb{R}_+$. Therefore, they corresponds to points of $\Phi(D)$ which are in $]0, +\infty[\times]0, 2\pi[$, that is

$$\int_D f = \int_{\Phi(D)\cap]0,+\infty[\times]0,2\pi[} f(\rho\cos\theta, \rho\sin\theta)\rho \ d\rho d\theta = \int_{\Phi(D)} f(\rho\cos\theta, \rho\sin\theta)\rho \ d\rho d\theta$$

because the difference between $\Phi(D)\cap]0, +\infty[\times]0, 2\pi[$ and $\Phi(D)$ are just points where $\rho = 0$ or $\theta = 0$ or $\theta = 2\pi$, in any case measure 0 sets.

Example 5.4.5: $(*)$

Q. *Compute*

$$\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}} \ dxdy.$$

A. We have

$$\begin{aligned}\int_{\mathbb{R}^2} e^{-\sqrt{x^2+y^2}} \ dxdy &= \int_{\rho\geqslant 0, \theta\in[0,2\pi]} e^{-\rho}\rho \ d\rho d\theta = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\rho}\rho \ d\theta\right) \ d\rho = 2\pi\int_0^{+\infty} \rho e^{-\rho} \ d\rho \\ &= 2\pi\left([-\rho e^{-\rho}]_{\rho=0}^{\rho=+\infty} + \int_0^{+\infty} e^{-\rho} \ d\rho\right) = 2\pi.\end{aligned}$$

Next one is a smart calculation of a very important (in Probability, Statistics, Physics, Engineering,. . .) integral.

Example 5.4.6: (∗∗) Gaussian integral

Q. Computing the double integral

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy,$$

deduce the Gaussian integral

$$\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}.$$

A. We have

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_0^{+\infty}\left(\int_0^{2\pi} e^{-\frac{\rho^2}{2}}\rho\, d\theta\right) d\rho = 2\pi\int_0^{+\infty} e^{-\frac{\rho^2}{2}}\rho\, d\rho = 2\pi\left[e^{-\frac{\rho^2}{2}}\right]_{\rho=0}^{\rho=+\infty} = 2\pi.$$

Now, this says that $e^{-\frac{x^2+y^2}{2}}$ is integrable on $\mathbb{R}^2$. By the RF we have

$$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-\frac{x^2+y^2}{2}}\, dx\right) dy = \int_{\mathbb{R}} e^{-\frac{x^2}{2}}\left(\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\, dx\right) dy = \left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx\right)^2.$$

Therefore,

$$\left(\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx\right)^2 = 2\pi,$$

and from this the conclusion follows.

Example 5.4.7: (∗∗∗) multidimensional Gaussian integral

Q. *Show that, if C is a $m\times m$ strictly positive and symmetric matrix, then*

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}C^{-1}x\cdot x}\, dx = \sqrt{(2\pi)^m \det C}. \tag{5.4.4}$$

A. To compute (5.4.4) notice first that, being C symmetric, it is diagonalizable: this means that there exists T invertible such that $T^{-1}CT = \mathrm{diag}(\sigma_1,\dots,\sigma_d)$. Furthermore, because C is symmetric, T is also orthogonal, that is $T^{-1} = T^t$ (transposed matrix). Therefore $C = TDT^{-1}$, hence

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}C^{-1}\vec{x}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}(TDT^{1-})^{-1}\vec{x}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}(TD^{-1}T^{1-})\vec{x}\cdot\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}T^{-1}\vec{x}\cdot T^{-1}\vec{x}}\, d\vec{x}.$$

Now, set $\vec{y} = T^{-1}\vec{x}$, in such a way that $\vec{x} = T\vec{y}$ and

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}T^t\vec{x}\cdot T^t\vec{x}}\, d\vec{x} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}\vec{y}\cdot\vec{y}}\,|\det T|\, d\vec{y} = \int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}\vec{y}\cdot\vec{y}}\, d\vec{y}.$$

Last = is justified because, being T orthogonal, $TT^t = \mathbb{I}$, hence $1 = \det(TT^t) = \det T\det T^t = (\det T)^2$ by which $|\det T| = 1$. Moreover,

$$D^{1-}\vec{y}\cdot\vec{y} = \sum_j \frac{1}{\sigma_j}y_j^2,$$

therefore

$$\int_{\mathbb{R}^m} e^{-\frac{1}{2}D^{-1}\vec{y}\cdot\vec{y}}\, d\vec{y} = \int_{\mathbb{R}^m}\prod_{j=1}^m e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j = \prod_{j=1}^m\int_{\mathbb{R}} e^{-\frac{y_j^2}{2\sigma_j}}\, dy_j \overset{x_j=\frac{y_j}{\sqrt{\sigma_j}}}{=} \prod_{j=1}^m\sqrt{\sigma_j}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\, dx = \sqrt{(2\pi)^m\sigma_1\cdots\sigma_m}.$$

To conclude just notice that
$$\sigma_1 \cdots \sigma_m = \det D = \det(T^{-1}CT) = \det T^{-1} \det C \det T = \det C.$$

5.4.2. Spherical and cylindrical coordinates. The analogous of polar coordinates for functions of three variables are *spherical coordinates*:
$$\begin{cases} x = \rho\cos\theta\sin\phi, \\ y = \rho\sin\theta\sin\phi, \\ z = \rho\cos\phi. \end{cases} \quad (\rho,\theta,\phi) \in [0,+\infty[\times[0,2\pi]\times[0,\pi].$$
Also in this case the change of variable is defined in the form
$$(x,y,z) = \Psi(\rho,\theta,\phi),$$
thus, referring to (5.4.1), $\Psi = \Phi^{-1}$. Hence,
$$\det(\Phi^{-1})' = \det\begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} = \rho^2\sin\phi.$$
Therefore, (5.4.1) becomes
$$\int_D f(x,y,z)\,dxdydz = \int_{\Phi_{sph}(D)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\,d\phi\,d\theta\,d\rho.$$
This type of change of variable is often useful when f has some spherical symmetry, that is it depends on $x^2+y^2+z^2$.

Example 5.4.8: (∗)

Q. *Using spherical coordinates, compute the volume of a sphere of radius* r.

A. We have
$$\begin{aligned}\lambda_3\left(\{x^2+y^2+z^2 \leqslant r^2\}\right) &= \int_{x^2+y^2+z^2\leqslant r^2} dxdydz = \int_{0\leqslant\rho\leqslant r,\ 0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\varphi\leqslant\pi} \rho^2\sin\varphi\,d\rho d\theta d\varphi \\ &= 2\pi\left(\int_0^\pi \sin\varphi\,d\varphi\right)\left(\int_0^r \rho^2\,d\rho\right) = \frac{4}{3}\pi r^3.\end{aligned}$$

When f has not a central symmetry but it is symmetric respect to some of the axes, a further variant of polar coordinates may be useful. Let first introduce this system of coordinates defined as
$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z. \end{cases} \quad (\rho,\theta,z) \in [0,+\infty[\times[0,2\pi]\times\mathbb{R}.$$
Also in this case the change of variables is defined in the form
$$(x,y,z) = \Psi(\rho,\theta,z), \quad \text{where } \Psi = \Phi^{-1}.$$
Being,
$$\det\Psi' = \det\begin{bmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho,$$
according to (5.4.1) we have
$$\int_D f(x,y,z)\,dxdydz = \int_{\Phi_{cil}(D)} f(\rho\cos\theta, \rho\sin\theta, z)\rho\,d\rho d\theta dz.$$

This change of variables is particularly useful in the case of functions symmetric respect to the z axis (that is depending on $x^2 + y^2$ that becomes ρ^2 in new coords).

Example 5.4.9: (∗∗)

Q. *Compute the volume of the rugby ball* $D = \{(x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \leqslant 1\}$ *by adapting cylindrical coordinates.*

A. Adapting the cylindrical coords $(x, y, z) = \Psi^{-1}(\rho, \theta, z) := (a\rho\cos\theta, a\rho\sin\theta, bz)$ we have

$$\det(\Psi^{-1})' = \det\begin{bmatrix} a\cos\theta & -a\rho\sin\theta & 0 \\ a\sin\theta & a\rho\cos\theta & 0 \\ 0 & 0 & b \end{bmatrix} = ba^2\rho,$$

therefore

$$\lambda_3(E) = \int_{\rho^2+\tilde{z}^2\leqslant 1,\ \rho\geqslant 0,\ \theta\in[0,2\pi],\ \tilde{z}\in\mathbb{R}} ba^2\rho \, d\rho d\theta dz = 2\pi a^2 b \int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0} \rho \, d\rho dz.$$

To compute the last integral we may use polar coords for $(\rho, z) = (r\cos\alpha, r\sin\alpha)$. Then

$$\int_{\rho^2+z^2\leqslant 1,\ \rho\geqslant 0} \rho \, d\rho dz = \int_{-\frac{\pi}{2}\leqslant\alpha\leqslant\frac{\pi}{2},\ 0\leqslant r\leqslant 1} (r\cos\alpha) r \, dr d\alpha = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\alpha \, d\alpha \int_0^1 r^2 \, dr = \frac{2}{3}.$$

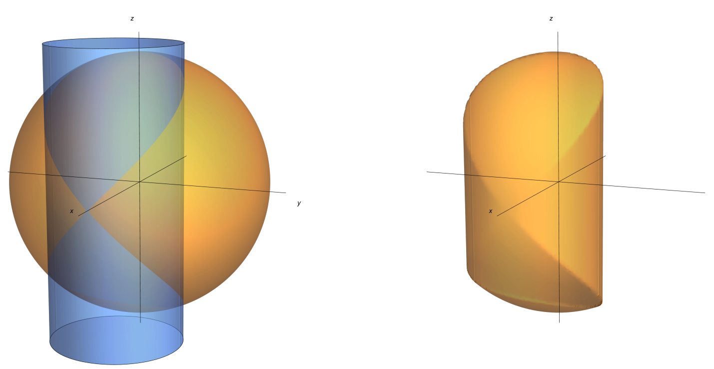
Moral: $\lambda_3(E) = \frac{4\pi}{3} a^2 b$.

One of the most delicate steps is characterizing the integration domain w.r.t. the new integration coordinates.

Example 5.4.10: (∗∗)

Q. *Compute the volume of* $D := \left\{(x, y, z) \ : \ x^2 + y^2 + z^2 \leqslant r^2,\ \left(x - \frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}\right\}$.

A. The set is clearly closed and bounded, thus $\lambda_3(D) < +\infty$. We can also have an idea of D: it is the intersection between the sphere centered at the origin with radius r and a cylinder centered at $(\frac{r}{2}, 0, 0)$, radius $\frac{r}{2}$ and axis parallel to the z−axis.



To compute the volume, we write

$$\lambda_3(D) = \int_D 1 \, dxdydz.$$

Here, a first difficulty is: which coordinates should we use, spherical coords? cylindrical coords? or any other system? Since the integrand function remains $\equiv 1$ whatever is the system of coordinates, we have to focus on the domain D. The first constraint gets easier if we use spherical coordinates (ρ, θ, ϕ), $(x, y, z) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$,

$$x^2 + y^2 + z^2 \leqslant r^2, \iff \rho^2 \leqslant r^2.$$

However, with this choice the second constraint becomes a mess:

$$\left(x - \frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}, \iff \left(\rho \cos\theta \sin\phi - \frac{r}{2}\right)^2 + \rho^2 \sin^2\theta \sin^2\phi \leqslant \frac{r^2}{4}$$

Another option could be "adapted" cylindrical coordinates as

$$(x, y, z) = \left(\frac{r}{2} + \rho \cos\theta, \rho \sin\theta, z\right).$$

In this way, the second constraint becomes simple:

$$\left(x - \frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}, \iff \rho^2 \leqslant \frac{r^2}{4},$$

but the first in more complex:

$$\left(\frac{r}{2} + \rho \cos\theta\right)^2 + \rho^2 \sin^2\theta + z^2 \leqslant r^2.$$

A better choice is a compromise choice: let us use standard $z-$axis based cylindrical coordinates

$$(x, y, z) = (\rho \cos\theta, \rho \sin\theta, z).$$

In this way

$$(x, y, z) \in D, \iff \begin{cases} \rho^2 + z^2 \leqslant r^2, \\ \\ \rho^2 - r\rho \cos\theta \leqslant 0. \end{cases} \overset{\rho \geqslant 0}{\iff} \begin{cases} \rho^2 + z^2 \leqslant r^2, \\ \\ \rho \leqslant r \cos\theta. \end{cases}$$

Therefore

$$\lambda_3(D) \overset{CV}{=} \int_{\rho \geqslant 0,\ \theta \in [0, 2\pi],\ z \in \mathbb{R}\ :\ \rho^2 + z^2 \leqslant r^2,\ \rho \leqslant r \cos\theta} \rho \, d\rho d\theta \, dz.$$

Now, we have to be careful of "hidden" constraints.

- The first condition $\rho^2 + z^2 \leqslant r^2$ must remind that $\rho \geqslant 0$, so it is actually an half-disk centered at $(0, 0)$ with radius r.
- The second condition,

  $$\rho \leqslant r \cos\theta,$$

  being $\rho \geqslant 0$ it must be $r \cos\theta \geqslant 0$, and since here $r > 0$ is a fixed parameter, $\cos\theta > 0$. For $\theta \in [0, 2\pi]$ this happens iff $\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$. Equivalently, instead of taking $\theta \in [0, 2\pi]$ and considering $\theta \in [-\pi, \pi]$ we have $\cos\theta \geqslant 0$ iff $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

So

$$\lambda_3(D) \overset{RF}{=} \int_{-\pi/2}^{\pi/2} \left(\iint_{\rho \geqslant 0,\ z \in \mathbb{R}\ :\ \rho^2 + z^2 \leqslant r^2,\ \rho \leqslant r \cos\theta} \rho \, d\rho \, dz \right) d\theta.$$

Focusing on the innermost double integral, we may notice that

$$\rho^2 + z^2 \leqslant r^2, \iff z^2 \leqslant r^2 - \rho^2, \iff |z| \leqslant \sqrt{r^2 - \rho^2}.$$

It must be $r^2 - \rho^2 \geqslant 0$, that is $\rho^2 \leqslant r^2$, which is automatically true if ρ must also fulfill $0 \leqslant \rho \leqslant r\cos\theta$. Thus, by the reduction formula we have

$$\begin{aligned}
\int_{\rho\geqslant 0,\ z\in\mathbb{R}\,:\,\rho^2+z^2\leqslant r^2,\ \rho\leqslant r\cos\theta} \rho\, d\rho\, dz \quad &\overset{RF}{=} \int_0^{r\cos\theta}\left(\int_{-\sqrt{r^2-\rho^2}}^{\sqrt{r^2-\rho^2}} \rho\, dz\right) d\rho \\
&= \int_0^{r\cos\theta} 2\rho\sqrt{r^2-\rho^2}\, d\rho = -\frac{2}{3}\int_0^{r\cos\theta} \partial_\rho (r^2-\rho^2)^{3/2}\, d\rho \\
&= -\frac{2}{3}\left[\left(r^2-\rho^2\right)^{3/2}\right]_{\rho=0}^{\rho=r\cos\theta} = -\frac{2}{3}\left(r^3(1-\cos^2\theta)^{3/2} - r^3\right) \\
&= \frac{2}{3}r^3\left(1-\sin^3\theta\right).
\end{aligned}$$

Therefore,

$$\lambda_3(D) = \int_{-\pi/2}^{\pi/2} \frac{2}{3}r^3\left(1-\sin^3\theta\right)\, d\theta = \frac{2}{3}\pi r^3 - \frac{2}{3}r^3 \underbrace{\int_{-\pi/2}^{\pi/2} \sin^3\theta\, d\theta}_{=0} = \frac{2}{3}\pi r^3.$$

($\int_{-\pi/2}^{\pi/2} \sin^3\theta\, d\theta = 0$ being the integral of an odd function over a symmetric interval around 0).

5.5. Barycenter, center of mass, inertia moments

Multiple integrals provide a precise definition to several different geometrical or physical entities. To fix ideas consider a set $D \subset \mathbb{R}^3$. We call **barycenter** of D the point $(\overline{x}, \overline{y}, \overline{z})$ defined as

$$\overline{x} = \frac{1}{\lambda_3(D)}\int_D x\, dx\, dy\, dz, \quad \overline{y} = \frac{1}{\lambda_3(D)}\int_D y\, dx\, dy\, dz, \quad \overline{z} = \frac{1}{\lambda_3(D)}\int_D z\, dx\, dy\, dz.$$

In other words, the barycenter is the point whose coordinates are the mean values of the coordnates of D. With special symmetries some of the coordinates of the barycenter may vanish. For instance, if D is symmetric with respect to the plane yz, that is $(x, y, z) \in D$ iff $(-x, y, z) \in D)$, then $\overline{x} = 0$. Indeed, if $\Phi(x, y, z) = (-x, y, z)$ we have $\Phi(D) = D$ and $|\det\Phi'| = 1$. Therefore, by change of variable,

$$\int_D x\, dxdydz = \int_{\Phi(D)} x\, dxdydz = \int_D (-x)|\det\Phi'(x, y, z)|\, dxdydz = -\int_D x\, dxdydz$$

from which it follows that $\int_D x\, dxdydz = 0$.

Example 5.5.1: ($**$)

Q. *Determine the barycenter of a spherical cap* $E := \{(x, y, z)\ :\ x^2 + y^2 + z^2 \leqslant r^2,\ z \geqslant h\}$ *with* $0 \leqslant h < r$.

A. By symmetry, we have $\overline{x} = \overline{y} = 0$. Let's compute

$$\overline{z} = \frac{1}{\lambda_3(D)}\int_D z\, dxdydz.$$

It seems convenient to slice D perpendicularly to the $z-$axis:

$$\begin{aligned}\lambda_3(D) &= \int_h^r \left(\int_{x^2+y^2\leqslant r^2-z^2} dxdy\right)\, dh = \int_h^r \pi(r^2-z^2)\, dz = \pi r^2(r-h) - \pi\left[\frac{z^3}{3}\right]_{z=h}^{z=r} \\ &= \pi(r-h)\left(r^2 - \frac{1}{3}(r^2+rh+h^2)\right).\end{aligned}$$

Similarly

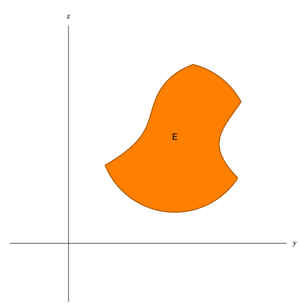
$$\begin{aligned}\int_D z\, dxdydz &= \int_h^r \left(\int_{x^2+y^2\leqslant r^2-z^2} z\, dxdy\right)\, dz = \int_h^r z\left(\int_{x^2+y^2\leqslant r^2-z^2} dxdy\right)\, dz = \int_h^r z\pi(r^2-z^2)\, dz \\ &= \pi r^2\left[\frac{z^2}{2}\right]_{z=h}^{z=r} - \pi\left[\frac{z^4}{4}\right]_{z=h}^{z=r} = \pi r^2\frac{r^2-h^2}{2} - \pi\frac{r^4-h^4}{4} = \pi\frac{r^2-h^2}{2}\left(r^2 - \frac{r^2+h^2}{2}\right) \\ &= \pi\frac{(r^2-h^2)^2}{4}.\end{aligned}$$

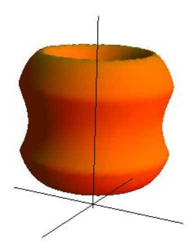
By this we get $\overline{z}$. In the case $h=0$ (that is when D is the half-sphere) we have $\overline{z} = \frac{3}{8}r$.

Let $D \subset \mathbb{R}^3$ by a domain obtained by a rotation around one of the axes of a plane set E. To fix ideas, let's assume that the rotation be around the $z-$axis of a domain E in the plane yz. This domain can be identified by $\{(0,y,z)\ :\ (y,z)\in E\} \subset \mathbb{R}^3$. The set D can be represented as

$$D = \{(y\cos\theta, y\sin\theta, z)\ :\ (y,z)\in E,\ \theta\in[0,2\pi]\} = \Phi(E\times[0,2\pi]),$$

where Φ is nothing but the cylindrical coordinates map.





By the formula of change of variables

$$\lambda_3(D) = \int_{E\times[0,2\pi]} |\Phi'(y,\theta,z)|\, dy\, d\theta\, dz = \int_{E\times[0,2\pi]} y\, dy\, d\theta\, dz = 2\pi\int_E y\, dydz,$$

that gives the **Pappo's Theorem**:

$$\boxed{\lambda_3(D) = 2\pi\lambda_2(E)\overline{y}.} \tag{5.5.1}$$

Example 5.5.2: (∗∗)

Q. *Compute the volume of a thorus* $\mathbb{T}_{r,R} := \left\{(x,y,z)\in\mathbb{R}^3\ :\ (\sqrt{x^2+y^2}-R)^2+z^2\leqslant r^2\right\}$ $(0<r<R)$.

A. According to Pappo's formula (5.5.1), we have

$$\lambda_3(\mathbb{T}_{r,R}) = 2\pi\lambda_2\left(\{(y-R)^2+z^2 \leqslant r^2\right)\overline{y} = 2\pi 1\pi r^2\overline{y} = 4\pi^2 r^2\overline{y}.$$

Here $\overline{y}$ it's the ordinate of the barycenter of the disk $E := \{(y-R)^2+z^2 \leqslant r^2\}$, so

$$\overline{y} = \frac{1}{\lambda_2(E)}\int_E y\,dydz = \frac{1}{\pi r^2}\int_{(y-R)^2+z^2\leqslant r^2} y\,dydz.$$

Changing to polar coord $y-R=\rho\cos\theta$, $z=\rho\sin\theta$, we have easily

$$\overline{y} = \frac{1}{\pi r^2}\int_0^{2\pi}\left(\int_0^r \rho(R+\rho\cos\theta)\,d\rho\right)\,d\theta = \frac{1}{\pi r^2}2\pi\frac{r^2}{2}R = R,$$

(as it is natural!). Hence $\lambda_3(\mathbb{T}_{r,R}) = 4\pi^2 r^2 R$.

If D represents a solid body, multiple integrals provide a method to compute the **mass** of the body. Assuming an inhomogeneous body, we introduce the **density of mass** $\varrho = \varrho(x,y,z)$. Intuitively, $\rho(x,y,z)$ is mass per unit of volume. So, a little cube of solid centered at (x,y,z) with sides dx, dy, dz has mass

$$\varrho(x,y,z)\cdot dxdydz.$$

Therefore, the mass of the solid is, by definition,

$$\mu(D) := \int_D \varrho(x,y,z)\,dxdydz.$$

In Physics it is important the **center of mass**: it is the point where the sum of all the forces acting on D could be applied to get the same effect. This point has coordinates (x_G, y_G, z_G) defined by

$$x_G = \frac{1}{\mu(D)}\int_D x\varrho(x,y,z)\,dxdydz,\quad y_G = \frac{1}{\mu(D)}\int_D y\varrho(x,y,z)\,dxdydz,\quad z_G = \frac{1}{\mu(D)}\int_D z\varrho(x,y,z)\,dxdydz.$$

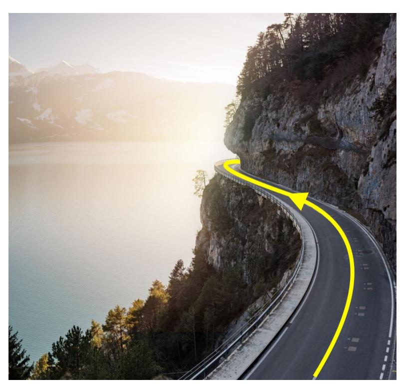
If the body is homogeneous (that is $\varrho \equiv \varrho_0 \in \mathbb{R}$) the center of mass coincide with the barycenter, as it is easy to verify.

Another important quantity for Physics is the **inertia moment with respect to some axis**. For instance, if the axis is the z one, this is defined by

$$I_z := \int_D (x^2+y^2)\varrho(x,y,z)\,dxdydz.$$

5.6. Green's formula

Green's formula is a remarkable application of multiple integrals to vector fields. The formula deals with circulations $\oint_{\vec{\gamma}} \vec{F}$ when $\vec{g}$ is the boundary of some plane region D. There is an important requirement: $\vec{\gamma}$ must turn around D in a *counter-clock wise* way. If this is quite intuitive for simple sets, as for a disk $D = B(\vec{x}, r]$, a precise definition is less evident for a generic D. We do not enter into this problem but we will accept an intuitive definition that might work as follows. Imagine D as a lake and $\vec{\gamma} = \partial D$ as the trajectory of a car turning around the lake. We say that $\vec{\gamma}$ is counter-clock wise oriented w.r.t. D if the lake is always at the left of the car.



Theorem 5.6.1

Let $\vec{F} = (f, g) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^1 vector field. Assume that the boundary of ∂D be described by a regular curve $\vec{\gamma}$ counterclockwise oriented, we write $\vec{\gamma} = \partial D$. Then,

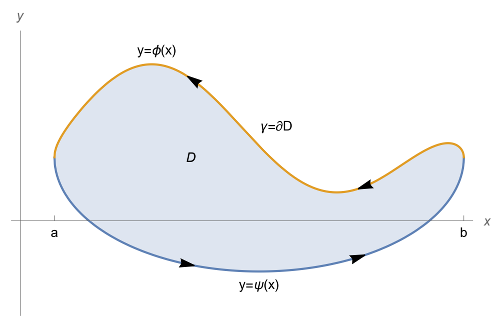
$$\oint_{\vec{\gamma}} \vec{F} = \int_D (\partial_y f - \partial_x g)\, dxdy. \tag{5.6.1}$$

Proof. (in a simplified setup) For simplicity, we assume that the domain D is the region delimited by two functions, that is,

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x \in [a, b],\ \phi(x) \leqslant y \leqslant \psi(y)\}$$

and $\phi(a) = \psi(a)$, $\phi(b) = \psi(b)$. In this case

$$\partial D = \mathrm{Graph}(\phi) \cup \mathrm{Graph}(\psi).$$



Therefore,

$$\vec{\gamma} = \vec{\gamma}_\psi + \vec{\gamma}_\phi, \quad \text{where } \vec{\gamma}_\psi(x) = (x, \psi(x)),\ x \in [a, b],\ \ \vec{\gamma}_\phi(x) = (x, \phi(x)),\ x \in [b, a].$$

With $x \in [b, a]$ we mean that x runs from b to a (right to left). With all these premises,

$$\begin{aligned}
\oint_\gamma \vec{F} &= \int_a^b \vec{F}(x, \psi(x)) \cdot (1, \psi'(x))\, dx + \int_b^a \vec{F}(x, \phi(x)) \cdot (1, \phi'(x))\, dx \\
&= \int_a^b f(x, \psi(x)) + g(x, \psi(x))\psi'(x)\, dx - \int_a^b f(x, \phi(x)) + g(x, \phi(x))\phi'(x)\, dx \\
&= -\int_a^b (f(x, \phi(x)) - f(x, \psi(x)))\ dx - \int_a^b (g(x, \phi(x))\phi'(x) - g(x, \psi(x))\psi'(x))\ dx.
\end{aligned}$$

Now, by the fundamental theorem of integral calculus,

$$f(x, \phi(x)) - f(x, \psi(x)) = \int_{\psi(x)}^{\phi(x)} \partial_y f(x, y)\, dy.$$

A bit more complicate the remaining term. By the fundamental theorem of integral calculus

$$\partial_x \int_{\psi(x)}^{\phi(x)} g(x, y)\, dy = g(x, \phi(x))\phi'(x) - g(x, \psi(x))\psi'(x) + \int_{\psi(x)}^{\phi(x)} \partial_x g(x, y)\, dy,$$

thus,

$$\int_a^b (g(x,\phi(x))\phi'(x) - g(x,\psi(x))\psi'(x))\ dx = \int_a^b \left(\partial_x \int_{\psi(x)}^{\phi(x)} g(x,y)\ dy\right)\ dx - \int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_x g(x,y)\ dy\right)\ dx$$

$$= \left[\int_{\psi(x)}^{\phi(x)} g(x,y)\ dy\right]_{x=a}^{x=b} - \int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_x g(x,y)\ dy\right)\ dx$$

$$= -\int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_x g(x,y)\ dy\right)\ dx$$

being $\phi(a) = \psi(a)$, $\phi(b) = \psi(b)$. In conclusion,

$$\oint_\gamma \vec{F} = \int_a^b \left(\int_{\psi(x)}^{\phi(x)} \partial_y f(x,y) - \partial_x g(x,y)\ dy\right)\ dx = \int_D (\partial_y f - \partial_x g)\ dxdy,$$

which is the conclusion. □

Green's formula has an interesting impact on the problem of establishing if an irrotational field is conservative:

Corollary 5.6.2

Let $\vec{F} = (f,g)$ be an irrotational field on D. Suppose moreover that every $\vec{\gamma} \subset D$ circuit is such that $+\vec{\gamma}$ or $-\vec{\gamma}$ is the counter-clock wise oriented boundary of an $\Omega \subset D$. Then $\vec{F}$ is conservative.

Proof. Since $\vec{F}$ is irrotational, $\partial_y f \equiv \partial_x g$ on D. Let $\gamma \subset D$ be a closed path. According to hypotheses, $\pm\vec{\gamma} = \partial\Omega$ for some $\Omega \subset D$. But then,

$$\oint_{\pm\vec{\gamma}} \vec{F} = \int_\Omega (\partial_y f - \partial_x g)\ dxdy = 0.$$

The conclusion follows now from Theorem 3.3.5. □

Remark 5.6.3

The assumption that *every $\vec{\gamma} \subset D$ circuit is such that $+\vec{\gamma}$ or $-\vec{\gamma}$ is the counter-clock wise oriented boundary of an $\Omega \subset D$* is, in fact, an assumption on the nature of D. Roughly speaking, it says that whenever you take a circuit $\vec{\gamma}$, whatever is delimited by $\vec{\gamma}$ must be contained in D.

Here is another nice consequence of the Green's formula

Corollary 5.6.4: Area formula

Let D be an open and bounded domain with $\partial D = \vec{\gamma}$, where $\vec{\gamma}(t) = (x(t), y(t))$ is a counterclockwise oriented circuit. Then

$$\lambda_2(D) = \oint_{\vec{\gamma}} (y,0) = -\oint_{\vec{\gamma}} (0,x).$$

The following shortening is commonly used:

$$\lambda_2(D) = \oint_\gamma y\ dx = -\oint_\gamma x\ dy.$$

Proof. Just notice that

$$\oint_{\vec{\gamma}} (y,0) = \int_D (\partial_y y - \partial_x 0)\ dxdy = \int_D 1\ dxdy = \lambda_2(D). \quad \square$$

Remark 5.6.5

This is curios and interesting. Returning to the initial interpretation of D as a lake and $\vec{\gamma}$ the road run by a car turning around the lake, the area formula says that you can compute the area of the lake just knowing the time trajectory of the car (position and velocity) turning around the lake!

Example 5.6.6: (∗)

Q. *Compute the area of a disk of radius r.*

A. Let $\vec{\gamma}(t) = r(\cos t, \sin t)$, $t \in [0, 2\pi]$. Then

$$\lambda_2(D) = \int_0^{2\pi} r \sin t \; d(r\cos t) = -r^2 \int_0^{2\pi} (\sin t)^2 \; dt.$$

Integrating by parts

$$-\int_0^{2\pi} (\sin t)^2 \; dt = [\sin t \cos t]_{t=0}^{t=2\pi} - \int_0^{2\pi} (\cos t)^2 \; dt = 2\pi - \int_0^{2\pi} (\sin t)^2 \; dt,$$

from which $-\int_0^{2\pi} (\sin t)^2 \; dt = -\pi$. Therefore $\lambda_2(D) = \pi r^2$ as well known.

5.7. Exercises

Exercise 5.7.1 (∗)**.** Compute

1. $\displaystyle\int_{0\leqslant y\leqslant 1,\ 0\leqslant x\leqslant 1-y^2} xe^y \; dxdy.$ 2. $\displaystyle\int_{0\leqslant y\leqslant 1-x^2} \frac{x}{2+y} \; dxdy.$ 3. $\displaystyle\int_{|y|\leqslant 1-x^2} \frac{1}{1+y} \; dxdy.$

4. $\displaystyle\int_{[0,1]\times[2,4]} \frac{1}{(x-y)^2} \; dxdy$ 5. $\displaystyle\int_{1\leqslant x\leqslant 2,\ \frac{1}{x}\leqslant y\leqslant x} \frac{x}{y} \; dxdy.$ 6. $\displaystyle\int_{[0,1]^2} e^{\max\{x^2,y^2\}} \; dxdy.$

7. $\displaystyle\int_{[0,+\infty[\times[1,+\infty[} e^{-xy^4} \; dxdy$ 8. $\displaystyle\int_{0\leqslant x\leqslant y\leqslant 1} x\sqrt{y^2-x^2} \; dxdy.$ 9. $\displaystyle\int_{|xy|\leqslant 1} \frac{x^2e^{-x^2}}{1+(xy)^2} \; dxdy.$

Exercise 5.7.2 (∗∗)**.** Compute

1. $\displaystyle\int_{[1,+\infty[^3} y^3z^8e^{-xy^2z^3} \; dxdydz.$ 2. $\displaystyle\int_{x\geqslant 0,\ y\geqslant 0,\ x+y+z\leqslant 1} xyz \; dxdydz.$ 3. $\displaystyle\int_{0\leqslant x,y\leqslant 1,\ 0\leqslant z\leqslant x^2} zy^2\sqrt{x^2+zy} \; dxdydz.$

Exercise 5.7.3 (∗)**.** Let $D := \{(x,y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0,\ x^2+y^2 \leqslant r^2\}$. Draw D and describe it in polar coords. Determine its barycenter and compute the integral

$$\int_D \frac{x+y}{x^2+y^2} \; dxdy.$$

Exercise 5.7.4 ($**$)**.** By a suitable use of special coordinates, compute the volume of the following sets:

1. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ z\geqslant x^2+y^2,\ z\leqslant 18-x^2-y^2\right\}$.
2. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1\right\},\ (a,b,c>0)$.
3. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2+z^2\leqslant 16,\ x^2+y^2\geqslant 4\right\}$
4. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ z\geqslant\sqrt{x^2+y^2},\ x^2+y^2+z^2\leqslant 1\right\}$
5. $\left\{(x,y,z) : 9(1-\sqrt{x^2+y^2})^2+4z^2\leqslant 1\right\}$.
6. $\left\{(x,y,z)\in\mathbb{R}^3\ :\ x^2+y^2\leqslant 4,\ 4x^2+4y^2+z^2\leqslant 64\right\}$
7. $\left\{(x,y,z)\ :\ x^2+y^2\leqslant 1,\ x^2+z^2\leqslant 1,\ y^2+z^2\leqslant 1\right\}$.

Exercise 5.7.5 ($**$)**.** Compute

1. $\displaystyle\int_{x^2+y^2\leqslant 4}\sqrt{4-x^2-y^2}\,dxdy$
2. $\displaystyle\int_{x^2+y^2\leqslant 1}\frac{1}{1+x^2+y^2}\,dxdy$.
3. $\displaystyle\int_{\mathbb{R}^2}e^{-(x^2+2y^2)}\,dxdy$.
4. $\displaystyle\int_{x^2+y^2\leqslant 16,\ -5\leqslant z\leqslant 4}\sqrt{x^2+y^2}\,dxdydz$.
5. $\displaystyle\int_{\mathbb{R}^3}\sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dxdydz$.
6. $\displaystyle\int_{[0,+\infty[^3}\frac{x}{1+(x^2+2y^2+3z^2)^2}\,dxdydz$.

Exercise 5.7.6 ($**$)**.** By using the suggested change of variables, compute

1. $\displaystyle\int_D xy\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 3,\ x\leqslant y\leqslant 3x\},\ (u=xy,\ v=\frac{y}{x})$.
2. $\displaystyle\int_D y^2\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant xy\leqslant 2,\ 1\leqslant xy^2\leqslant 2\}.\ (u=xy,\ v=xy^2)$.
3. $\displaystyle\int_D\sqrt{x^2-y^2}\,dxdy,\ D=\{(x,y)\in\mathbb{R}^2\ :\ 1\leqslant x^2-y^2\leqslant 2,\ x\leqslant y\leqslant 2x\}.\ (u=x^2-y^2,\ v=\frac{y}{x})$.

Exercise 5.7.7 ($**$)**.** Let $a>1$ and

$$D_a:=\left\{(x,y)\in\mathbb{R}^2\ :\ \frac{1}{ax}\leqslant y\leqslant\frac{1}{x},\ x^2\leqslant y\leqslant ax^2\right\}.$$

Draw D_a. Then, using the change of variables $u=xy$, $v=\frac{y}{x}$, compute

$$I(a):=\int_{E_a}\frac{x^2}{y}e^{xy}\,dxdy.$$

Exercise 5.7.8 ($**+$)**.** Let

$$f(x,y):=\frac{x^{3/2}}{\sqrt{y-x}}e^{-(xy)^{3/2}},\ \ (x,y)\in D:=\{(x,y)\in\mathbb{R}^2\ :\ 0\leqslant x\leqslant y\}.$$

Use the change of variables $(u,v):=(xy,x/y)$ to compute $\int_D f$.

Exercise 5.7.9 ($**+$)**.** Let

$$f(x,y):=\frac{\log(xy)}{y(x+y^2)^7},\ (x,y)\in D:=\{(x,y)\in[0,+\infty[^2\ :\ xy\geqslant 1\}.$$

Use the change of variables $(u,v):=(xy,\frac{y^2}{x})$ to compute $\int_D f$.

Exercise 5.7.10 ($**$). Applying Green's formula compute

1) $\oint_\gamma (xy, 3x+2y),\ \ \gamma \equiv \partial[-1,1]^2.$

2) $\oint_\gamma (\cos x + 6y^2, 3x - e^{-y^2}),\ \ \gamma \equiv \partial B(0,1].$

3) $\oint_\gamma \left(x^2+y, xy\right),\ \ \gamma(t) := (1+\cos t, \sin t),\ t \in [0,2\pi].$

4) $\oint_\gamma (x^3 - y^3, x^3 + y^3),\ \ \gamma = \partial\left(B(0,r] \cap [0,+\infty[^2\right).$

Exercise 5.7.11 ($**$). Compute the area delimitated by the following curves:

1) $t(\cos t, \sin t),\ t \in [0,2\pi].$ 2) $(\sin t + \sin^2 t, -\cos t - \sin t \cos t),\ t \in [0,2\pi].$ 3) $(\cos^2 t, \sin^2 t),\ t \in [0,2\pi].$

4) $(\cos^3 t, \sin^3 t),\ t \in [0,2\pi].$ 5) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ (a, b > 0).$

CHAPTER 6

Surface integrals

A surface is a "two" dimensional object in $\mathbb{R}^3$. Intuitively, a surface is like a curved sheet of paper. We may imagine that such an object has volume 0 in $\mathbb{R}^3$ but it should have some positive area. The problem is that area λ_2 is defined only for objects in $\mathbb{R}^2$, so *how a concept of area could be defined?*

The concept of area of a surface embedded in $\mathbb{R}^3$ is just, as we will see, a particular case of the concept of *surface integral*, an operation we will introduce in this Chapter. Through this, many important entities used in Physics and Engineering, as *fluxes of a vector fields through a surface*, can be rigorously defined and computed. This shows, once more, how much important is Integration in applications of Mathematics.

Chapter requirements: multiple integration, vector calculus (in particular, vector product)

Learning objectives:

- (basic $*$) concept of parametric surface, being able to determine a parametrization of a surface in simple cases, calculus of the area of a surface in simple cases, concept of flux of a field through a surface.
- (intermediate $**$) applying divergence theorem to compute outward fluxes.
- (advanced $***$) Stokes' theorem, abstract reasoning.

6.1. Parametric Surfaces

The first step is to give a precise definition of what is a surface in $\mathbb{R}^3$. There are two natural ways to define a surface in the space:

- through a *Cartesian equation*;
- through a *parametrization*.

The first method is perhaps the most simple and general. Let us see some examples.

Example 6.1.1: ($*$)

The following are surfaces embedded in $\mathbb{R}^3$:

- a *plane* $ax + by + cz = r$ where $a, b, c, r \in \mathbb{R}$, and $(a, b, c) \neq \vec{0}$;
- a *spherical surface*, as for example
$$x^2 + y^2 + z^2 = r^2.$$
- a *paraboloid*,
$$z = ax^2 + by^2,\ a, b > 0.$$
- a *cone*,
$$z^2 = ax^2, by^2,\ a, b > 0.$$

If a Cartesian equation is a natural way to describe an embedded surface, also a parametrization is pretty natural. Everybody, even a kid, knows a remarkable example of this: our planet. Indeed, nobody uses Cartesian coordinates to identify points on the surface of the planet. Rather, we use *latitude* and *longitude*, which are, in fact, *spherical coordinates*:

Example 6.1.2: (∗)

Let $\mathbb{S}_r := \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = r^2\}$ be the spherical surface of radius $r > 0$. Points $(x, y, z) \in \mathbb{S}_r$ can be also described through

$$(x, y, z) = (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi) \,,$$

where $(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$. Referring $\theta = 0$ to the Greenwich meridian, ϕ is the so-called *colatitude*, that is *latitude* $= \phi - \frac{\pi}{2}$.

We formalize now the idea presented in the previous example. We had a set $\mathscr{M} \subset \mathbb{R}^3$ such that

$$\mathscr{M} = \Phi(D), \text{ where } \Phi = \Phi(u, v) \ : \ D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3.$$

We may expect that some technical condition is needed to ensure $\mathscr{M}$ be a true surface, that is a *two dimensional object*. Indeed, if, for example, Φ is a constant function, say $\Phi(u, v) = (0, 0, 0)$ for all $(u, v) \in D = \mathbb{R}^3$, clearly

$$\Phi(D) = \{\vec{0}\}$$

is not a surface but just a singleton. So, what is the right condition to ask on Φ in order $\mathscr{M} = \Phi(D)$ be a true surface? The idea is suggested from the example of coordinates on Earth. Let us consider the standard spherical parametrization

$$\Phi(\theta, \phi) := (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi) \,.$$

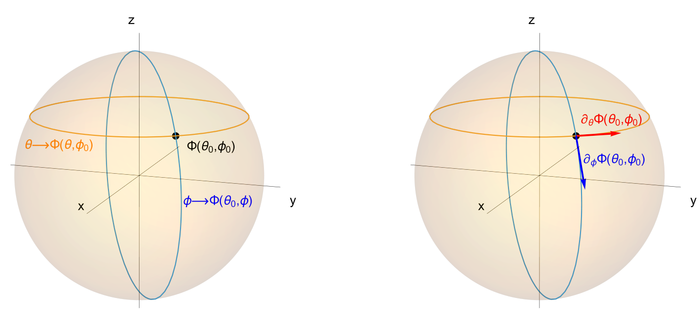
We may notice, if we fix the longitude $\theta = \theta_0$, the line

$$\phi \longmapsto \Phi(\theta_0, \phi),$$

is a meridian while, for a fixed co-latitude $\phi = \phi_0$, the line

$$\theta \longmapsto \Phi(\theta, \phi_0),$$

is a parallel.



Latitude and longitude are good coordinates on the planet because locally, around to a point (θ_0, ϕ_0), they works are cartesian coordinates x, y on the plane xy. In other words, the longitude axis $\phi = \phi_0$ and latitude axis $\theta = \theta_0$ are *perpendicular*. Indeed, by taking the parallel $\phi = \phi_0$, that is the curve $\theta \longmapsto \Phi(\theta, \phi_0)$, its tangent vector at $\theta = \theta_0$ is

$$\partial_\theta \Phi(\theta_0, \phi_0) = (-r \sin\theta_0 \sin\phi_0, r \cos\theta_0 \sin\phi_0, 0)$$

Similarly, the tangent vector to the meridian $\theta = \theta_0$ at $\phi = \phi_0$ is

$$\partial_\phi \Phi(\theta_0, \phi_0) = (r \cos\theta_0 \cos\phi_0, r \sin\theta_0 \cos\phi_0, -r \sin\phi_0).$$

These two vectors are perpendicular because

$$\partial_\theta \Phi \cdot \partial_\phi \Phi = -r^2(\sin\theta_0 \sin\phi_0)(\cos\theta_0 \cos\phi_0) + r^2(\cos\theta_0 \sin\phi_0)(\sin\theta_0 \cos\phi_0) \equiv 0.$$

The fact that the two vectors $\partial_\theta \Phi$ and $\partial_\phi \Phi$ are not parallel is the key to say that, at least "locally" around point $\Phi(\theta_0, \phi_0)$, the set $\Phi(D)$ looks similar to a two dimensional plane. This yields to the following definition.

Definition 6.1.3: parametric surface

Let $\Phi = \Phi(u, v) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$. We say that Φ is an **immersion** if

$$\partial_u \Phi(u, v),\ \partial_v \Phi(u, v) \text{ are linearly independent } \forall (u, v) \in D.$$

The set $\mathscr{M} := \Phi(D)$ is called **parametric surface described by** Φ, which is called **parametrization** of $\mathscr{M}$.

Remark 6.1.4

$$\partial_u \Phi,\ \partial_v \Phi \text{ linearly independent } \iff \text{rank} \begin{bmatrix} \partial_u \Phi \\ \\ \partial_v \Phi \end{bmatrix} = 2$$

Let us review some of the above examples.

Example 6.1.5: $(*)$

Q. Check that a cartesian plane $ax + by + cz = r$, with $(a, b, c) \neq \vec{0}$, is a parametric surface.

A. Assume, for example $c \neq 0$. Then

$$ax + by + cz = r, \iff z = \frac{1}{c}(r - ax - by).$$

Define

$$\Phi = \Phi(x, y) := \left(x, y, \frac{1}{c}(r - ax - by)\right),\ (x, y) \in D = \mathbb{R}^2.$$

Then Φ is a parametrization of $\mathscr{M} := \{(x, y, z)\ : ax + by + cz = r\}$. Indeed:

$$\text{rank} \begin{bmatrix} \partial_x \Phi \\ \\ \partial_y \Phi \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & -\frac{a}{c} \\ \\ 0 & 1 & -\frac{b}{c} \end{bmatrix} = 2,$$

clearly. Thus Φ is an immersion and by construction $\mathscr{M} = \Phi(D)$.

Example 6.1.6: $(*)$

Q. *Show that* $\mathscr{M} := \{(x, y, z)\ :\ z = ax^2 + by^2\}$, $a, b > 0$ *is a parametric surface.*

A. Let

$$\Phi = \Phi(x, y) = (x, y, ax^2 + by^2),\ (x, y) \in \mathbb{R}^2 =: D.$$

Then

$$\text{rank} \begin{bmatrix} \partial_x \Phi \\ \\ \partial_y \Phi \end{bmatrix} = \text{rank} \begin{bmatrix} 1 & 0 & 2ax \\ \\ 0 & 1 & 2by \end{bmatrix} = 2,$$

clearly. Thus Φ is an immersion and $\mathscr{M} = \Phi(D)$ is a parametric surface.

The previous examples suggests a general fact:

Proposition 6.1.7

The graph of a regular function is a parametric surfaces. Precisely, if $f = f(x, y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f \in \mathsf{C}^1(D)$, then

$$\mathscr{M} := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z = f(x, y)\right\},$$

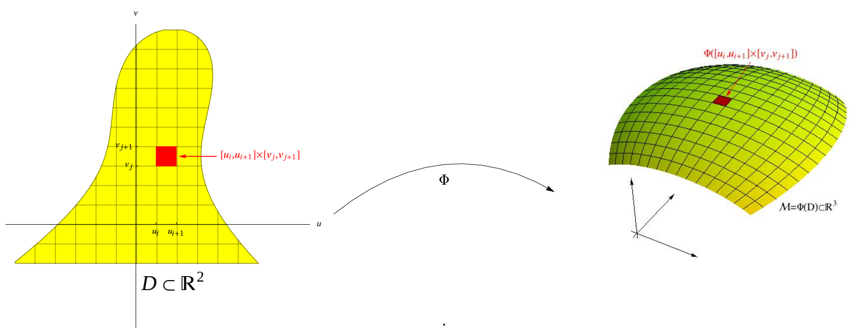
is a parametric surface with standard parametrization $\Phi(x, y) := (x, y, f(x, y))$.

Proof. It is sufficient to notice that

$$\operatorname{rank}\begin{bmatrix} \partial_x \Phi \\ \\ \partial_y \Phi \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & \partial_x f \\ \\ 0 & 1 & \partial_y f \end{bmatrix} = 2. \quad \square$$

6.2. Area of a parametric surface

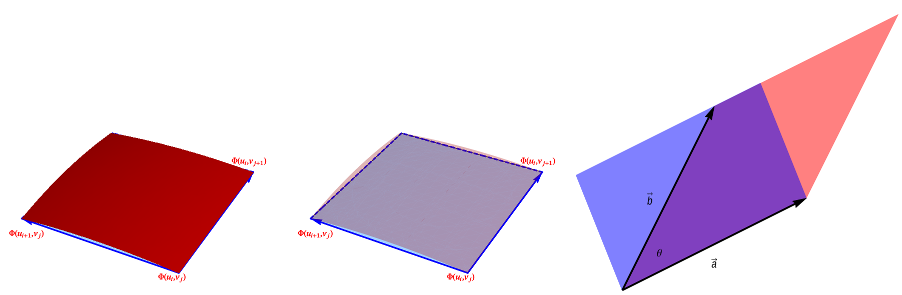
Now we are acquainted with the definition of *parametric surface*, let us see how to use this to define a proper concept of *area for* $\mathscr{M} = \Phi(D)$, where Φ is an immersion on D.



A natural idea consists in dividing D in small rectangles $[u_i, u_{i+1}] \times [v_j, v_{j+1}]$. It seems then reasonable that

$$\text{Area}(\mathscr{M}) = \sum_{i,j} \text{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right).$$

Now, if $u_i \sim u_{i+1}$ and $v_j \sim v_{j+1}$ we could say that $\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])$ is almost a parallelogram of sides $\Phi(u_i, v_{j+1}) - \Phi(u_i, v_j)$ and $\Phi(u_{i+1}, v_j) - \Phi(u_i, v_j)$.



Then,

$$\Phi(u_{i+1}, v_j) - \Phi(u_i, v_j) \approx \partial_u \Phi(u_i, v_j)(u_{i+1} - u_i), \quad \Phi(u_i, v_{j+1}) - \Phi(u_i, v_j) \approx \partial_v \Phi(u_i, v_j)(v_{j+1} - v_j),$$

so

$$\text{Area}\left(\Phi([u_i, u_{i+1}] \times [v_j, v_{j+1}])\right) \approx \text{Area}\left(\left[\partial_u \Phi(u_i, v_j)\ \partial_v \Phi(u_i, v_j)\right]\right)(u_{i+1} - u_i)(v_{j+1} - v_j),$$

where we settled $[\vec{a}\ \vec{b}]$ the parallelogram of sides $\vec{a}$ and $\vec{b}$. Here we notice that $\mathrm{Area}[\vec{a},\vec{b}] \neq 0$ iff $\vec{a}$ and $\vec{b}$ are linearly independent, which is true here because Φ is supposed to be an immersion. Elementary geometry yields the formula

$$\mathrm{Area}[\vec{a},\vec{b}] = \|\vec{a}\|\|\vec{b}\| \sin\theta,$$

where θ is the angle formed by $\vec{a}$ and $\vec{b}$. By recalling the vector product,

$$\mathrm{Area}[\vec{a},\vec{b}] = \|\vec{a}\|\|\vec{b}\| \sin\theta = \|\vec{a}\times\vec{b}\|, \quad \text{where } \vec{a}\times\vec{b} = \det\begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Returning to the initial problem

$$\mathrm{Area}\left(\Phi([u_i,u_{i+1}]\times[v_j,v_{j+1}])\right) \approx \|\partial_u\Phi(u_i,v_i)\times\partial_v\Phi(u_i,v_i)\|(u_{i+1}-u_i)(v_{j+i}-v_j)$$

hence

$$\mathrm{Area}(\mathscr{M}) \approx \sum_{i,j} \|\partial_u\Phi(u_i,v_i)\times\partial_v\Phi(u_i,v_i)\|(u_{i+1}-u_i)(v_{j+i}-v_j) \approx \int_D \|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\, dudv.$$

This informal argument justifies the

Definition 6.2.1

Let $\mathscr{M} = \Phi(D)$ be a parametric surface. We call **area of** $\mathscr{M}$ the quantity

$$\sigma_2(\mathscr{M}) := \int_D \|\partial_u\Phi(u,v)\times\partial_v\Phi(u,v)\|\, dudv \equiv \int_D \|\partial_u\Phi\times\partial_v\Phi\|. \tag{6.2.1}$$

Example 6.2.2: $(*)$

Q. *Compute the area of a spherical surface of radius* r, $\mathbb{S}^2_r := \{(x,y,z)\in\mathbb{R}^3 \ :\ x^2+y^2+z^2=r^2\}$.

A. Recalling the natural parametrization of the spherical surface,

$$\Phi(\theta,\phi) := r\left(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\right), \ (\theta,\phi)\in[0,2\pi]\times[0,\pi],$$

we have

$$\partial_\theta\Phi = r(-\sin\theta\sin\phi, \cos\theta\sin\phi, 0), \quad \partial_\varphi\Phi = r(\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi).$$

Shortening $S=\sin\theta$, $C=\cos\theta$, $s=\sin\phi$, $c=\cos\phi$,

$$\partial_\theta\Phi\times\partial_\varphi\Phi = \det\begin{bmatrix} i & j & k \\ -rSs & rCs & 0 \\ rCc & rSc & -rs \end{bmatrix} = r^2(-Cs^2, -Ss^2, -S^2sc - C^2sc) = -r^2(Cs^2, Ss^2, sc),$$

thus

$$\|\partial_\theta\Phi\times\partial_\varphi\Phi\| = r^2\sqrt{C^2s^4+S^2s^4+s^2c^2} = r^2\sqrt{s^2(s^2+c^2)} = r^2\sqrt{s^2} = r^2|s| = r^2|\sin\phi|.$$

Therefore

$$\sigma_2(\mathbb{S}^2_r) = \int_{0\leqslant\theta\leqslant 2\pi,\ 0\leqslant\phi\leqslant\pi} r^2|\sin\phi|\, d\theta d\phi = 2\pi r^2\int_0^\pi \sin\phi\, d\phi = 4\pi r^2.$$

Apparently, formula (6.2.1) depends on the parametrization Φ. Were this be the case, this would be disturbing. Fortunately, it may be proved that, under natural assumptions, $\sigma_2(\mathscr{M})$ does not depend on a specific parametrization (see next section).

6.3. Surface Integral

We may see at the area of a parametric surface as

$$\sigma_2(\mathscr{M}) = \int_{\mathscr{M}} 1\, d\sigma_2,$$

where

$$\boxed{d\sigma_2 = \|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|\, dudv,}$$

is called **area element**. This leads to a natural extension:

Definition 6.3.1

Let $f : \mathscr{M} \longrightarrow \mathbb{R}$ be continuous on $\mathscr{M} = \Phi(D)$ parametric surface. We set

$$\int_{\mathscr{M}} f\, d\sigma_2 := \int_D f(\Phi(u,v))\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|\, dudv, \tag{6.3.1}$$

provided integral exists and it is finite.

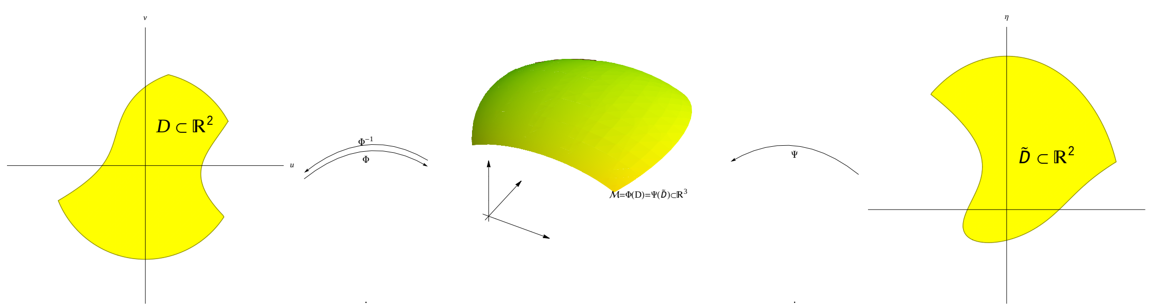
Apparently, the surface integral $\int_{\mathscr{M}} f\, d\sigma$ depends on the specific parametrization Φ. In particular, if $f \equiv 1$, area of a parametric surface depends on the parametrization. Of course, this is not the case if we can pass from one parametrization to another one regularly:

Proposition 6.3.2

Let $\mathscr{M} = \Phi(D) = \Psi(\widetilde{D})$, with $\Phi, \Psi \in \mathscr{C}^1$ immersions such that $\Phi^{-1} \circ \Psi$ be a diffeomorphism (that is, it is differentiable with its inverse). Then

$$\int_D f(\Phi(u,v))\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|\, dudv = \int_{\widetilde{D}} f(\Psi(\xi,\eta))\|\partial_\xi \Psi(\xi,\eta) \times \partial_\eta \Psi(\xi,\eta)\|\, d\xi d\eta \tag{6.3.2}$$

Proof. Call $x = (u,v)$ and $y = (\xi,\eta)$ and set $\Gamma := \Phi^{-1} \circ \Psi$ be the change of parametrization.



Then, by chain rule,

$$\Psi(y) = \Phi(\Phi^{-1}(\Psi(y)) = \Phi(\Gamma(y)), \implies \Psi'(y) = \Phi'(\Gamma(y))\Gamma'(y),$$

that is

$$[\partial_\xi \Psi\ \partial_\eta \Psi] = [\partial_u \Phi(\Gamma)\ \partial_v \Phi(\Gamma)]\Gamma'.$$

Therefore

$$\begin{aligned}\|\partial_\xi \Psi \times \partial_\eta \Psi\| &= \mathrm{Area}[\partial_\xi \Psi\ \partial_\eta \Psi] = \mathrm{Area}\left([\partial_u \Phi(\Gamma)\ \partial_v \Phi(\Gamma)]\Gamma'\right) = \mathrm{Area}\left([\partial_u \Phi(\Gamma)\ \partial_v \Phi(\Gamma)]\right) |\det \Gamma'(y)| \\ &= \|\partial_u \Phi(\Gamma) \times \partial_v \Phi(\Gamma)\| |\det \Gamma'|.\end{aligned}$$

To finish, by the change of variables

$$\int_{\widetilde{D}} f(\Psi(y))\|\partial_\xi(y)\Psi \times \partial_\eta \Psi(y)\|\, dy = \int_{\widetilde{D}} f(\Phi(\Gamma(y)))\|\partial_u \Phi(\Gamma(y)) \times \partial_v \Phi(\Gamma(y))\| |\det \Gamma'(y)|\, dy$$

$$\overset{x=\Gamma(y)}{=} \int_D f\left(\Phi\left(x\right)\right) \|\partial_u \Phi(x) \times \partial_v \Phi(x)\|\, dx. \quad \square$$

An important particular case is the following:

Proposition 6.3.3

Let $\varphi = \varphi(x, y) : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^1 and let

$$\mathscr{M} := \{(x, y, \varphi(x, y)) \ : \ (x, y) \in D\}. \text{ (graph of } \varphi).$$

Then

$$\int_{\mathscr{M}} f\, d\sigma_2 = \int_D f(x, y, \varphi(x, y)))\sqrt{1 + \|\nabla\varphi(x, y)\|^2}\, dxdy. \tag{6.3.3}$$

In particular, we have the **area of a graph formula**,

$$\sigma_2(\mathscr{M}) = \int_D \sqrt{1 + \|\nabla\varphi\|^2}\, dxdy. \tag{6.3.4}$$

Proof. Since $\mathscr{M}$ is the graph of φ, we have the standard parametrization $\Phi : D \longrightarrow \mathbb{R}^3$ is given by $\Phi(x, y) = (x, y, \varphi(x, y))$. Then

$$\partial_u \Phi = (1, 0, \partial_x \varphi),\ \partial_v \Phi = (0, 1, \partial_y \varphi), \implies \partial_x \Phi \times \partial_y \Phi = \det \begin{bmatrix} i & j & k \\ 1 & 0 & \partial_x \varphi \\ 0 & 1 & \partial_y \varphi \end{bmatrix} = (-\partial_x \varphi, -\partial_y \varphi, 1),$$

so

$$\|\partial_x \Phi \times \partial_y \Phi\| = \sqrt{(\partial_x \varphi)^2 + (\partial_y \varphi)^2 + 1} = \sqrt{1 + \|\nabla\varphi\|^2}.$$

From this, formulas (6.3.3) and (6.3.4) follow. $\square$

Example 6.3.4: (∗∗)

Q. *By using area of graph formula, compute the area of spherical surface* $\mathbb{S}^2_r$.

A. We can see the spherical surface $\mathbb{S}^2_r$ as twice the area of an half sphere. This last is a graph: for instance in the half plane $z \geqslant 0$, $z = \sqrt{r^2 - (x^2 + y^2)} =: \varphi(x, y)$. Then

$$\sigma_2(\mathbb{S}^2_r) = 2\int_{x^2+y^2 \leqslant r^2} \sqrt{1 + \|\nabla\varphi(x, y)\|^2}\, dxdy.$$

Being $\nabla\varphi = -\frac{(x,y)}{\sqrt{r^2-(x^2+y^2)}}$ we have

$$\begin{aligned}\sigma_2(\mathbb{S}^2(r)) &= 2\int_{x^2+y^2 \leqslant r^2} \sqrt{1 + \frac{x^2 + y^2}{r^2 - (x^2 + y^2)}}\, dxdy = 2\int_{0 \leqslant \rho \leqslant r,\ 0 \leqslant \theta \leqslant 2\pi} \sqrt{1 + \frac{\rho^2}{r^2 - \rho^2}}\rho\, d\rho d\theta \\ &= 4\pi r\int_0^r \frac{\rho}{\sqrt{r^2 - \rho^2}}\, d\rho = 4\pi r\left[-\sqrt{r^2 - \rho^2}\right]_{\rho=0}^{\rho=r} = 4\pi r^2.\end{aligned}$$

As for multidimensional integrals, also surface integrals are used to compute significant quantities. For example, if $\varrho(x,y,z)$ is a mass density on $\mathscr{M}$ per unit of area, the **total mass** of $\mathscr{M}$ is

$$\mu(\mathscr{M}) := \int_{\mathscr{M}} \varrho \, d\sigma_2.$$

The **barycentre** of a surface $\mathscr{M}$ is the point

$$(\overline{x}, \overline{y}, \overline{z}) = \frac{1}{\sigma_2(\mathscr{M})} \left(\int_{\mathscr{M}} x \, d\sigma_2, \int_{\mathscr{M}} y \, d\sigma_2, \int_{\mathscr{M}} z \, d\sigma_2 \right).$$

while the **center of mass** of $\mathscr{M}$ is the point

$$\frac{1}{\mu(\mathscr{M})} \left(\int_{\mathscr{M}} x\varrho \, d\sigma_2, \int_{\mathscr{M}} y\varrho \, d\sigma_2, \int_{\mathscr{M}} z\varrho \, d\sigma_2 \right).$$

Example 6.3.5: ($*$)

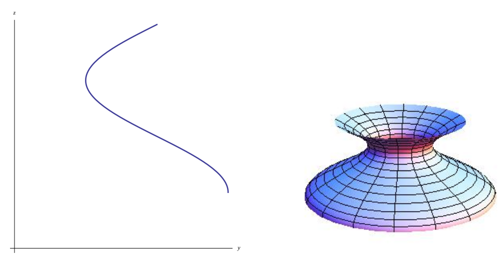
Q. *Compute the barycenter of an half sphere.*

A. The half sphere is the graph of $\beta : D = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leqslant r^2\} \longrightarrow \mathbb{R}$, $\varphi(x,y) = \sqrt{r^2 - (x^2+y^2)}$. By simmetry it is evident that $\int_{\mathscr{M}} x \, d\sigma_2 = \int_{\mathscr{M}} y \, d\sigma_2 = 0$. It remains

$$\int_{\mathscr{M}} z \, d\sigma_2 = \int_{x^2+y^2 \leqslant r^2} z \frac{r}{\sqrt{r^2 - (x^2+y^2)}} \, dxdy = r \int_{x^2+y^2 \leqslant r^2} dxdy = \pi r^3.$$

Being $\sigma_2(\mathscr{M}) = 2\pi r^2$ we obtain the barycenter as the point $\frac{1}{2\pi r^2}\left(0,0,\pi r^3\right) = \left(0,0,\frac{r}{2}\right)$.

6.3.1. Guldino's formula. Consider a surface $\mathscr{M}$ obtained by rotating of a plane curve respect to one of the axes. To fix ideas let $\gamma = (y(t), z(t)) \in \mathsf{C}^1([a,b])$ be a curve in the yz plane. We assume $y(t) > 0$.



The rotation surface is described by

$$\mathscr{M} = \{(y(t)\cos\theta, y(t)\sin\theta, z(t)) \ : \ t \in [a,b], \ \theta \in [0,2\pi]\} = \Phi([a,b] \times [0,2\pi]),$$

where $\Phi(t,\theta) = (y(t)\cos\theta, y(t)\sin\theta, z(t))$. Let's prove the

Theorem 6.3.6: Guldino

Let $\gamma(t) = (y(t), z(t)) \in \mathsf{C}^1([a,b])$ with $\|\gamma'\| \neq 0$ and let

$$\mathscr{M} := \{(y(t)\cos\theta, y(t)\sin\theta, z(t)) \ : \ t \in [a,b], \ \theta \in [0,2\pi]\}.$$

Then

$$\sigma_2(\mathscr{M}) = 2\pi \int_a^b y(t)\|\gamma'(t)\| \, dt. \tag{6.3.5}$$

PROOF. Notice that

$$\partial_t \Phi = (y' \cos\theta, y' \sin\theta, z'),\ \partial_\theta \Phi = (-y \sin\theta, y\cos\theta, 0),$$

hence the area element is

$$\begin{aligned}\|\partial_t \Phi \times \partial_\theta \Phi\| &= \left\| \det \begin{bmatrix} i & j & k \\ y'\cos\theta & y'\sin\theta & z' \\ -y\sin\theta & y\cos\theta & 0 \end{bmatrix} \right\| = \|(yz'\cos\theta, yz'\sin\theta, yy'(\cos^2\theta + \sin^2\theta))\| \\ &= \sqrt{y^2 z'^2(\cos^2\theta + \sin^2\theta) + y^2 y'^2} = \sqrt{y^2(y'^2 + z'^2)} \overset{y\geqslant 0}{=} y\|\gamma'\|.\end{aligned}$$

Therefore,

$$\sigma_2(\mathscr{M}) = \int_{t\in[a,b],\ \theta\in[0,2\pi]} y\|\gamma'\|\, dt d\theta = 2\pi \int_a^b y\|\gamma'\|\, dt. \quad \square$$

Example 6.3.7: $(*)$

Q. *Compute the area of a torus* $\mathbb{T}_{R,r}$ *of radius* $0 < r < R$.

A. The torus is obtained by rotating a circumference of radius r at distance R around some axis. Taking the section in the plane yz with $y \geqslant 0$,

$$\gamma\ :\ (y(t), z(t)) = (R + r\cos t, r\sin t),\ t \in [0, 2\pi].$$

By Guldino's formula

$$\sigma_2(\mathbb{T}_{R,r}) = 2\pi \int_0^{2\pi} (R + r\cos t)\|\gamma'\|\, dt.$$

Now, $\gamma' = (-r\sin t, r\cos t)$ hence clearly $\|\gamma'\| = r$. Therefore

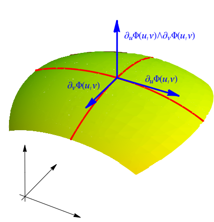
$$\sigma_2(\mathbb{T}_{R,r}) = 2\pi \int_0^{2\pi} (R + r\cos t) r\, dt = 4\pi^2 r R. \quad \square$$

6.4. Flux of a vector field

One of the most important applications of surface integrals is to give a correct definition of *flux of a vector field through a surface*. Let $\mathscr{M} = \Phi(D)$ be a parametric surface and let $\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a *vector field* defined on $\Omega \supset \mathscr{M}$. Our aim is to define

$$\int_{\mathscr{M}} \vec{F} \cdot \vec{n}\, d\sigma_2, \quad \text{where } \vec{n}(\vec{x}) = \text{unit normal vector to } \mathscr{M} \text{ at point } \vec{x}. \tag{6.4.1}$$

Of course, the main problem is how to define the normal $\vec{n}(\vec{x})$. Assume that $\vec{x} = \Phi(u, v)$. Then, vectors $\partial_u \Phi(u, v)$ and $\partial_v \Phi(u, v)$ are tangent to $\mathscr{M}$.



Therefore, by the properties of the vector product,

$$\partial_u \Phi(u,v) \times \partial_v \Phi(u,v) \perp \partial_u \Phi(u,v), \partial_v \Phi(u,v).$$

It is therefore natural to set

$$\vec{n}_\Phi(u,v) := \frac{\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)}{\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|}. \tag{6.4.2}$$

This yields to the following definition.

Definition 6.4.1: flux

Let $\vec{F} : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a continuous vector field on $\Omega \supset \mathscr{M} = \Phi(D)$ parametric surface. We call **flux of $\vec{F}$ through $\mathscr{M}$**

$$\langle \vec{F} \rangle_{\mathscr{M}} := \int_{\mathscr{M}} \vec{F} \cdot \vec{n}_\Phi \, d\sigma_2. \tag{6.4.3}$$

Warning 6.4.2

The integral (6.4.3) **depends** on the specific parametrization. Indeed, if $\mathscr{M} = \Phi(D)$, then setting $\Psi(u,v) := \Phi(v,u)$, we have

$$\mathscr{M} = \Psi(\widetilde{D}), \quad \text{where } \widetilde{D} := \{(v,u) \ : \ (u,v) \in D\},$$

and since,

$$\partial_u \Psi = \partial_v \Phi, \quad \partial_v \Psi = \partial_u \Phi, \quad \Longrightarrow \quad \partial_u \Psi \times \partial_v \Psi = \partial_v \Phi \times \partial_u \Phi = -\partial_u \Phi \times \partial_v \Phi,$$

we have

$$\vec{n}_\Psi = -\vec{n}_\Phi.$$

Thus,

$$\int_{\mathscr{M}} \vec{F} \cdot \vec{n}_\Psi \, d\sigma_2 = -\int_{\mathscr{M}} \vec{F} \cdot \vec{n}_\Phi \, d\sigma_2.$$

Notice that

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \vec{F}(\Phi) \cdot \frac{\partial_u \Phi \times \partial_v \Phi}{\|\partial_u \Phi \times \partial_v \Phi\|} \|\partial_u \Phi \times \partial_v \Phi\| \, dudv = \int_D \vec{F}(\Phi) \cdot (\partial_u \Phi \times \partial_v \Phi) \, dudv.$$

It is easy to check that

$$\boxed{\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \det \left[\vec{F}(\Phi(u,v)) \ \partial_u \Phi(u,v) \ \partial_v \Phi(u,v) \right] \, dudv.} \tag{6.4.4}$$

Example 6.4.3: $(**)$

Q. *Compute the flux of $\vec{F} = (x,y,z)$ through the paraboloid $z = x^2 + y^2$ included between $z = 1$ and $z = 2$.*

A. On a $\mathscr{M} = \Phi(D)$ où $\Phi(u,v) = (u, v, u^2+v^2)$ et $(u,v) \in D = \{1 \leqslant u^2+v^2 \leqslant 2\}$. Par la (6.4.4)

$$\begin{aligned}\langle \vec{F} \rangle_{\mathscr{M}} &= \int_{1\leqslant u^2+v^2\leqslant 2} \det \begin{bmatrix} u & 1 & 0 \\ v & 0 & 1 \\ u^2+v^2 & 2u & 2v \end{bmatrix} \, dudv = \int_{1\leqslant u^2+v^2\leqslant 2} \left(-2u^2-2v^2-(u^2+v^2)\right) \, dudv \\ &= -3 \int_{1\leqslant u^2+v^2\leqslant 2} (u^2+v^2) \, dudv = -3 \int_0^{2\pi} \left(\int_1^{\sqrt{2}} \rho^2 \cdot \rho \, d\rho \right) d\theta = -6\pi \left[\frac{\rho^4}{4} \right]_{\rho=1}^{\rho=\sqrt{2}} = -\frac{9}{2}\pi.\end{aligned}$$

A special but important case is when $\mathscr{M}$ is described as graph of a regular function, that is

$$\mathscr{M} := \{(x, y, \varphi(x,y)) \ : \ (x,y) \in D \subset \mathbb{R}^2\}.$$

In this case a natural parametrization is $\Phi(x,y) := (x, y, \varphi(x,y))$ and

$$\vec{n} = \frac{(1,0,\partial_x\varphi) \times (0,1,\partial_y\varphi)}{\|(1,0,\partial_x\varphi) \times (0,1,\partial_y\varphi)\|},$$

and being

$$(1,0,\partial_x\varphi) \times (0,1,\partial_y\varphi) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial_x\varphi \\ 0 & 1 & \partial_y\varphi \end{bmatrix} = (-\partial_x\varphi, -\partial_y\varphi, 1)$$

we obtain

$$\vec{n} = \frac{(-\partial_x\varphi, -\partial_y\varphi, 1)}{\sqrt{1+\|\nabla\varphi\|^2}}.$$

The (6.4.4) take the form

$$\langle \vec{F} \rangle_{\mathscr{M}} = \int_D \vec{F} \cdot (-\partial_x\varphi, -\partial_y\varphi, 1) \, dxdy, \quad (\vec{F} = \vec{F}(x, y, \varphi(x,y))). \tag{6.4.5}$$

6.4.1. Outward flux. An important case of flux is the following. Suppose that the parametric surface $\mathscr{M}$ is the boundary of an open domain Ω, that is

$$\mathscr{M} = \partial\Omega.$$

At any point $\vec{x} \in \mathscr{M} = \partial\Omega$ we may expect that $\vec{n}(\vec{x})$ can point only in two directions: either *inward* or *outward* of Ω. Formalizing this intuitive idea is not easy, but, yet, this concept is quite natural. We denote by $\vec{n}_e$ the outward normal. For illustrative purposes, here we consider a particular important case showing how $\vec{n}_e$ can be formally defined:

Proposition 6.4.4

Let $\Omega := \{(x,y,z) \in \mathbb{R}^3 \ : \ g(x,y,z) < 0\}$ with $g \in \mathrm{C}^1$ submersion on $\partial\Omega = \{g = 0\}$ (that is, $\nabla g \neq \vec{0}$ on $\partial\Omega$). Then

$$\vec{n}_e(x,y,z) = \frac{\nabla g(x,y,z)}{\|\nabla g(x,y,z)\|}, \ \forall (x,y,z) \in \partial\Omega.$$

Proof. Since g is a submersion on $\partial\Omega$, $\vec{n}_e$ is well defined for every $(x,y,z) \in \partial\Omega$. To show that $\vec{n}_e$ is orthogonal to the surface at any point $(x,y,z) \in \partial\Omega$ we argue as follows. Let $\vec{\gamma} = \vec{\gamma}(t)$ be any regular curve on $\partial\Omega$ passing through point (x,y,z) at some t_0 (that is, $\vec{\gamma}(t_0) = (x,y,z)$). Since $\vec{\gamma}(t) \in \partial\Omega$, we have

$$g(\vec{\gamma}(t)) \equiv 0, \implies 0 \equiv \frac{d}{dt} g(\vec{\gamma}(t)) = \nabla g(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t),$$

according to the total derivative formula 2.2.5. Taking $t = t_0$ we get

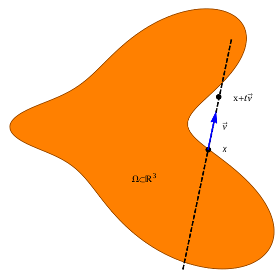
$$\nabla g(x, y, z) \cdot \gamma'(t_0) = 0, \quad \Longleftrightarrow \quad \nabla g(x, y, z) \perp \vec{\gamma}'(t_0).$$

This happens no matter who is $\vec{\gamma}$. Since vectors $\vec{\gamma}'(t_0)$ are tangent to $\partial\Omega$ and they are arbitrarily chosen we have that $\nabla g(x, y, z)$ is normal to $\partial\Omega$ at (x, y, z) (this would require some technicality that we skip here). So $\vec{n}_e$ is unit normal vector to $\partial\Omega$.

To show that ∇g (hence $\vec{n}_e$) is pointing outward, consider the straight line passing through point $(x, y, z) \in \partial\Omega$ with direction $\nabla g(x, y, z)$, that is

$$\vec{\gamma}(t) := (x, y, z) + t\nabla g(x, y, z).$$

We claim that for $t > 0$ and small, $\vec{\gamma}(t) \notin \Omega$, thus this line leaves Ω at least for a while.



To show this, let

$$\phi(t) := g(\vec{\gamma}(t)).$$

Clearly, $\phi(0) = g(x, y, z) = 0$ because $(x, y, z) \in \partial\Omega$. Now, by the total derivative formula

$$\phi'(t) = \nabla g(\vec{\gamma}(t)) \cdot \vec{\gamma}'(t), \quad \Longrightarrow \quad \phi'(0) = \nabla g(x, y, z) \cdot \nabla g(x, y, z) = \|\nabla g(x, y, z)\|^2 > 0.$$

Therefore, for some $\delta > 0$, we have $\phi' > 0$ on $[0, \delta]$, thus $\phi \nearrow$ on $[0, \delta]$ and since $\phi(0) = 0$ this means $\phi(t) = g(\vec{\gamma}(t)) > 0$. Thus, in particular, $\vec{\gamma}(t) \notin \Omega$ for $t \in [0, \delta]$ as claimed. $\square$

Example 6.4.5: $(*)$

In the case of the sphere $\Omega = \{(x, y, z) \in \mathbb{R}^3 \;:\; x^2 + y^2 + z^2 < r^2\}$, $g(x, y, z) = x^2 + y^2 + z^2 - r^2$, hence $\nabla g = (2x, 2y, 2z) = \vec{0}$ iff $(x, y, z) = 0_3 \notin \partial\Omega$. Therefore

$$\vec{n}_e(x, y, z) = \frac{\nabla g}{\|\nabla g\|} = \frac{2(x, y, z)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}.$$

It is easy to check directly that $\vec{n}_e$ is outward.

The flux computed w.r.t. the outward normal is also called **outward flux of $\vec{F}$ from** Ω:

$$\langle \vec{F} \rangle_{\partial\Omega} := \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

6.5. Divergence theorem

The divergence theorem allows to transform the computation of an outward flux (which is, in general, not an easy task) into a volume integral (in general much easier).

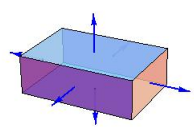
Theorem 6.5.1: Gauss

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set equipped with outward normal. Let $\vec{F} = (f, g, h) : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}^3$ be such that $\vec{F} \in \mathrm{C}^1(\Omega) \cap \mathrm{C}(\partial\Omega)$. Then

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_\Omega \mathrm{div}\vec{F}. \tag{6.5.1}$$

where $\mathrm{div}\,\vec{F} = \partial_x f + \partial_y g + \partial_z h$ is called **divergence of** $\vec{F}$.

Proof. The proof in the general case of Ω is hard and technical. We will do on a very simple case that shows the basic idea. Let's consider Ω be a parallelepiped $\Omega =]a_1, b_1[\times]a_2, b_2[\times]a_3, b_3[$ and let's call $\mathcal{M}_1^\pm$, $\mathcal{M}_2^\pm$ and $\mathcal{M}_3^\pm$ the opposite sides.



Then

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\mathcal{M}_1^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_1^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_2^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_2^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_3^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_3^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

Let's take the outward flux by $\mathcal{M}_1^\pm$ that can be described as

$$\mathcal{M}_1^+ = \{(b_1, y, z) \;:\; y \in [a_2, b_2],\; z \in [a_3, b_3]\}, \quad \mathcal{M}_1^- = \{(a_1, y, z) \;:\; y \in [a_2, b_2],\; z \in [a_3, b_3]\}.$$

It is clear that $\vec{n}_e = (1, 0, 0)$ on $\mathcal{M}_1^+$ while $\vec{n}_e = (-1, 0, 0)$ on $\mathcal{M}_1^-$. Therefore

$$\begin{aligned}
&\int_{\mathcal{M}_1^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_1^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2 \\
&\quad = \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f, g, h) \cdot (1, 0, 0) \, dydz + \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} (f, g, h) \cdot (-1, 0, 0) \, dydz \\
&\quad = \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} f(b_1, y, z) \, dydz - \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} f(a_1, y, z) \, dydz \\
&\quad = \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} \left(f(b_1, y, z) - f(a_1, y, z)\right) \, dydz.
\end{aligned}$$

Now, by the fundamental formula of integral calculus

$$f(b_1, y, z) - f(a_1, y, z) = \int_{a_1}^{b_1} \partial_x f(x, y, z) \, dx,$$

hence

$$\int_{\mathcal{M}_1^+} \vec{F} \cdot \vec{n}_e \, d\sigma_2 + \int_{\mathcal{M}_1^-} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{(y,z)\in[a_2,b_2]\times[a_3,b_3]} \left(\int_{a_1}^{b_1} \partial_x f(x, y, z) \, dx \right) dydz = \int_\Omega \partial_x f.$$

Similarly

$$\int_{\mathcal{M}_2^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathcal{M}_2^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \partial_y g, \quad \int_{\mathcal{M}_3^+} \vec{F}\cdot\vec{n}_e\, d\sigma_2 + \int_{\mathcal{M}_3^-} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \partial_z h.$$

Summing up these formulas we obtain finally

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma = \int_\Omega \left(\partial_x f + \partial_y g + \partial_z h\right) = \int_\Omega \mathrm{div}\vec{F}. \quad \square$$

Example 6.5.2: $(**)$

Q. *Compute the outward flux by* $\Omega = \left\{(x,y,z)\in\mathbb{R}^3 \ : \ \frac{\sqrt{x^2+(y/2)^2}-3}{4} + z < 1\right\}$ *of the vector field* $\vec{F}(x,y,z) = (x,y,z^2)$.

A. According to the divergence theorem,

$$\int_{\partial\Omega} \vec{F}\cdot\vec{n}_e\, d\sigma_2 = \int_\Omega \mathrm{div}\vec{F}\, dxdydz = \int_\Omega (1+1+2z)\, dxdydz = 2\int_\Omega dxdydz + 2\int_\Omega z\, dxdydz.$$

The second integral vanishes being Ω invariant by symmetry $(x,y,z)\longmapsto(x,y,-z)$ while the integrand is odd. The first integral is instead the volume of Ω: by using adapted cylindrical coordinates,

$$\begin{cases} x = \rho\cos\theta, \\ \frac{y}{2} = \rho\sin\theta, \\ z = z, \end{cases} \iff \begin{cases} x = \rho\cos\theta, \\ y = 2\rho\sin\theta, \\ z = z. \end{cases}$$

we have

$$\Omega_{cyl} = \left\{(\rho,\theta,z)\in\mathbb{R}_+\times[0,2\pi]\times\mathbb{R} \ : \ \frac{\rho-3}{4} + z^2 \leqslant 1, \iff \rho \leqslant 4(1-z^2)\right\}.$$

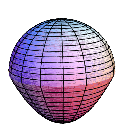
Notice that this last condition forces $z\in[-1,1]$ whence

$$\begin{aligned}\int_\Omega dxdydz &= \int_{\Omega_{cyl}} 2\rho\, d\rho d\theta dz = 2\int_{-1}^1 dz\left(\int_0^{2\pi} d\theta\left(\int_0^{4(1-z^2)} \rho\, d\rho\right)\right) = 2\pi\int_{-1}^1 16(1-z^2)^2\, dz \\ &= 2\pi\int_{-1}^1 16(1-2z^2+z^4)\, dz = 2\pi\left(32 - \frac{64}{3} + \frac{32}{5}\right). \quad \square\end{aligned}$$

Example 6.5.3: $(**)$

Q. *Let* $\Omega := \{(x,y,z)\in\mathbb{R}^3 \ : \ x^2+y^2 < z < 1+\sqrt{1-(x^2+y^2)}\}$. *Compute the outward flux by* Ω *of the vector field* $\vec{F}(x,y,z) := (x,y,x^2+y^2)$, *computing also all the part of the flux relative to the several portions of* $\partial\Omega$.

A. Ω is the region included between the paraboloid $z = x^2+y^2$ and the sphere centered at $(0,0,1)$ with radius 2.



By the divergence theorem,

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma = \int_{\Omega} \mathrm{div}\vec{F} \, dxdydz = \int_{\Omega} (1+1+0) \, dxdydz = 2\lambda_3(\Omega).$$

Let's compute $\lambda_3(\Omega)$. In cylindrical coordinates

$$\lambda_3(\Omega) = \int_{\Omega} dxdydz \overset{CV}{=} \int_{\Omega_{cyl}} \rho \, d\rho d\theta dz,$$

where

$$\Omega_{cyl} = \left\{ (\rho, \theta, z) \in [0, +\infty[\times [0, 2\pi] \times \mathbb{R} \; : \; \rho^2 \leqslant z \leqslant 1 + \sqrt{1-\rho^2} \right\}.$$

Of course one need that $1 - \rho^2 \geqslant 0$, that is $0 \leqslant \rho \leqslant 1$ and also $\rho^2 \leqslant 1 + \sqrt{1-\rho^2}$, always true being $\rho \leqslant 1$. Therefore

$$\begin{aligned}
\lambda_3(\Omega) &= \int_{0\leqslant\rho\leqslant1,\ 0\leqslant\theta\leqslant2\pi,\ \rho^2\leqslant z\leqslant1+\sqrt{1-\rho^2}} \rho \, d\rho d\theta dz \overset{RF}{=} \int_0^{2\pi} \left(\int_0^1 \left(\int_{\rho^2}^{1+\sqrt{1-\rho^2}} \rho \, dz \right) d\rho \right) d\theta \\
&= 2\pi \int_0^1 \rho \left(1 + \sqrt{1-\rho^2} - \rho^2 \right) d\rho = 2\pi \left(\left[\frac{\rho^2}{2} \right]_0^1 + \left[-\frac{1}{3}(1-\rho^2)^{3/2} \right]_0^1 - \left[\frac{\rho^4}{4} \right]_0^1 \right) \\
&= 2\pi \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{4} \right) = \frac{7}{6}\pi.
\end{aligned}$$

We now compute the component of the flux outgoing by the paraboloid. This is described by the equation $z = x^2 + y^2$ and the part interior to Ω is $z > x^2 + y^2$, that is $g(x, y, z) := x^2 + y^2 - z < 0$. Therefore,

$$\vec{n}_e = \frac{\nabla g}{\|\nabla g\|} = \frac{(2x, 2y, -1)}{\sqrt{1 + 4(x^2+y^2)}}.$$

Hence, calling $\mathscr{P}$ the paraboloid, we have

$$\int_{\mathscr{P}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\mathscr{P}} (x, y, x^2+y^2) \cdot \frac{(2x, 2y, -1)}{\sqrt{1+4(x^2+y^2)}} \, d\sigma_2 = \int_{\mathscr{P}} \frac{x^2+y^2}{\sqrt{1+4(x^2+y^2)}} \, d\sigma_2.$$

Now, $\mathscr{P}$ is the graph of $(x, y) \longmapsto x^2 + y^2$ on the domain $D = x^2 + y^2 \leqslant 1$: therefore

$$\begin{aligned}
\int_{\mathscr{P}} \frac{x^2+y^2}{\sqrt{1+4(x^2+y^2)}} \, d\sigma_2 &= \int_D \frac{x^2+y^2}{\sqrt{1+4(x^2+y^2)}} \sqrt{1+4(x^2+y^2)} \, dxdy = \int_D (x^2+y^2) \, dxdy \\
&\overset{CV+RF}{=} \int_0^1 \left(\int_0^{2\pi} \rho^3 \, d\theta \right) d\rho = \frac{\pi}{2}.
\end{aligned}$$

The remaining component of the flux (that one relative to the half sphere) can be now deduce by difference by the total outward flux.

The remainder of the section is devoted to illustrate some remarkable applications of the divergence theorem.

6.5.1. Gauss theorem on central fields. A beautiful application of the divergence theorem is the Gauss' theorem on outward flux of a central field. This result is very important for gravitation and electric forces. In general, a **central field centered at** $\vec{x}_*$ is a vector field $\vec{F} : \mathbb{R}^3 \backslash \{\vec{x}_*\} \longrightarrow \mathbb{R}^3$ according to

$$\vec{F}(\vec{x}) := k \frac{\vec{x} - \vec{x}_*}{\|\vec{x} - \vec{x}_*\|^3}, \ \forall \vec{x} \in \mathbb{R}^3 \backslash \{\vec{x}_*\}.$$

This is the case of gravitational and electric force fields. The Gauss theorem states that the outward flux through a closed surface $\partial\Omega$ is zero if the center $\vec{x}_*$ lies outside the surface, and $4\pi k$ if $\vec{x}_*$ lies inside it.

Theorem 6.5.4: Gauss

Let $\vec{F}$ be a central field centered at $\vec{x}_*$. Then, for any open set $\Omega \subset \mathbb{R}^3$,

$$\langle \vec{F} \rangle_{\partial\Omega} = \begin{cases} 0, & \text{if } \vec{x}_* \in \Omega^c \backslash \partial\Omega, \\ 4\pi k, & \text{if } \vec{x}_* \in \Omega. \end{cases}$$

Proof. For computational simplicity we assume $\vec{x}_* = \vec{0}$.

Case $\vec{x}_* = \vec{0} \in \Omega^c \backslash \partial\Omega$. Then $\vec{F} \in \mathrm{C}^1(\Omega) \cap \mathrm{C}(\partial\Omega)$ and the divergence theorem applies:

$$\langle \vec{F} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\Omega} \mathrm{div}\vec{F} \, dxdydz.$$

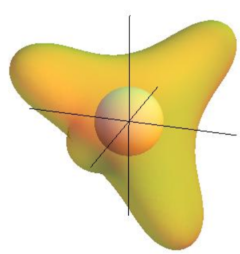
Now: the key point is that $\mathrm{div}\vec{F} = 0$. Indeed,

$$\begin{aligned} \mathrm{div}\vec{F} &= k \left(\partial_x \frac{x}{(x^2+y^2+z^2)^{3/2}} + \partial_y \frac{y}{(x^2+y^2+z^2)^{3/2}} + \partial_z \frac{z}{(x^2+y^2+z^2)^{3/2}} \right) \\ &= l \left(\frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{(x^2+y^2+z^2) - 3y^2}{(x^2+y^2+z^2)^{5/2}} + \frac{(x^2+y^2+z^2) - 3z^2}{(x^2+y^2+z^2)^{5/2}} \right) = 0. \end{aligned}$$

Therefore we deduce $\langle \vec{F} \rangle_{\partial\Omega} = 0$.

Case $\vec{x}_* = \vec{0} \in \Omega$. In this case, we cannot say that $\vec{F} \in \mathrm{C}^1(\Omega)$ because of the singularity of $\vec{F}$ at $\vec{0} \in \Omega$, so we cannot apply the divergence theorem on Ω. However we can take out a suitably small ball $B(\vec{0}, \rho] \subset \Omega$ (it exists because Ω is open and $\vec{0} \in \Omega$) and define

$$\widetilde{\Omega} := \Omega \backslash B(\vec{0}, \rho].$$



Now, $\partial\widetilde{\Omega} = \partial\Omega \cup \partial B(\vec{0}, \rho]$ and $\vec{F} \in \mathrm{C}^1(\widetilde{\Omega}) \cap \mathrm{C}(\partial\widetilde{\Omega})$. By the divergence theorem, we have:

$$0 = \int_{\widetilde{\Omega}} \mathrm{div}\vec{F} = \int_{\partial\widetilde{\Omega}} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 - \int_{\partial B(\vec{0},\rho]} \vec{F} \cdot \vec{n}_e \, d\sigma_2.$$

Notice that in the second integral $\vec{n}_e$ stand for the outward normal unit vector by $B(\vec{0}, \rho]$ (therefore the outward normal by the ball is inward for $\widetilde{\Omega}$, this motivates the $-$ in front of the integral). Now,

$$\int_{\partial B(\vec{0},\rho]} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = \int_{\partial B(\vec{0},\rho]} k \frac{(x,y,z)}{\|(x,y,z)\|^3} \cdot \frac{(x,y,z)}{\|(x,y,z)\|} \, d\sigma_2 = k \int_{\partial B(\vec{0},\rho]} \frac{1}{\|(x,y,z)\|^2} \, d\sigma_2 = \frac{k}{\rho^2} 4\pi\rho^2 = 4\pi k,$$

so we conclude that $\int_{\partial\Omega} \vec{F} \cdot \vec{n}_e \, d\sigma_2 = 4\pi k$. □

The electric field generated by a point charge q at point $\vec{x}_*$ is

$$\vec{E}(\vec{x}) = \frac{q}{\varepsilon} \frac{\vec{x} - \vec{x}_*}{\|\vec{x} - \vec{x}_*\|^3}$$

where ε is the *permittivity*. According to Gauss' theorem,

$$\langle \vec{E} \rangle_{\partial\Omega} = \begin{cases} 4\pi \frac{q}{\varepsilon}, & \text{if } \vec{x}_* \in \Omega, \\ \\ 0, & \text{if } \vec{x}_* \in \Omega^c \backslash \partial\Omega. \end{cases}$$

Suppose now that there are two charges q_1 and q_2 at points $\vec{x}_{*,1}$ and $\vec{x}_{*,2}$. The electric force at $\vec{x}$ is obtained by summing the two forces generated by the two charges. Since a flux depends linearly on the vector field to which is applied,

$$\langle \vec{E} \rangle_{\partial\Omega} = \langle \vec{E}_1 \rangle_{\partial\Omega} + \langle \vec{E}_2 \rangle_{\partial\Omega} = \frac{4\pi}{\varepsilon} \left(q_1 1_\Omega(\vec{x}_{*,1}) + q_2 1_\Omega(\vec{x}_{*,2}) \right)$$

where $1_\Omega(\vec{x})$ is the **unit function**, equals to 1 if $\vec{x} \in \Omega$, and 0 elsewhere. In general, for N charges at points $\vec{x}_{*,j}$, we have

$$\langle \vec{E} \rangle_{\partial\Omega} = \frac{4\pi}{\varepsilon} \sum_{j=1}^{N} q_j 1_\Omega(\vec{x}_{*,j}).$$

Suppose now that the charge is continuously distributed in the space. This can be described through a **charge density** $\varrho = \varrho(x, y, z)$, that yields the charge per unit of volume. In other words, if we consider a small parallelepiped centered at (x, y, z) with sides dx, dy, dz, the total charge of the parallelepiped is, approximately, $\varrho(x, y, z)\ dxdydz$. Then, we may imagine that extending the previous argument we obtain

$$\langle \vec{E} \rangle_{\partial\Omega} = \frac{4\pi}{\varepsilon} \int_\Omega \varrho(x, y, z)\ dxdydz.$$

On the other hand, by divergence thm,

$$\langle \vec{E} \rangle_{\partial\Omega} = \int_{\partial\Omega} \vec{E} \cdot \vec{n}_e\ d\sigma = \int_\Omega \mathrm{div}\vec{E},$$

so

$$\int_\Omega \mathrm{div}\vec{E} = \int_\Omega \frac{4\pi}{\varepsilon} \varrho.$$

Now this argument holds for every Ω, so we might expect that, because of the arbitrary nature of Ω, the previous equation holds iff

$$\boxed{\mathrm{div}\vec{E} = \frac{4\pi}{\varepsilon} \varrho.}$$

This is the first of Maxwell's equations, the fundamental equations of the electro-magnetic field.

6.5.2. Gradient theorem. Anothe important consequence of the divergence theorem is the following

Corollary 6.5.5

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set equipped with outward normal and let $f : \Omega \cup \partial\Omega \longrightarrow \mathbb{R}$ be a function $f \in \mathsf{C}^1(\Omega) \cap \mathsf{C}(\partial\Omega)$. Then

$$\int_\Omega \nabla f = \int_{\partial\Omega} f \vec{n}_e\ d\sigma_2. \tag{6.5.2}$$

PROOF. The identity (6.5.2) involves two vectors:

$$\int_\Omega \nabla f = \int_\Omega (\partial_x f, \partial_y f, \partial_z f) = \left(\int_\Omega \partial_x f, \int_\Omega \partial_y f, \int_\Omega \partial_z f\right),$$

$$\int_{\partial\Omega} f\vec{n}_e \, d\sigma_2 = \left(\int_{\partial\Omega} f(\vec{n}_e)_x \, d\sigma_2, \int_{\partial\Omega} f(\vec{n}_e)_y \, d\sigma_2, \int_{\partial\Omega} f(\vec{n}_e)_z \, d\sigma_2\right).$$

So, we have to check it component by component. We do this for the first component, the others are similar: we have to check that

$$\int_\Omega \partial_x f = \int_{\partial\Omega} f(\vec{n}_e)_x$$

where $(\vec{n}_e)_x$ is the first component of $\vec{n}_e$. To this aim introduce the auxiliary vector field $\vec{F} := (f, 0, 0)$. Then, by the divergence theorem,

$$\mathrm{div}\vec{F} = \partial_x f + \partial_y 0 + \partial_z 0 = \partial_x f,$$

from which,

$$\int_\Omega \partial_x f = \int_\Omega \mathrm{div}\vec{F} = \int_{\partial\Omega} \vec{F}\cdot\vec{n}_e = \int_{\partial\Omega} (f,0,0)\cdot\vec{n}_e = \int_{\partial\Omega} f(\vec{n}_e)_x. \quad \square$$

A nice application of this Corollary is the well known

Corollary 6.5.6: Archimedean principle

A body immersed in an incompressible homogeneous fluid experiences a buoyant force equal to the weight of the displaced fluid.

PROOF. Assume that the fluid has a constant density per unit of volume ϱ and let $p = p(x, y, z)$ the pressure (weight force per unit of surface) exercised by the fluid at point (x, y, z). For simplicity we may assume that the z axis corresponds to the vertical axis and $z = 0$ correspond to the surface of the fluid (assumed to be flat). Therefore, if p_0 (constant) is the atmospheric pressure on the surface of the fluid,

$$p(x, y, z) = -\varrho g z + p_0.$$

Now, assume that the body is represented by an open and bounded set $\Omega \subset \{z \leqslant 0\}$. The resultant of the pressure forces on the body is therefore

$$-\int_{\partial\Omega} p\vec{n}_e \, d\sigma_2.$$

The $-$ gives account of the fact that are counted positively pressures inward directed to Ω. Now, by the gradient thm

$$-\int_{\partial\Omega} p\vec{n}_e \, d\sigma_2 = -\int_\Omega \nabla p \, dxdydz = -\int_\Omega (0, 0, -\varrho g) \, dxdydz = \varrho g \lambda_3(\Omega)\vec{k},$$

that is the resultant of the pressure forces is upward oriented with intensity $\varrho g \lambda_3(\Omega)$ which is nothing but the weight of the body. $\square$

6.6. Stokes' formula

Green's formula transforms a plane circulation along $\vec{\gamma} \subset \mathbb{R}^2$ into a plane integral on a domain D such that its boundary coincides with the circuit $\vec{\gamma}$. *Stokes' formula* is the extension of Green's formula to the case of a circulation along $\vec{\gamma} \subset \mathbb{R}^3$. In this case we might expect that, under good assumptions, $\vec{\gamma}$ is the **edge** of a surface, like the edge of a chips. This seems intuitively clear, but not easy to be formalized. For example, we cannot say that $\vec{\gamma} = \partial\mathscr{M}$. This because, if $\mathscr{M}$ is a surface, reasonably $\partial\mathscr{M} = \mathscr{M}$ is a surface (2 dimensional object) and not a curve (1 dimensional object).

Definition 6.6.1: edge

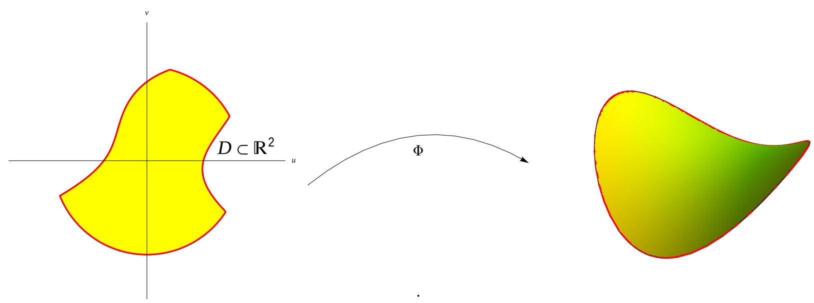
Let $\mathscr{M} := \Phi(D) \subset \mathbb{R}^3$ be a parametric surface. We call **edge** of $\mathscr{M}$ the set

$$\partial\mathscr{M} := \Phi(\partial D).$$

We say that $\partial\mathscr{M}$ is **counterclockwise oriented** if ∂D is counterclockwise oriented.

Warning 6.6.2

When $\mathscr{M}$ is a parametric surface, $\partial\mathscr{M}$ is not the boundary of $\mathscr{M}$ but its edge.



We have the

Theorem 6.6.3: Stokes'

Let $\mathscr{M}$ be a parametric surface with edge counter-clock wise oriented and let $\vec{F} : D \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a C^1 vector field on $D \supset \mathscr{M}$. Then

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_{\mathscr{M}} (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma_2, \tag{6.6.1}$$

where

$$\nabla \times \vec{F} := \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{bmatrix} = \left(\partial_y h - \partial_z g, \partial_z f - \partial_x h, \partial_x g - \partial_y f\right).$$

In other words: the circulation of $\vec{F}$ is the flux of $\nabla \times \vec{F}$ (called **curl** of $\vec{F}$) through $\mathscr{M}$.

Proof. Let's limit to a special (but important) case: $\mathscr{M}$ be the graph of a regular function,

$$\mathscr{M} = \{(x, y, \varphi(x, y)) \ : \ (x, y) \in D \subset \mathbb{R}^2\}, \text{ with } \partial D \equiv \gamma.$$

Assume that $\gamma = \gamma(t) = (x(t), y(t))$ be counter-clock wise oriented. Then

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_a^b (f, g, h) \cdot (x, y, \varphi(x, y))'.$$

Noticed that

$$(x, y, \varphi(x, y))' = (x', y', \partial_x\varphi x' + \partial_y\varphi y'),$$

we have

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_a^b (f x' + g y' + h(\partial_x\varphi x' + \partial_y\varphi y')) = \int_a^b (f + h\partial_x\varphi, g + h\partial_y\varphi) \cdot (x', y') = \oint_{\partial D} (f + h\partial_x\varphi, g + h\partial_y\varphi).$$

By Green formula then

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_D \left(\partial_x \left(g + h\partial_y\varphi\right) - \partial_y \left(f + h\partial_x\varphi\right)\right).$$

Now, recall that every of f, g, h is evaluated at $(x, y, \varphi(x, y))$. Therefore

$$\partial_x \left(g + h\partial_y\varphi\right) = \partial_x g + \partial_z g\partial_x\varphi + (\partial_x h + \partial_z h\partial_x\varphi)\partial_y\varphi + h\partial_{xy}\varphi,$$

$$\partial_y \left(f + h\partial_x\varphi\right) = \partial_y f + \partial_z f\partial_y\varphi + (\partial_y h + \partial_z h\partial_y\varphi)\partial_x\varphi + h\partial_{xy}\varphi,$$

and by taking their difference we obtain the quantity

$$(\partial_x g - \partial_y f) + \partial_x\varphi(\partial_z g - \partial_y h) + \partial_y\varphi(\partial_x h - \partial_z f) = \nabla \times (f, g, h) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1),$$

hence

$$\oint_{\partial\mathscr{M}} \vec{F} = \int_D (\nabla \times (f, g, h)) \cdot (-\partial_x\varphi, -\partial_y\varphi, 1)\, dxdy \overset{(6.4.5)}{=} \int_{\mathscr{M}} (\nabla \times (f, g, h)) \cdot \vec{n}\, d\sigma_2. \quad \square$$

Example 6.6.4: $(**)$

Q. *Compute the circulation of* $\vec{F}(x, y, z) := (x^2 z, xy^2, z^2)$ *along* $\gamma = \{x + y + z = 1,\ x^2 + y^2 = 9\}$.

A. Clearly γ is an ellipse that we may think as edge of

$$\mathscr{M} = \Phi(D),\ \ \Phi : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3,\ \ \Phi(x, y) = (x, y, 1 - (x + y)),\ \ D := \{(x, y) : x^2 + y^2 \leqslant 9\}.$$

It is easy to take the normal unit vector to $\mathscr{M}$ in such a way that $\gamma = \partial\mathscr{M}$ be counter-clock wise oriented: $\vec{n} = \frac{(1,1,1)}{\sqrt{3}}$. Moreover, being $\mathscr{M}$ graph of $f(x, y) := 1 - (x + y)$ $(x, y) \in D$, we have

$$d\sigma_2 = \sqrt{1 + \|\nabla f(x, y)\|^2}\, dxdy = \sqrt{1 + \|(-1, -1)\|^2}\, dxdy = \sqrt{3}\, dxdy.$$

Therefore being $\nabla \times \vec{F} = (0, x^2, y^2)$, by the curl theorem

$$\oint_\gamma \vec{F} = \int_{\mathscr{M}} \nabla \times \vec{F} \cdot \vec{n}\, d\sigma = \int_{x^2+y^2\leqslant 9} (0, x^2, y^2) \cdot \frac{(1, 1, 1)}{\|(1, 1, 1)\|} \sqrt{3}\, dxdy = \int_{x^2+y^2\leqslant 9} (x^2 + y^2)\, dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^3 \rho^3\, d\rho = 2\pi \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=3} = \pi\frac{3^4}{2}.$$

6.7. Exercices

Exercise 6.7.1 $(**)$**.** For each of the following surfaces, determine a parametrization and check if these are parametric surfaces:

i) $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3\ :\ x + y + z = 1,\ x^2 + y^2 \leqslant 1\}$.
ii) $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3\ :\ x^2 + y^2 - z^2 = 1\}$.
iii) $\mathscr{M} := \{(x, y, z) \in \mathbb{R}^3\ :\ z^2 = xy + 1\}$

Exercise 6.7.2 $(**)$**.** Compute the area of each of the following surfaces:

- $\mathscr{M} := \left\{(x; y; z) \in \mathbb{R}^3\ :\ x \geqslant 0,\ z \geqslant 0,\ x + z = 2,\ x^2 + y^2 - 2x \leqslant 0\right\}$.
- $\mathscr{M} := \left\{(x, y, z)\ : z \geqslant 0,\ x^2 + y^2 + z^2 = r^2,\ \left(x - \frac{r}{2}\right)^2 + y^2 \leqslant \frac{r^2}{4}\right\}$ (Viviani's vault).
- $\mathscr{M}$ rotation surface around the $x-$axis of the graph of the function $f(x) = \alpha x^2$, $x \in [0, a]$, $a, \alpha > 0$.
- $\mathscr{M}$ rotation surface around the $z-$axis of the graph of the function $f(y) = 2 - \cosh y$, $y \in [0, a]$, a such that $f \geqslant 0$.

Exercise 6.7.3 ($**$)**.** Let $\mathscr{M}$ be the rotation surface around the z−axis of the curve $y = ze^{-z}$, $z \in [0, h]$ $(h > 0)$. Compute the outward flux of $\vec{F}(x, y, z) := (x + y^2, y + x^2, z)$ through $\mathscr{M}$.

Exercise 6.7.4 ($**$)**.** Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 \ : \ 1 \leqslant z \leqslant 2, \ \sqrt{x^2 + y^2} \leqslant z\}$. Draw Ω and compute the outward flux of $\vec{F}(x, y, z) := (x^3, 0, z^2)$ by Ω.

Exercise 6.7.5 ($**$)**.** Let $\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ (x - z)^2 + (y + z)^2 \leqslant z^2, \ 0 \leqslant z \leqslant b\right\}$ $(b > 0)$. Compute the outward flux of $\vec{F}(x, y, z) = (xy, x - y, yz)$ by Ω.

Exercise 6.7.6 ($**$)**.** Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant z \leqslant 1\}$. Draw Ω, compute its volume and the area of $\partial\Omega$. Hence, compute the outward flux of $\vec{F}(x, y, z) := (x, y, z)$ by Ω, computing the components of the flux by each part of $\partial\Omega$.

Exercise 6.7.7 ($**$)**.** Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1, \ a \leqslant z \leqslant 1\}$ with $0 < a < 1$. Draw Ω, compute its volume and its barycenter. Compute the outward flux of $\vec{F}(x, y, z) := (x^2, -y^2, z)$ by Ω and each of its components on the several parts of $\partial\Omega$.

Exercise 6.7.8 ($**$)**.** Let $\Omega := \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leqslant e^y, 0 \leqslant y \leqslant a\}$ with $a > 0$. Draw Ω and compute outward flux and relative components of $\vec{F}(x, y, z) := (-z, (x^2 + z^2)y, x)$ by Ω.

Exercise 6.7.9 ($**$)**.** Let $\mathscr{M}$ the rotation surface around the z−axis of the curve $z = 4 - \cosh(y - 1)$, $y \in [1, 2]$. Calculate its area. Let now

$$\vec{F}(x, y, z) := \frac{(x, y, z)}{\|(x, y, z)\|^3} = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}, \ \ (x, y, z) \in \mathbb{R}^3 \backslash \{0_3\}.$$

Compute the flux of $\vec{F}$ through $\mathscr{M}$ (hint: introduce an auxiliary disk $\left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 = r^2, \ z = a\right\}$ and apply suitably the divergence theorem. . .).

Exercise 6.7.10 ($**$)**.** Let $\mathscr{M}$ the rotation surface around the z−axis of the curve $y = z^2$ $z \in [0, h]$ $(h > 0)$. Compute the area of $\mathscr{M}$ and the flux of $\vec{F}(x, y, z) := (xz, yz, -z)$ through $\mathscr{M}$.

Exercise 6.7.11 ($**$)**.** Let $\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \ : \ z^2 + 6 \leqslant x^2 + y^2 \leqslant 5z\right\}$. Is Ω is a rotation solid? Draw Ω, compute its volume and the outward flux of $\vec{F}(x, y, z) := (x^2, y, z^2)$ by Ω.

Exercise 6.7.12 ($**$)**.** Let $D := \{(x, y) \in \mathbb{R}^2 : x \geqslant 0, \ \sqrt{1 + x^2} \leqslant y \leqslant 2\}$ and let Ω the solid obtained by rotating D around the y−axis. Compute the area of $\partial\Omega$ and the outward flux by Ω of $\vec{F}(x, y, z) := (x^2, y, z^2)$.

Exercise 6.7.13 ($**$)**.** Let $\Omega := \{(x, y, z) \in [0, +\infty[^3 : \ 1 \leqslant x^2 + y^2 \leqslant 4 - z^2\}$. Compute the outward flux by Ω of $\vec{F}(x, y, z) := (x, y, z^2)$. Determine in particular the component of this flux on $\partial\Omega \cap \{x^2 + y^2 + z^2 = 4\}$.

Exercise 6.7.14 ($**$)**.** Compute the outward flux by $\Omega = \{(x, y, z) \in \mathbb{R}^3 \ : \ \max\{|x|, |y|, |z|\} \leqslant 1\}$ of the vector field $\vec{F}(x, y, z) = \frac{(x,y,z)}{|(x,y,z)|^3}$.

Exercise 6.7.15 ($**$)**.** Applying the Stokes' theorem, compute the $\oint_\gamma \vec{F}$ in the following cases:

$$1) \ \gamma \equiv \begin{cases} x^2 + y^2 = 1, \\ x + z = 1, \end{cases} \quad \vec{F} = (y - z, z - x, x - y). \qquad 2) \ \gamma \equiv \begin{cases} x + y + z = 1, \\ x^2 + y^2 = 9, \end{cases} \quad \vec{F} = (x^2 z, xy^2, z^2)$$

$$3) \ \gamma \equiv \begin{cases} x^2 + z^2 = 1, \\ y = x^2, \end{cases} \quad \vec{F} = (x^3 + z^3, y^2, x^3 + z^3). \qquad 4) \ \gamma \equiv \begin{cases} x^2 + y^2 = 1, \\ z = xy, \end{cases} \quad \vec{F} = (y, z, x)$$

$$5) \ \gamma \equiv \begin{cases} x^2 + y^2 + z^2 = 1, \\ z = \sqrt{x^2 + y^2}. \end{cases} \quad \vec{F} = (y^2 + z^2, z^2 + x^2, x^2 + y^2).$$

CHAPTER 7

Holomorphic functions

7.1. Introduction

We studied the differentiability for functions $\vec{F}: D \subset \mathbb{R}^d \longrightarrow \mathbb{R}^m$. In particular, we realized that the definition of a derivative is not straightforward:

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{F}(\vec{x}+\vec{h})-\vec{F}(\vec{x})}{\vec{h}}$$

does not make sense because we cannot divide by vectors. In this Chapter we discuss again differentiability on functions $f = f(z) : D \subset \mathbb{C} \longrightarrow \mathbb{C}$. Apparently, $\mathbb{C}$ is "like" $\mathbb{R}^2$. However, the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

now makes sense because the division is well defined in $\mathbb{C}$. $\mathbb{C}-$differentability is incredibly different from $\mathbb{R}$ or $\mathbb{R}^2$ differentiability. This because of a number of non trivial exceptional properties. For instance:

- a $\mathbb{C}-$differentiable function is automatically C^∞, that is it has derivatives of any order (this is completely false with $\mathbb{R}-$differentiability; for example, $f(x) = x|x|$ is differentiable at $x = 0$, but $f'(x) = 2x\,\mathrm{sgn}x = 2|x|$ is not);
- a bounded $\mathbb{C}-$differentiable function is necessarily constant (also this is false in $\mathbb{R}$: take $f(x) = \sin x$);

Most of the important elementary functions (like $\exp, \sin, \cos, \sinh, \cosh$) can be naturally extended to $\mathbb{C}$ and they are $\mathbb{C}-$differentiable functions.

This chapter aims to introduce to these ideas. Since theory is particularly complicate, we will do a number of compromises. As usual, proofs will be limited to the simplest and helpful for the understanding. Definitions will be given with some little restriction that helps in presenting the theory quickly.

Chapter requirements: $\mathbb{R}^2-$differential, one variable calculus (differential and integral); vector fields; Gauss-Green formula.

Learning objectives:

- (basic $*$) basic complex variable functions (exp, sin, cos, etc), power series, $\mathbb{C}-$differentiability.
- (intermediate $**$) Cauchy-Riemann equations, classifying zeros and isolated singularities.
- (advanced $***$) use of $\mathbb{C}-$arc integrals to compute generalized integrals (e.g.: Fourier integrals).

7.2. Elementary functions

In this chapter we deal with functions of complex variable. Simple examples of this type of functions are powers z^n, combination of powers, that is polynomials $c_0 + c_1 z + \cdots + c_n z^n$, fractions of polynomials, that is *rational functions*. Other simple examples are $\mathrm{Re}\, z, \mathrm{Im}\, z, |z|, \bar{z}$. A important class that encompasses polynomials is the class of *power series* or, improperly said, infinite degree polynomials, that is functions of type

$$\sum_{k=0}^{\infty} c_k (z-z_0)^k.$$

Before we can formalize these functions, we need to say few words on *series of complex numbers*

7.2.1. Series in $\mathbb{C}$. As for real series, a series $\sum_{k=0}^{\infty} w_k$ where $(w_k) \subset \mathbb{C}$ is defined as the limit (if any) of finite sums:

$$\sum_{k=0}^{\infty} w_k := \lim_n \sum_{k=0}^{n} w_k \in \mathbb{C}.$$

In this case we say that the series is **convergent**, otherwise we say the series is **not convergent**.

Example 7.2.1: geometric series

The geometric series $\sum_{k=0}^{\infty} z^k$ converges iff $|z| < 1$ and

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \ \forall z \in \mathbb{C}, \ |z| < 1. \tag{7.2.1}$$

Proof. With the same algebraic passages of the real case,

$$s_n := \sum_{k=0}^{n} z^k = \begin{cases} n+1, & z = 1, \\ \frac{1-z^{n+1}}{1-z}, & z \neq 1. \end{cases}$$

From this it is clear that, for $z = 1$, $|s_n| \longrightarrow +\infty$, so $\lim_n s_n$ is not finite. For $z \neq 1$ we have

$$\lim_n s_n = \frac{1}{1-z}(1 - \lim_n z^n).$$

For $|z| < 1$, since $|z^n| = |z|^n \longrightarrow 0$ we have

$$\exists \lim_{n \to +\infty} s_n = \frac{1}{1-z},$$

and this proves, at once, the (7.2.1). For $|z| > 1$ we have $|z^n| = |z|^n \longrightarrow +\infty$, therefore $\lim_n s_n$ does not exists as a finite limit in $\mathbb{C}$. Finally, if $|z| = 1$ but $z \neq 1$, writing z in the trigonometric representation $z = \cos\theta + i \sin\theta$ we have $z^n = \cos(n\theta) + i\sin(n\theta)$. If $z \neq 1$, that is $\theta \neq k2\pi$, $\nexists \lim_n \cos(n\theta), \lim_n \sin(n\theta)$, so the cannot be $\lim_n z^n$. This concludes the proof of (7.2.1).

Apart for exceptional cases, it is impossible in general to compute the partial sums $s_n := \sum_{k=0}^{n} w_k$ in a way useful to determine $\lim_n s_n$. This is already known for real series. In that case, there is a number of convergence tests for constant sign series and generic series. Complex series can be considered as general "sign" series. For this type of series it is not easy to characterize convergence. Nonetheless, there are two important results that can be effective when studying convergence. The first one is the

Proposition 7.2.2: necessary condition of convergence

Let $(w_k) \subset \mathbb{C}$. Then,

$$\sum_k w_k \text{ convergent} \implies w_k \longrightarrow 0.$$

This test is mainly used to disprove convergence: whenever $w_k \not\longrightarrow 0$, we can exclude convergence. A very important sufficient condition for convergence is the following:

Proposition 7.2.3: absolute convergence test

Let $(w_k) \subset \mathbb{C}$. Then
$$\sum_k |w_k| \text{ convergent } \implies \sum_k w_k \text{ convergent}$$

The absolute convergence test is very important because it allows to use positive terms series tests.

Warning 7.2.4

It must be clear that, as for real series, *absolute convergence is a sufficient but not necessary condition to ensure convergence*. This means that there are series for which $\sum_k |w_k|$ is not convergent but $\sum_k w_k$ is convergent. An example of this is the series
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1},$$
which is convergent (by Leibniz's test), but not absolutely convergent because
$$\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{k+1} \right| = \sum_{k=0}^{\infty} \frac{1}{k+1},$$
which is divergent (hence not convergent).

7.2.2. Power series. A **power series** is a series of the form
$$\sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

Here, z_0 is called **center**, $(c_k) \subset \mathbb{C}$ are called **coefficients**. Notice that here we use the agreement that $(z - z_0)^0 \equiv 1$.

We are interested to determine for which $z \in \mathbb{C}$ the series converges. Trivially, *a power series is always convergent at its center*, because $(z_0 - z_0)^k = 0$ for every $k \geqslant 1$, so the infinite sum reduces to c_0. In general, convergence is ruled by a couple of results straight consequence of root and ratio tests.

Proposition 7.2.5

Let $(c_k) \subset \mathbb{C}$ and set
$$R := \frac{1}{\lim_k |c_k|^{1/k}} \in [0, +\infty] \quad \left(\frac{1}{0+} := +\infty, \; \frac{1}{+\infty} := 0 \right)$$
Then,

- if $|z - z_0| < R$, the series is convergent at z,
- if $|z - z_0| > R$, the series is not convergent at z.

In other words, the series converges in the open disk $B(z_0, R[$, while it is not convergent outside of the closed disk $B(0, R]$, that is on $B(0, R]^c$. The number R is called **radius of convergence**.

Proof. We consider the absolute convergence of the power series, that is
$$\sum_{k=0}^{\infty} |c_k (z - z_0)^k| = \sum_{k=0}^{\infty} |c_k||z - z_0|^k =: \sum_{k=0}^{\infty} a_k.$$
Let $\ell := \lim_k a_k^{1/k}$. According to the root test we have

- if $\ell < 1$, the series $\sum_k a_k$ converges, so $\sum_k c_k(z-z_0)^k$ is absolutely convergent, hence convergent by absolute convergence test.
- if $\ell > 1$, the series $\sum_k a_k$ is divergent and $a_k \longrightarrow +\infty$. By this last, $|c_k(z-z_0)^k| \longrightarrow +\infty$, so $c_k(z-z_0)^k \not\longrightarrow 0$, from which the series $\sum_k c_k(z-z_0)^k$ cannot be convergent.

Now, notice that

$$\ell = \lim_k a_k^{1/k} = \lim_k |c_k|^{1/k}|z-z_0| < 1, \iff |z-z_0| < \frac{1}{\lim_k |c_k|^{1/k}}.$$

From this, the conclusion follows.

Proposition 7.2.6

Let $(c_k) \subset \mathbb{C}$ and set

$$R := \lim_k \frac{|c_k|}{|c_{k+1}|} \in [0, +\infty]$$

Then,

- if $|z-z_0| < R$, the series is convergent at z,
- if $|z-z_0| > R$, the series is not convergent at z.

Proof. As in the previous proof, we apply the ratio test to the series

$$\sum_{k=0}^{\infty} |c_k||z-z_0|^k =: \sum_k a_k.$$

Let

$$\ell = \lim_k \frac{a_{k+1}}{a_k} = \lim_k \frac{|c_{k+1}||z-z_0|^{k+1}}{|c_k||z-z_0|^k} = |z-z_0| \lim_k \frac{|c_{k+1}|}{|c_k|}.$$

According to the ratio test we have

- if $\ell < 1$, the series $\sum_k a_k$ converges, so $\sum_k c_k(z-z_0)^k$ is absolutely convergent, hence convergent by absolute convergence test.
- if $\ell > 1$, the series $\sum_k a_k$ is divergent and $a_k \longrightarrow +\infty$. By this last, $|c_k(z-z_0)^k| \longrightarrow +\infty$, so $c_k(z-z_0)^k \not\longrightarrow 0$, from which the series $\sum_k c_k(z-z_0)^k$ cannot be convergent.

To finish, we notice that

$$\ell < 1, \iff |z-z_0| < \lim_k \frac{|c_k|}{|c_{k+1}|}.$$

Example 7.2.7: $(*)$ Geometric series

Q. *Determine the radius of convergence of the geometric series* $\sum_{k=0}^{\infty} z^k$.

A. Here $c_k = 1$ for every k, therefore

$$R = \lim_k \frac{|c_k|}{|c_{k+1}|} = 1.$$

7.2.3. Exponential. The first remarkable example of function defined through a power series is the *exponential*. The starting point is the identity

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \; x \in \mathbb{R}.$$

We use the r.h.s. series to define the exponential for a complex number:

Proposition 7.2.8

Let
$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$
Then, exp is well defined for every $z \in \mathbb{C}$ and it fulfills the following properties:

i) $\exp(x) = e^x$ for every $x \in \mathbb{R}$.
ii) in particular, $\exp(0) = 1$.
iii) $\exp(z+w) = \exp(z)\exp(w)$, $\forall z, w \in \mathbb{C}$.
iv) $\overline{\exp(z)} = \exp(\bar{z})$, $\forall z \in \mathbb{C}$.
v) $\exp(z) \neq 0$, $\forall z \in \mathbb{C}$.

Proof. i) Let $c_n = \frac{1}{n!}$, then
$$R = \lim_n \frac{|c_n|}{|c_{n+1}|} = \lim_n \frac{(n+1)!}{n!} = \lim_n n = +\infty.$$
In particular, the series converges for every $z \in \mathbb{C}$.
i) and ii) evident.
iii) We have
$$\exp(z+w) = \sum_n \frac{(z+w)^n}{n!} = \sum_n \sum_{k=0}^{n} \frac{1}{n!} \begin{pmatrix} n \\ k \end{pmatrix} z^{n-k} w^k = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)!k!}.$$
On the other side
$$\exp(z)\exp(w) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{j+m=n} \frac{z^j w^m}{j!m!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k} w^k}{(n-k)!k!}.$$
By this, the group identity follows.
iv) We have
$$\overline{\exp z} = \overline{\sum_{n=0}^{\infty} \frac{z^n}{n!}} = \sum_{n=0}^{\infty} \frac{\overline{z^n}}{n!} = \sum_{n=0}^{\infty} \frac{\bar{z}^n}{n!} = \exp \bar{z}.$$
v) If, for some $z \in \mathbb{C}$, $\exp(z) = 0$, then $0 = \exp(z)\exp(-z) = \exp(z-z) = \exp(0) = 1$, which is impossible.

Since for $x \in \mathbb{R}$ we have $\exp(x) = e^x$, we use the notation
$$e^z := \exp(z), \ z \in \mathbb{C}.$$
Exponential leads naturally to the definition of **hyperbolic functions**,
$$\sinh z := \frac{e^z - e^{-z}}{2}, \ \ \cosh z := \frac{e^z + e^{-z}}{2}.$$
Easily, hyperbolic functions are power series,
$$\sinh z = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \right) = \frac{1}{2}\left(\sum_{n=2k} \frac{z^n - z^n}{n!} + \sum_{n=2k+1} \frac{z^n + z^n}{n!} \right) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!},$$
and, similarly,
$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}.$$

7.2.4. Trigonometric functions. We now extend sin and cos to complex numbers.

Definition 7.2.9

We set
$$\sin_{\mathbb{C}} z := \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos_{\mathbb{C}} z = \frac{e^{iz} + e^{-iz}}{2}, \; z \in \mathbb{C}.$$

We notice that, from the definition,
$$\sin_{\mathbb{C}} z = -i \sinh(iz), \quad \cos_{\mathbb{C}} z = \cosh(iz).$$
Of course, it is not evident that sin and cos coincide with the standard trigonometric function when computed on real numbers. This is a consequence of the following

Proposition 7.2.10

The following properties hold:

i) $\sin_{\mathbb{C}}$ and $\cos_{\mathbb{C}}$ are power series,
$$\sin_{\mathbb{C}} z = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}, \quad \cos_{\mathbb{C}} z = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}, \;\; \forall z \in \mathbb{C}.$$

ii) In particular, $\sin_{\mathbb{C}} x = \sin x$ and $\cos_{\mathbb{C}} x = \cos x$ for every $x \in \mathbb{R}$. For this reason, we just write $\sin_{\mathbb{C}} \equiv \sin$ and $\cos_{\mathbb{C}} \equiv \cos$.

iii) The fundamental identity holds
$$\cos^2 z + \sin^2 z = 1, \; \forall z \in \mathbb{C}.$$

iv) (Euler formula)
$$e^{iz} = \cos z + i \sin z.$$

Proof. i) For example,
$$\cos_{\mathbb{C}} z = \cosh(iz) = \sum_{j=0}^{\infty} \frac{(iz)^{2j}}{(2j)!} = \sum_{j=0}^{\infty} i^{2j} \frac{z^{2j}}{(2j)!} = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}.$$
Same conclusion for $\sin_{\mathbb{C}} z$.
ii) We remind that, for $x \in \mathbb{R}$,
$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = \sin_{\mathbb{C}} x.$$
Same argument for $\cos_{\mathbb{C}} x = \cos x$ for every $x \in \mathbb{R}$.
iii) By the definitions,
$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = \frac{1}{4}\left(e^{2iz} + e^{-2iz} + 2 - (e^{2iz} + e^{-2iz} - 2)\right) = 1.$$
iv) By the definitions,
$$\cos z + i \sin z = \frac{e^{iz} + e^{-iz}}{2} + i \frac{e^{iz} - e^{-iz}}{2i} = \frac{2e^{iz}}{2} = e^{iz}.$$

The Euler formula is very important and justifies the use of
$$\rho e^{i\theta} = \rho(\cos\theta + i \sin\theta),$$
for the trigonometric representation of complex numbers.

7.2.5. Logarithm. $\mathbb{C}-$logatirhm can be defined but, differently from the previous examples, it is a slightly more complicated function. Since log arises as the inverse of exp, we start with the equation

$$e^z = w,$$

where w is given and z is unknown. Since $e^z \neq 0$ for every $z \in \mathbb{C}$, such equation cannot have solutions if $w = 0$. If $w \neq 0$, setting $z = x + iy$,

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y),$$

thus, if $w = \rho e^{i\theta}$, we have

$$e^z = w, \iff e^x e^{iy} = \rho e^{i\theta}, \iff \begin{cases} e^x = \rho, \\ \\ y = \theta + k2\pi, \ k \in \mathbb{Z}. \end{cases} \iff \begin{cases} x = \log \rho, \\ \\ y = \theta + k2\pi, \ k \in \mathbb{Z}. \end{cases}$$

Therefore, for $\rho > 0$ *the equation $e^z = w$ has infinitely many solutions*,

$$\boxed{z_k = \log \rho + i(\theta + k2\pi), \ k \in \mathbb{Z}.}$$

These numbers are called **logarithms of** z.

As we see, for complex number, $\log w$ makes sense for every $w \neq 0$. In particular, also when $w \in \mathbb{R}$, $w < 0$.

Example 7.2.11: $(*)$

Q. *Compute* $\log(-1)$.

A. Since $-1 = 1 \cdot e^{i\pi}$, we have $\log(-1) = \log 1 + i(\pi + k2\pi) = i\pi(2k+1)$, $k \in \mathbb{Z}$.

Example 7.2.12: $(*)$

Q. *Solve the following equations:*

i) $e^z = 1$;
ii) $e^z = i$;
iii) $\sinh z = 0$.
iv) $\cosh z = i$.

A. i),ii) Solutions are

$$z_k = \log|1| + i(\arg 1 + k2\pi) = 0 + ik2\pi = ik2\pi, \ k \in \mathbb{Z},$$

for the first case, while

$$z_k = \log|i| + i(\arg i + k2\pi) = \log 1 + i\left(\frac{\pi}{2} + k2\pi\right) = i\left(\frac{\pi}{2} + k2\pi\right), \ k \in \mathbb{Z}.$$

iii) We write

$$\sinh z = 0, \iff \frac{e^z - e^{-z}}{2} = 0, \iff e^{2z} - 1 = 0, \iff e^{2z} = 1.$$

Solutions are

$$2z_k = \log|1| + ik2\pi, \ k \in \mathbb{Z}, \iff z_k = ik\pi, \ k \in \mathbb{Z}.$$

iv) We have

$$\cosh z = 1, \iff e^z + e^{-z} = 2, \iff e^{2z} + 1 = 2e^z, \iff (e^z - 1)^2 = 0, \iff e^z = 1.$$

Thus

$$z_k = \log|1| + ik2\pi = ik2\pi, \ k \in \mathbb{Z}.$$

Example 7.2.13: $(*)$

Q. *Solve the equation,*

$$\sin z = i.$$

A. Recalling that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ we have

$$\sin z = i, \iff e^{iz} - e^{-iz} = 2i^2 = -2, \iff e^{i2z} - 1 = -2e^{iz}, \iff (e^{iz})^2 + 2e^{iz} + 1 = 2,$$

that is

$$(e^{iz}+1)^2 = 2, \iff e^{iz} + 1 = \pm\sqrt{2}, \iff e^{iz} = -1 \pm \sqrt{2}.$$

Now,

$$e^{iz} = -1 + \sqrt{2}, \iff iz = \log(\sqrt{2}-1) + ik2\pi, \iff z = 2k\pi - i\log(\sqrt{2}-1),\ k \in \mathbb{Z}.$$

Similarly

$$e^{iz} = -1 - \sqrt{2}, \iff iz = \log(1+\sqrt{2}) + i\left(-\pi + k2\pi\right), \iff z = -\pi + k2\pi - i\log(1+\sqrt{2}),\ k \in \mathbb{Z}.$$

As for roots, the multiplicity of solutions of $e^z = w$ does not allow to define a funcion

Definition 7.2.14

The **principal logarithm** is the function

$$\log_{\mathbb{C}} z := \log|z| + i\arg z.$$

Here, $\arg z \in [0, 2\pi[$.

7.3. $\mathbb{C}$−differentiability

In this section we introduce and discuss the concept of $\mathbb{C}$−differentiable function. Differently from the case of functions of several variables, the definition of $\mathbb{C}$−derivative works similarly to the definition of $\mathbb{R}$−derivative.

Definition 7.3.1

Let $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$, D open. We say that f is $\mathbb{C}$−**differentiable at point** $z \in D$ if

$$\exists\, f'(z) := \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \in \mathbb{C}.$$

A function differentiable at every point $z \in D$ with $f' \in \mathsf{C}(D)$ is called **holomorphic on** D. We write $f \in \mathsf{H}(D)$.

Example 7.3.2: $(*)$

Q. *Check that power z^n, $n \in \mathbb{N}$ is holomorphic on $\mathbb{C}$ and the usual relation holds:*

$$(z^n)' = nz^{n-1},\ \forall n \geqslant 1.$$

A. It is the same calculation as for real powers. Indeed, according to Newton binomial formula

$$(z+h)^n = z^n + nz^{n-1}h + \sum_{j=2}^{n} \binom{n}{j} z^{n-j}h^j.$$

Therefore

$$\frac{(z+h)^n - z^n}{h} = nz^{n-1} + \sum_{j=2}^{n} \binom{n}{j} z^{n-j} h^{j-1} \overset{h \longrightarrow 0}{\longrightarrow} nz^{n-1}.$$

Example 7.3.3: (∗∗)

Q. *Check that* Re z*,* Im z*,* $|z|$ *and* $\bar{z}$ *are not differentiable at any point* $z \in \mathbb{C}$*.*

A. Let us discuss the case of Re z, the others are similar and left as useful exercise. Notice that

$$\frac{\mathrm{Re}\,(z+h) - \mathrm{Re}\,z}{h} = \frac{\mathrm{Re}\,h}{h}.$$

Now, let $z = a + ib$ and $h = x + i0$, $x \in \mathbb{R}$. Clearly,

$$h \longrightarrow 0, \iff x \longrightarrow 0.$$

But then,

$$\frac{\mathrm{Re}\,h}{h} = \frac{x}{x+i0} = 1 \overset{h \longrightarrow 0}{\longrightarrow} 1.$$

Take now $h = 0 + iy$. In this case,

$$\frac{\mathrm{Re}\,h}{h} = \frac{0}{iy} \equiv 0 \longrightarrow 0.$$

Thus, limit

$$\lim_{h \to 0} \frac{\mathrm{Re}\,(z+h) - \mathrm{Re}\,z}{h} \text{ does not exist.}$$

$\mathbb{C}-$derivative fulfills the same rules of calculus of the $\mathbb{R}-$derivative, as:

- if $\exists f'(z), g'(z)$ then $\exists (f \pm g)'(z) = f'(z) \pm g'(z)$.
- if $\exists f'(z), g'(z)$ then $\exists (f \cdot g)'(z) = f'(z)g(z) + f(z)g'(z)$.
- $\exists f'(z), g'(z)$ and $g(z) \neq 0$, then $\exists \left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$.
- (chain rule) if $\exists f'(z)$ and $g'(f(z))$ then $\exists (g \circ f)'(z) = g'(f(z))f'(z)$.

From these rules some simple functions turns out to be differentiable:

- every polynomial $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ is holomorphic on $\mathbb{C}$.
- every rational function $f(z) := \frac{p(z)}{q(z)}$, where p, q are polynomials, is holomorphic where defined, that is on $D := \{z \in \mathbb{C} \,:\, q(z) \neq 0\}$.

Power series are important cases of holomorphic functions.

Theorem 7.3.4

Let $f(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n$ be a power series with radius of convergence $R > 0$. Then

$$\exists f'(z) = \sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1}, \; \forall z \in \mathbb{C} \,:\, |z - z_0| < R. \tag{7.3.1}$$

In particular, $f \in \mathsf{H}(B(z_0, R[)$.

Basically, the theorem says that a power series can be differentiated term by term, that is,

$$f'(z) = \left(\sum_{n=0}^{\infty} c_n (z - z_0)^n\right)' = \sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1}.$$

The proof is technical and of no interest here.

Example 7.3.5: $(*)$

In particular, we have:

- $(e^z)' = \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$.
- $(\sinh z)' = \cosh z$, $(\cosh z)' = \sinh z$.
- $(\sin z)' = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = \cos z$. Similarly $(\cos z)' = -\sin z$.

Example 7.3.6: $(*)$

Q. *By differentiating the geometric series, prove that*

$$\frac{1}{(z-1)^2} = \sum_{n=1}^{\infty} nz^{n-1}, \ \forall |z| < 1.$$

A. Recall that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \ |z| < 1.$$

Differentiating side by side,

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \left(\sum_{n=0}^{\infty} z^n\right)' = \sum_{n=1}^{\infty} nz^{n-1}.$$

Since the radius of the geometric series is $R = 1$, the previous formula holds for every $z \ : \ |z| < 1$.

7.4. Cauchy–Riemann conditions

Let $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ and set $u := \operatorname{Re} f$, $v := \operatorname{Im} f$ in such a way $f = u + iv$. Real and imaginary part u, v of f are real valued function of complex variable $z = x + iy$. Since z is determined by (x, y), we will consider u, v directly as functions of (x, y), that is

$$u = u(x, y), \quad v = v(x, y),$$

or two real variables. In this way

$$f(x + iy) = u(x, y) + iv(x, y).$$

Example 7.4.1: $(*)$

For example:

- if $f(z) = z$, then $u + iv = f = x + iy$, so $u(x, y) = x$ and $v(x, y) = y$.
- if $f(z) = z^2$, then
$$u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy, \implies u(x, y) = x^2 - y^2, \ v(x, y) = 2xy.$$
- if $f(z) = e^z$, then
$$u + iv = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos x + i \sin y) = e^x \cos y + ie^x \sin y,$$
from which $u(x, y) = e^x \cos y$, and $v(x, y) = e^x \sin y$.

A natural question arises: *under which conditions on u and v is f $\mathbb{C}$−differentiable?* We already seen that even extremely simple and nice functions are not $\mathbb{C}$−differentiable. For example, $f(z) = \operatorname{Re} z$. Here, since $f(x + iy) = x = u + iv$, we see that $u(x, y) = x$ while $v(x, y) \equiv 0$. In particular, u and v are polynomials.

Nonetheless, $u+iv$ is not $\mathbb{C}-$differentiable. This leads to suspect that something deep happens. This is the content of the following

Theorem 7.4.2: Cauchy–Riemann equations

Let $f = u+iv$. Then, the following properties are equivalent:

i) f is $\mathbb{C}-$differentiable at point $z = x+iy$.

ii) u, v, are $\mathbb{R}^2-$differentiable at point (x, y) and they fulfill the following conditions (called **Cauchy-Riemann equations**):

$$\begin{cases} \partial_x u(x,y) = \partial_y v(x,y) \\ \\ \partial_y u(x,y) = -\partial_x v(x,y). \end{cases} \tag{7.4.1}$$

In this case

$$f'(x+iy) = (\partial_x - i\partial_y)u(x,y) \equiv i(\partial_x - i\partial_y)v(x,y).$$

Proof. Let $z = x+iy$ and $h = \xi + i\eta$. f is differentiable if and only if

$$f(z+h) - f(z) = f'(z)h + o(h).$$

Let $f'(z) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$. Then, writing $f = u+iv$, previous identity is equivalent to

$$(u(x+\xi, y+\eta) - u(x,y)) + i\,(v(x+\xi, y+\eta) - v(x,y)) = (\alpha+i\beta)(\xi+i\eta) + o(\xi+i\eta),$$

that is, separating real and imaginary parts,

$$\begin{cases} u(x+\xi, y+\eta) - u(x,y) = (\alpha\xi - \beta\eta) + o(\xi,\eta) = (\alpha, -\beta)\cdot(\xi,\eta) + o(\xi,\eta), \\ \\ v(x+\xi, y+\eta) - V(x,y) = (\beta\xi + \alpha\eta) + o(\xi,\eta) = (\beta, \alpha)\cdot(\xi,\eta) + o(\xi,\eta). \end{cases}$$

That is, again: f is $\mathbb{C}-$differentiable iff u and v are differentiable and

$$\nabla u = (\alpha, -\beta), \;\; \nabla v = (\beta, \alpha), \;\; \Longleftrightarrow \;\; \begin{cases} \partial_x u = \partial_y v, \\ \\ \partial_y u = -\partial_x v. \end{cases}$$

In this case

$$f'(x+iy) = \alpha + i\beta = \partial_x u - i\partial_y u \equiv \partial_y v + i\partial_x v. \quad \square$$

Example 7.4.3: $(*)$

Q. *Show that $f(z) = \bar{z}$ is never $\mathbb{C}-$differentiable.*

A. Here, $f(z) = \bar{z} = u+iv$, then $u(x,y) = x$, $v(x,y) = -y$. Clearly u, v are $\mathsf{C}^1(\mathbb{R}^2)$, so they are $\mathbb{R}^2-$differentiable. However, since

$$\partial_x u = 1, \;\; \partial_y v = -1,$$

the first of the CR conditions is always false.

Example 7.4.4: $(**)$

Q. *Show that the principal logarithm is holomorphic on $D = \mathbb{C}\backslash\mathbb{R}_+$ (where $\mathbb{R}_+ = \{x+i0 \;:\; x \geqslant 0\}$) and*

$$\log_{\mathbb{C}}'(z) = \frac{1}{z}, \;\forall z \in \mathbb{C}\backslash\mathbb{R}_+.$$

A. Recall that the principal logarithm is defined as

$$\log_{\mathbb{C}}(z) := \log|z| + i\arg(z) =: u + iv,$$

where,

$$u(x,y) = \log\sqrt{x^2+y^2}, \quad v(x,y) = \arg(x+iy) = \begin{cases} \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right), & y \geqslant 0, \\ 2\pi - \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right), & y < 0. \end{cases}$$

Notice that

$$\partial_x u = \frac{x}{x^2+y^2}, \quad \partial_y u = \frac{y}{x^2+y^2}, \quad \Longrightarrow \quad \partial_x u, \partial_y u \in \mathsf{C}(\mathbb{R}^2\backslash\mathbb{R}_+),$$

so u is $\mathbb{R}^2-$differentiable because of the differentiability test. Similarly,

$$\partial_x v = \begin{cases} -\dfrac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\dfrac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = -\dfrac{y^2}{\sqrt{y^2}(x^2+y^2)} = -\dfrac{y}{x^2+y^2}, & y \geqslant 0, \\ \dfrac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\dfrac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \dfrac{y^2}{\sqrt{y^2}(x^2+y^2)} = -\dfrac{y}{x^2+y^2}, & y < 0. \end{cases}$$

From this $\partial_x v \in \mathsf{C}$, and, similarly, $\partial_y v \in \mathsf{C}$. Therefore, according to the differentiability test, also v is differentiable. Furthermore, we immediately have

$$\partial_x v = -\partial_y u$$

and, similarly, $\partial_y v = \partial_x u$. Thus, the CR equations are fulfilled, so $\log_{\mathbb{C}}$ is $\mathbb{C}-$differentiable on $\mathbb{C}\backslash\mathbb{R}_+$ and

$$f'(z) = \left(\partial_x - i\partial_y\right) u = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}. \quad \square$$

Warning 7.4.5

A common mistake is to think that f is $\mathbb{C}-$differentiable iff CR equations hold, forgetting of u, v differentiability. Exercise 7.10.6 shows that this can be false!

Check List 7.4.6: How can I check $\mathbb{C}-$differentiability?

To check $\mathbb{C}-$differentiability of $f = f(z)$ at some point $z = x + iy$ you can either

- apply the definition, computing $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$
- or apply the CR equations. In this case, if $f = u + iv$ ($u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$), you have
  (1) check $\mathbb{R}^2-$differentiability of both u and v at point (x, y) (here you can use whatever you know about $\mathbb{R}^2-$differentiability, differentiability test included);
  (2) check that the CR equations are fulfilled at point (x, y).

7.5. Cauchy Theorem

We may look at a function $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ as a *vector field*. Indeed, introducing real and imaginary parts,

$$f(x+iy) = u(x,y) + iv(x,y),$$

Identifying complex numbers $x+iy$ with points (x,y), we could associate to f a planar vector field $\vec{F}(x,y) := (u(x,y), v(x,y))$. We notice that, if f is $\mathbb{C}-$differentiable on D, then, from Cauchy-Riemann equations,

$$\partial_y u \equiv \partial_x v, \iff \vec{F} \text{ irrotational.}$$

This opens a deep linkage between holomorphic functions and vector fields. We start introducing *line integrals*:

Definition 7.5.1: (line integral)

Let $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a continuous function, $\gamma : [a,b] \longrightarrow \mathbb{C}$ be a regular (namely C^1) path, $\gamma \subset D$. We call **line integral** of f along γ the integral

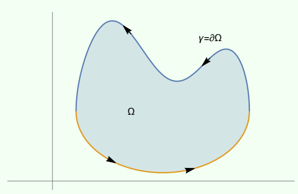
$$\int_\gamma f(z) \equiv \int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

If γ is a circuit ($\gamma(a) = \gamma(b)$), the line integral of f along γ is called **circulation** of f along γ and it is denoted by $\oint_\gamma f$.

The mother of all the consequences of holomorphicity is the following fundamental theorem:

Theorem 7.5.2: Cauchy

Let $f \in \mathsf{H}(D)$, and $\gamma \subset D$ a counter-clock wise oriented circuit such that $\gamma = \partial\Omega$ with $\Omega \subset D$ open.



Then

$$\oint_\gamma f = 0.$$

More in general, the same conclusion holds if $f \in \mathsf{H}(D\backslash\{z_*\}) \cap \mathsf{C}(D)$.

Proof. If $f = u + iv$ and $\gamma = \alpha + i\beta$, then

$$\oint_\gamma f = \int_a^b (u+iv)(\alpha' + i\beta')\, dt = \int_a^b (u\alpha' - v\beta') + i\int_a^b (v\alpha' + u\beta') = \oint_\gamma (u,-v) + i\oint_\gamma (v,u).$$

Now, according to Green's formula,

$$\int_\gamma (u,-v) = \int_\Omega (\partial_x(-v) - \partial_y u) = 0,$$

being, by the CR equations, $\partial_y u = -\partial_x v$. Similarly, $\oint_\gamma (v,u) = 0$, and the conclusion follows.
The unique (apparently irrelevant) point is that it is assumed that f might not be $\mathbb{C}-$differentiable at some point z_* but it must be at least therein continuous.

Let $\gamma = \partial\Omega$ with $\Omega \subset D$. If $z_* \notin \Omega$ then Cauchy's theorem applies and we conclude. If $z_* \in \Omega$, let $\varepsilon > 0$ small enough such that $B(z_*, \varepsilon[\subset \Omega$. Applying Cauchy's theorem on $\Omega \backslash B(z_*, \varepsilon]$ we have

$$0 = \oint_{\partial(\Omega \backslash B(z_*,\varepsilon])} f = \oint_{\partial\Omega} f - \oint_{\partial B(z_*,\varepsilon]} f.$$

Since f is continuous at z_*, $|f(z)| \leqslant |f(z_*)| + 1$ when $z \in B(z_*, \varepsilon]$ provided ε is small enough. Therefore

$$\left|\oint_{\partial\Omega} f\right| = \left|\oint_{\partial B(z_*,\varepsilon]} f\right| \leqslant \int_0^{2\pi} |f(z_* + \varepsilon e^{it})||i\varepsilon e^{it}|\, dt \leqslant (|f(z_*)| + 1)2\pi\varepsilon.$$

Letting $\varepsilon \longrightarrow 0$, we conclude that $\left|\oint_{\partial\Omega} f\right| \leqslant 0$, from which the conclusion follows.

Remark 7.5.3

What if the circulation wind off a singularity of f? In general, but not always, the circulation might be $\neq 0$. For example, let us check that

$$\oint_{\partial B(z_0,r]} \frac{1}{(z - z_0)^m} = \begin{cases} i2\pi, & m = 1, \\ 0, & m \neq 1. \end{cases} \tag{7.5.1}$$

Proof. We parametrize $\partial B(z_0, r]$ with

$$\gamma(t) = z_0 + re^{it} = z_0 + r(\cos t + i \sin t),\ t \in [0, 2\pi].$$

This γ is counter-clock wise oriented. We have

$$\oint_{\partial B(z_0,r]} \frac{1}{(z - z_0)^m} = \int_0^{2\pi} \frac{1}{(re^{it})^m} ire^{it}\, dt = \frac{i}{r^{m-1}} \int_0^{2\pi} e^{i(m-1)t}\, dt.$$

Now, if $m = 1$ we have

$$\int_0^{2\pi} e^{i(m-1)t}\, dt = \int_0^{2\pi} 1\, dt = 2\pi,$$

while, if $m \neq 1$,

$$\int_0^{2\pi} e^{i(m-1)t}\, dt = \left[\frac{e^{i(m-1)t}}{i(m-1)}\right]_{t=0}^{t=2\pi} = 0,$$

and from this the conclusion follows.

7.6. Analiticity of holomorphic functions

In this section we prove a fundamental property of holomorphic functions: *locally — that is, in a suitable neighborhood of each point in its domain — every holomorphic function can be expressed as a power series*. We are not saying that f is *globally* (that is on its domain of definition) a power series, but *locally* (in a neighborhood of each point of D) this is true. This property is called *analiticity*.

For several reasons, this fact is quite surprising. First, because power series are indefinitely differentiable. So, just requiring that f is $\mathbb{C}$−differentiable implies that f has derivatives of any order at each point of its domain! This about: this is entirely false with real functions. If $\exists f'(x)$, there might not be $f''(x)$. An amazing consequence of this is the Liouville theorem: *any holomorphic function on $\mathbb{C}$ which is bounded is automatically constant!* Think about: in $\mathbb{R}$, $\sin x$ is bounded and infinitely many times differentiable, but it is not constant!

7.6.1. Cauchy formula. The proof of analiticity is based on the Cauchy formula that represents any holomorphic function through a suitable line integral. We start by a special case:

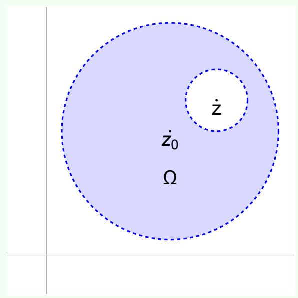
Lemma 7.6.1: logarithmic indicator

Let $z_0 \in \mathbb{C}$ be fixed. Then

$$\oint_{\partial B(z_0,r]} \frac{1}{w-z}\, dw = \begin{cases} i2\pi, & |z-z_0| < r. \\ 0, & |z-z_0| > r. \end{cases} \tag{7.6.1}$$

Proof. Let $f(w) = \frac{1}{w-z}$. If $|z-z_0| > r$, then $f \in \mathsf{H}(B(z_0,r])$ so the integral vanishes by the Cauchy theorem.
Suppose now that $|z-z_0| < r$, and consider a ball centered at z, $B(z,\rho]$ with radius ρ small enough to have $B(z,\rho] \subset B(z_0,r]$ and let

$$\Omega := B(w,r[\backslash B(z,\rho].$$



We have $f \in H(\mathbb{C}\backslash\{z\})$ so, in particular $f \in H(\Omega)$. Notice that Ω has a counter-clock wise oriented boundary made of circles $\partial B(z_0,r]$ and $-\partial B(z,\rho]$. The $-$ here comes from the counter-clock wise orientation respect to Ω. By Cauchy's theorem we have,

$$\oint_{\partial\Omega} \frac{1}{w-z}\, dw = 0.$$

Therefore

$$0 = \oint_{\partial B(z_0,r[} \frac{1}{w-z}\, dw - \oint_{\partial B(z,\rho[} \frac{1}{w-z}\, dw = \oint_{\partial B(z_0,r[} \frac{1}{w-z}\, dw - i2\pi,$$

from which (7.6.1) follows.

Theorem 7.6.2

Let $f \in \mathsf{H}(D)$ and $B(z_0,r[\subset D$. Then

$$f(z) = \frac{1}{i2\pi} \oint_{\partial B(z_0,r]} \frac{f(w)}{w-z}, \ \forall z \in B(z_0,r[. \tag{7.6.2}$$

Proof. Let

$$g(z) := \begin{cases} \frac{f(z)-f(z_0)}{z-z_0}, & z \neq z_0, \\ f'(z_0), & z = z_0. \end{cases}$$

Clearly g is well defined, $\mathbb{C}$–differentiable for $z \in D\backslash\{z_0\}$ and it is also continuous at $z = z_0$, being

$$\lim_{z\to z_0} g(z) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) = g(z_0).$$

Then, Cauchy's theorem applies and

$$\oint_{\partial B(z_0,r]} g = 0.$$

On the other hand,

$$\oint_{\partial B(z_0,r]} g = \oint_{\partial B(z_0,r]} \frac{f(w)-f(z)}{w-z}\,dw = \oint_{\partial B(z_0,r]} \frac{f(w)}{w-z}\,dw - f(z)\underbrace{\oint_{\partial B(z_0,r]} \frac{1}{w-z}\,dw}_{=i2\pi},$$

because of logarithmic indicator formula.

7.6.2. Analiticity. One of the most important consequences of the Cauchy formula is the *analiticity* of an holomorphic function.

Corollary 7.6.3

Let $f \in \mathsf{H}(D)$, $D \subset \mathbb{C}$ open. If $B(z_0, r[\subset D$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \forall z \in B(z_0, r[,$$

where

$$c_n = \frac{1}{i2\pi} \oint_{\partial B(z_0,r]} \frac{f(w)}{(w-z_0)^{n+1}}\,dw. \tag{7.6.3}$$

We say that f is **analytic** on D.

Proof. According to the Cauchy formula (7.6.2),

$$f(z) = \frac{1}{i2\pi} \oint_{\partial B(z_0,r[} \frac{f(w)}{w-z}\,dw.$$

Now, notice that

$$\frac{1}{w-z} = \frac{1}{(w-z_0)+(z_0-z)} = \frac{1}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}},$$

and recall of the geometric sum $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$, for $|q| < 1$. We have

$$\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n,$$

provided

$$\left|\frac{z-z_0}{w-z_0}\right| < 1, \iff |z-z_0| < |w-z_0| = r, \ \forall w \in \partial B(z_0, r].$$

Hence, returning on Cauchy's formula, we have

$$f(z) = \frac{1}{i2\pi} \oint_{\partial B(z_0,r[} \frac{f(w)}{w-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n dw = \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{i2\pi} \oint_{\partial B(z_0,r[} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)}_{=:c_n} (z-z_0)^n,$$

as claimed (the swap between $\sum$ and $\oint$ can be justified formally, we accept here).

As we know, power series are $\mathbb{C}-$differentiable, and since $f'(z) = \sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1}$ is still a power series, also f' is differentiable, so also f'', f''' and, in general there is $f^{(n)}$ for every $n \in \mathbb{N}$. So, we may conclude that the sole existence of a derivative implies the existence of all the derivatives of f. This is completely false for functions of real variable. Actually, the corollary says something more: f is, at least in a neighborhood of z_0, a power series. The next example shows that we can have functions of real variable which have all the derivatives of any order but still they are not power series.

Example 7.6.4: (∗∗∗)

Q. *Show that the function*

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

is C^∞ *but not analytic.*

A. Easily, f is continuous at $x = 0$. Let us consider $f'(x)$. For $x \neq 0$

$$f'(x) = e^{-1/x^2}\left(\frac{2x}{x^4}\right) = \frac{2}{x^3}e^{-1/x^2},$$

so

$$\lim_{x\to 0}|f'(x)| = \lim_{x\to 0}\frac{2}{|x|^3}e^{-1/x^2} \overset{y:=\frac{1}{x^2}}{=} 2\lim_{y\to+\infty}\frac{y^{3/2}}{e^y} = 0.$$

By a well known test, there exists $f'(0) = 0$. In a similar way, one proves that $\exists f^{(n)}(0) = 0$ for every $n \geqslant 2$. If f were analytic, in some neighborhood of $x = 0$ we would have

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} 0 \equiv 0,$$

which is manifestly false.

This example shows that $\mathbb{C}-$differentiability is something unprecedented with $\mathbb{R}-$differentiability. To give a taste of some other exceptional consequence of $\mathbb{C}-$differentiability, we now present another remarkable result:

Theorem 7.6.5: Liouville

A bounded holomorphic function on $\mathbb{C}$ is necessarily constant.

Proof. Assume $|f(z)| \leqslant M$ for all $z \in \mathbb{C}$. We know

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad \text{where } c_n = \frac{1}{i2\pi}\oint_{\partial B(0,r[}\frac{f(w)}{w^{n+1}}\,dw.$$

Then, for $n \geqslant 1$,

$$|c_n| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(re^{it})}{(re^{it})^{n+1}}ire^{it}\right|\,dt = \frac{1}{2\pi}\int_0^{2\pi}\frac{M}{r^{n+1}}r\,dt = \frac{M}{r^n}.$$

Since we can take r arbitrarily, letting $r \longrightarrow +\infty$ we see that $|c_n| \leqslant 0$ for every $n \geqslant 1$. Thus $f(z) \equiv c_0$.

7.7. Isolated singularities

An isolated singularity for a holomorphic function is a point where the function is not defined while being well defined and holomorphic all around the point itself. Let's formalize this idea:

Definition 7.7.1

Let $f \in \mathsf{H}(D\backslash\{z_*\})$ (with $z_* \in D$). We say that z_* is an **isolated singularity** if

$$\exists r > 0, : \ f \in \mathsf{H}(B(z_*, r]\backslash\{z_*\}).$$

The set $B_*(z_*, r] := B(z_*, r]\backslash\{z_*\}$ is called **punctured neighborhood of** z_*.

We classify three main types of singularities depending on the behavior of f around the singularity:

Definition 7.7.2

Let z_* be an isolated singularity for f. We say that

- z_* is a **removable singularity**, if f can be extended also at $z = z_*$ as an holomorphic function.
- z_* is a **pole of order** n if $(z - z_*)^n f(z)$ has a removable singularity but $(z - z_*)^{n-1} f(z)$ does not.
- z_* is an **essential singularity**, if z_* is not of the previous two types.

Example 7.7.3: removable singularity

Q. Check that $f(z) := \frac{e^z - 1}{z}$ has a removable isolated singularity at $z_* = 0$.

A. Clearly, $f \in \mathsf{H}(\mathbb{C}\backslash\{0\})$. In particular, $f \in \mathsf{H}(B_*(0, r])$ for any $r > 0$, and this confirms that $z_* = 0$ is an isolated singularity. Since

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad f(z) = \frac{e^z - 1}{z} = \frac{1}{z} \sum_{k=1}^{\infty} \frac{z^k}{k!} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} =: g(z).$$

The power series g has radius of convergence $R = +\infty$, so $g \in \mathsf{H}(\mathbb{C})$. So, f can be extended with g also at z_* being a holomorphic function.

Example 7.7.4: n−th order pole

Q. Let $f(z) = \frac{1}{z^n}$. Check that $z_* = 0$ is a pole of order n for f.

A. Just notice that $z^n f(z) \equiv 1 \in \mathsf{H}(\mathbb{C})$ but $z^{n-1} f(z) = \frac{1}{z}$ is not defined and holomorphic at $z_* = 0$.

Example 7.7.5: essential singularity

Q. Let $f(z) := e^{1/z}$. Then, $z_* = 0$ is an essential singularity.

A. To have a removable singularity, in particular $\lim_{z\to 0} f(z)$ should exists finite. But,

$$\lim_{x\to 0+} f(x) = e^{+\infty} = +\infty, \quad \lim_{x\to 0-} f(x) = e^{-\infty} = 0.$$

To be an n−th order pole, $z^n f(z) = z^n e^{1/z}$ must have a removable singularity. But, again,

$$\lim_{x\to 0+} x^n f(x) = \lim_{x\to 0+} x^n e^{1/x} = \lim_{y\to+\infty} \frac{e^y}{y^n} = +\infty,$$

while

$$\lim_{x\to 0-} x^n f(x) = \lim_{x\to 0-} x^n e^{1/x} = \lim_{y\to -\infty} \frac{e^y}{y^n} = 0.$$

So, again, $z^n f(z)$ cannot have a removable singularity at $z = 0$, this for every $n \in \mathbb{N}$.

Poles are very important standard singularities. They can be characterized in many equivalent ways. These characterizations are useful and they can be used to classify a pole in different cases.

Proposition 7.7.6

Let z_* be an isolated singularity for f. The following properties are equivalent:

i) z_* is a pole of order n for f.

ii) there exists a punctured neighborhood $B_*(z_*, r]$ such that

$$f(z) = \sum_{k=-n}^{\infty} c_k (z - z_*)^k, \ \forall z \in B_*(z_*, r], \tag{7.7.1}$$

with $c_{-n} \neq 0$.

Proof. i) $\Longrightarrow$ ii). Since z_* is a pole of order n, $(z - z_*)^n f(z) = g(z)$ can be extended to be holomorphic also at z_*. In particular $g \in \mathsf{H}(B(z_*, r])$ for some $r > 0$. Being g analytic in $B(z_*, r[$, we have

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_*)^k,$$

so

$$f(z) = \frac{1}{(z - z_*)^n} g(z) = \sum_{k=0}^{\infty} a_k (z - z_*)^{k-n} = \sum_{k=-n}^{\infty} a_{k+n} (z - z_*)^k =: \sum_{k=-n}^{\infty} c_k (z - z_*)^k \ (c_k := a_{n+k}).$$

Let's check that $c_{-n} \neq 0$. If $c_{-n} = a_0 = 0$, then

$$(z - z_*)^{n-1} f(z) = (z - z_*)^{n-1} \sum_{k=-n+1} c_k (z - z_*)^k = \sum_{k=-n+1} c_k (z - z_*)^{k+n-1} = \sum_{j=0}^{\infty} c_{j-n+1} (z - z_*)^j,$$

would have a removable singularity, thus z_* wouldn't be a pole of order n.

ii) $\Longrightarrow$ i) Assume (7.7.1) holds. Then

$$(z - z_*)^n f(z) = \sum_{k=-n}^{\infty} c_k (z - z_*)^{k+n} = \sum_{j=0}^{\infty} c_{j-n} (z - z_*)^j := g(z).$$

Now, since g is a power series convergent in the puctured neighborhood $B_*(z_*, r]$ and trivially also at $z = z_*$, we have that $g \in \mathsf{H}(B(z_*, r])$, so $(z - z_*)^n f(z)$ has a removable singularity. With the same argument

$$(z - z_*)^{n-1} f(z) = \sum_{k=-n}^{\infty} c_k (z - z_*)^{k+n-1} = \sum_{j=-1}^{\infty} c_{j-n+1} (z - z_*)^j = \frac{c_{-n}}{z - z_*} + \sum_{j=0}^{\infty} c_{j-n+1} (z - z_*)^j.$$

If $(z - z_*)^{n-1} f(z)$ had a removable singularity at z_*, then also

$$\frac{c_{-n}}{z - z_*} = (z - z_*)^{n-1} f(z) - \sum_{j=0}^{\infty} c_{j-n+1} (z - z_*)^j$$

should have a removable singularity at z_*. But this is possible if and only if $c_{-n} = 0$, which is not true by ii). Therefore, $(z-z_*)^n f(z)$ has a removable singularity at z_* but $(z-z_*)^{n-1} f(z)$ does not, that is, z_* is a pole of order n.

How can we identify a pole on a concrete function $f = f(z)$? An important and frequent case is when f is the ratio of two holomorphic functions,

$$f(z) = \frac{N(z)}{D(z)},$$

with $N, D \in \mathsf{H}(B(z_*, r])$. To make z_* an isolated sigularity for f, we will assume that $D(z_*) = 0$ and $D(z) \neq 0$ for all $z \in B_*(z_*, r]$. In this case, by analiticity,

$$N(z) = \sum_{k=0}^{\infty} a_k (z-z_*)^k, \quad D(z) = \sum_{k=0}^{\infty} b_k (z-z_*)^k.$$

Now, excluding $N \equiv 0$, there will be an index k_1 such that $a_k = 0$ for $k < k_1$ and $a_{k_1} \neq 0$ and, similarly, an index k_2 such that $b_k = 0$ for $k < k_2$ and $b_{k_2} \neq 0$. Then

$$f(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=k_1}^{\infty} a_k (z-z_*)^k}{\sum_{k=k_2}^{\infty} b_k (z-z_*)^k} = \frac{(z-z_*)^{k_1} \overbrace{\sum_{j=0}^{\infty} a_{k_1+j} (z-z_*)^j}^{=:N_1(z)}}{(z-z_*)^{k_2} \underbrace{\sum_{j=0}^{\infty} b_{k_2+j} (z-z_*)^j}_{=:D_1(z)}} = (z-z_*)^{k_1-k_2} \frac{N_1(z)}{D_1(z)}.$$

We notice that N_1, D_1 are both power series convergent in $B(z_*, r[$ so they are holomorphic and, moreover, $D_1(z_*) = b_{k_2} \neq 0$, so $\frac{N_1}{D_1} \in \mathsf{H}(B(z_*, r[)$. From this it follows that:

- if $k_1 \geqslant k_2$, then f can be extended at z_* as an holomorphic function, that is z_* **is a removable singularity**.
- if $k_1 < k_2$, then setting $n := k_2 - k_1$ we have

  $$(z-z_*)^n f(z) = (z-z_*)^{k_2-k_1} \cdot (z-z_*)^{k_1-k_2} \frac{N_1(z)}{D_1(z)} = \frac{N_1(z)}{D_1(z)},$$

  has a removable singularity while

  $$(z-z_*)^{n-1} f(z) = \frac{1}{z-z_*} \frac{N_1(z)}{D_1(z)}$$

  does not: z_* **is a pole of order** n.

Great, we have a test! But now, **how do we determine k_1 and k_2?** Have we to write the power series of N and D? The answer is no, we don't need to do that! We see the argument on N, the same applies to D. We recall that, being $N \in \mathsf{H}(B(z_*, r])$ we have

$$N(z) = \sum_{k=0}^{\infty} a_k (z-z_*)^k = \sum_{k=0}^{\infty} \frac{N^{(k)}(z_*)}{k!} (z-z_*)^k.$$

Therefore

$$\boxed{k_1 := \min\{k \in \mathbb{N} \,:\, N^{(k)}(z_*) \neq 0\} =: \mathrm{ord}(N, z_*)}$$

is **the first order of derivative of N at z_*** not vanishing. We can now summarize this argument into the following

Check List 7.7.7: how to classify an isolated dingularity

If
$$f(z) = \frac{N(z)}{D(z)},\ N, D \in B(z_*, r],\ D(z_*) = 0,\ D(z) \neq 0,\ \forall z \in B_*(z_*, r],$$
(1) compute $\mathrm{ord}(N, z_*)$, $\mathrm{ord}(D, z_*)$
(2) if $\mathrm{ord}(N, z_*) \geqslant \mathrm{ord}(D, z_*)$, then z_* **is a removable singularity**.
(3) if $\mathrm{ord}(N, z_*) < \mathrm{ord}(D, z_*)$, then z_* **is a pole of order** $n := \mathrm{ord}(D, z_*) - \mathrm{ord}(N, z_*)$**.**

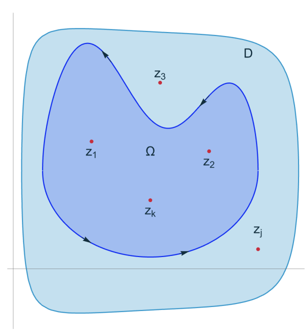
Example 7.7.8: $(**)$

Q. *Determine and classify the singularities of the function*
$$f(z) := \frac{1}{e^{iz} - 1}.$$

A. Clearly, f is holomorphic on its natural domain $D := \{z \in \mathbb{C} \ :\ e^{iz} - 1 \neq 0\}$. Now,
$$e^{iz} = 1, \iff iz = \log 1 + ik2\pi,\ k \in \mathbb{Z}, \iff z_k = k2\pi,\ k \in \mathbb{Z}.$$
So, $f \in \mathsf{H}(\mathbb{C} \backslash \{k2\pi \ :\ k \in \mathbb{Z}\})$. Since $B_*(k2\pi, r] \subset D$ for every $r < \pi$, points z_k are isolated singularities. We notice that $f = \frac{N}{D}$ where $N \equiv 1$ and $D(z) = e^{iz} - 1$. Now, clearly $\mathrm{ord}(N, z_k) = 0$. About the denominator we have $D(z_k) = 0$ and since $D'(z) = ie^{iz} \neq 0$ for every z ($e^w \neq 0$ always), we deduce that $\mathrm{ord}(D, z_k) = 1$. Therefore, z_k are all poles of order 1.

7.8. Laurent Residues theorem

The Cauchy theorem says that if $f \in \mathsf{H}(D)$ and $\gamma \subset D$ is a circuit such that $\gamma = \partial\Omega$, with $\Omega \subset D$, then
$$\oint_\gamma f = 0.$$
The goal of this section is to extend this formula to the case when there are *singular points* of f in Ω, that is points where f is not holomorphic.



Theorem 7.8.1: (Laurent)

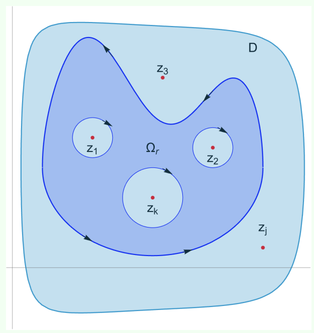
Let $f \in \mathsf{H}(D \backslash \{z_1, \ldots, z_N\})$, where z_j are poles. Let $\gamma \subset D \backslash \{z_1, \ldots, z_N\}$ be a circuit $\gamma = \partial\Omega$ with $\Omega \subset D$ open. Then,
$$\oint_\gamma f = i2\pi \sum_{z_k \in \Omega} \mathrm{res}(f, z_k),$$

where, $\text{res}(f, z_*)$ is called **residue of f at z_*** is given by

$$\boxed{\text{res}(f, z_*) = \lim_{z \to z_*} \frac{\partial_z^{n-1}\left((z - z_*)^n f(z)\right)}{(n-1)!}}. \tag{7.8.1}$$

(here ∂_z^{n-1} stands for the $n - 1$ th $\mathbb{C}$–derivative)

Proof. For each $z_j \in \Omega$ there is a disk $B(z_j, r_j] \subset \Omega$. Since the z_j are finitely many, we can always assume $r_j \equiv r$ and r small enough such that $B(z_k, r] \cap B(z_j, r] = \emptyset$ for $k \neq j$.



Let

$$\Omega_r := \Omega \setminus \bigcup_{z_k \in \Omega} B(z_k, r]$$

According to the Cauchy theorem,

$$\oint_{\partial \Omega_r} f = 0, \iff \oint_{\partial \Omega} f - \sum_{z_k \in \Omega} \oint_{\partial B(z_k, r]} f = 0, \iff \oint_\gamma f = \sum_{z_k \in \Omega} \oint_{\partial B(z_k, r]} f.$$

Let now focus on a single circulation $\oint_{\partial B(z_*, r]} f$ with $z_* \in \Omega$ pole of order n. We know that

$$f(z) = \sum_{k=-n}^{\infty} c_k (z - z_*)^k,$$

so

$$\oint_{\partial B(z_*, r]} f = \oint_{\partial B(z_*, r]} \sum_{k=-n}^{\infty} c_k (z - z_*)^k = \sum_{k=-n}^{\infty} c_k \oint_{\partial B(z_*, r]} (z - z_*)^k.$$

(the switch between the infinite sum and line integral can be justified formally, we accept here). Now, for $k \geqslant 0$, by the Cauchy theorem we have

$$\oint_{\partial B(z_*, r]} (z - z_*)^k = 0.$$

For $k < 0$, by the formula (7.5.1), we have

$$\oint_{\partial B(z_*, r]} (z - z_*)^k = \oint_{\partial B(z_*, r]} \frac{1}{(z - z_*)^{-k}} = \begin{cases} 0, & -k \neq 1, \\ i2\pi, & -k = 1, \iff k = -1. \end{cases}$$

Therefore,

$$\oint_{\partial B(z_*, r]} f = \sum_{k=-n}^{\infty} c_k \oint_{\partial B(z_*, r]} (z - z_*)^k = i2\pi c_{-1}.$$

To determine c_{-1} we notice that since

$$(z-z_*)^n f(z) = \sum_{k=-n}^{\infty} c_k (z-z_*)^{k+n} = \sum_{j=0}^{\infty} c_{j-n}(z-z_*)^j =: g(z).$$

Notice that here c_{-1} is the coefficient of $(z-z_*)^{n-1}$, so

$$c_{-1} = \frac{g^{(n-1)}(z_*)}{(n-1)!} = \lim_{z\to z_*} \frac{g^{(n-1)}(z)}{(n-1)!} = \lim_{z\to z_*} \frac{\partial_z^{n-1}\left((z-z_*)^n f(z)\right)(z)}{(n-1)!} =: \operatorname{res}(f, z_*).$$

From this the conclusion follows.

Remark 7.8.2: I

$f = \frac{N}{D}$ and z_* is a first order pole with $\operatorname{ord}(N, z_*) = 0$ and $\operatorname{ord}(D, z_*) = 1$, the calculation of the residue is particularly easy:

(7.8.2) $$\operatorname{res}(f, z_*) = \lim_{z\to z_*}(z-z_*)f(z) = \lim_{z\to z_*}(z-z_*)\frac{N(z)}{D(z)} = \lim_{z\to z_*}\frac{N(z)}{\frac{D(z)-D(z_*)}{z-z_*}} = \frac{N(z_*)}{D'(z_*)}.$$

7.9. Applications to generalized integrals

Cauchy and Laurent theorems apply to calculus of generalized integrals. In this section we illustrate some methods.

7.9.1. Rational functions. It is well known that, to compute an integral

$$\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\,dx,$$

where p, q are polynomials, there is an algorithm based on the Hermite decomposition of $\frac{p}{q}$ in simple fractions.

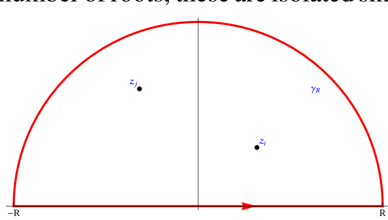
Here, we show an alternative and faster method based on complex integration. We assume $q \neq 0$ on $\mathbb{R}$, so the unique concern about integrability is the behavior of $\frac{p}{q}$ at $\pm\infty$. We know that, if m, n are resp. the degree of p and the degree of q, then

$$\frac{p(x)}{q(x)} \sim_{\pm\infty} \frac{a_m x^m}{b_n x^n} = \frac{C}{x^{n-m}},$$

from which

(7.9.1) $$\exists \int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\,dx \iff n-m>1, \iff n \geqslant m+2.$$

To compute the integral we introduce the function $f(z) := \frac{p(z)}{q(z)}$. Since q is a polynomial, the roots of q are singular points for f. Since there is only a finite number of roots, these are isolated singularities, and $f \in \mathsf{H}(\mathbb{C}\backslash\{z_1, \ldots, z_N\})$.



Now,

$$\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\,dx = \lim_{r\to+\infty}\int_{-r}^{r}\frac{p(x)}{q(x)}\,dx = \lim_{r\to+\infty}\int_{[-r,r]} f.$$

so we can see $\int_{[-r,r]} f$ as part of a line integral of f along a closed counter clock wise oriented circuit γ_r made of the segment $[-r,r]$ and the half circle $\sigma_r(t) = re^{it}$ $t \in [0,\pi]$. For r large enough, all singularities of the upper half plane will be contained in the interior of the circuit γ_r. According to Laurent residues theorem,

$$\oint_{\gamma_r} f = i2\pi \sum_{z_j \,:\, \operatorname{Im} z_j > 0} \operatorname{res}(f, z_j).$$

Notice that the r.h.s. is independent of r , so

$$i2\pi \sum_{z_j \,:\, \operatorname{Im} z_j > 0} \operatorname{res}(f, z_j) = \lim_{r\to+\infty} \oint_{\gamma_r} f = \int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\, dx + \lim_{r\to+\infty} \int_{\sigma_r} f.$$

We notice that

$$\left|\int_{\sigma_r} f\right| \leqslant \int_0^{\pi} |f(re^{it})||ire^{it}|\, dt = r\int_0^{\pi} |f(re^{it})|\, dt.$$

Now,

$$|f(z)| = \frac{|p(z)|}{|q(z)|} \sim \frac{|a_m||z|^m}{|b_n||z|^n} = \frac{c}{|z|^{n-m}} \implies |f(re^{it})| \leqslant \frac{c}{r^{n-m}},$$

from which

$$\left|\int_{\sigma_r} f\right| \leqslant r\int_0^{\pi} \frac{c}{r^{n-m}}\, dt = \frac{2\pi c}{r^{n-m-1}} \longrightarrow 0,\ r \to +\infty,$$

because, by (7.9.1), $n > m+1$. This proves that

$$\boxed{\int_{-\infty}^{+\infty} \frac{p(x)}{q(x)}\, dx = i2\pi \sum_{z_j \,:\, \operatorname{Im} z_j > 0} \operatorname{res}\left(\frac{p}{q}, z_j\right).} \tag{7.9.2}$$

The nice feature of this formula is that it does not involve any integration operation, which is often tricky and never trivial, but only the calculation of a finite number of residues, for which we have a formula.

Example 7.9.1: (∗∗)

Q. *Compute*

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\, dx.$$

A. Let $f(x) := \frac{1}{1+x^4}$. Clearly $f \in \mathscr{C}(\mathbb{R})$ and because $f \sim_{\pm\infty} \frac{1}{x^4}$ it is integrable at $\pm\infty$. Thus, the proposed integral converges. The singularities of f are the zeroes of $q(z) = 1 + z^4$. Now, $q(z) = 0$ iff $z^4 = -1$, that is $z_k = e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}$, $k = 0, 1, 2, 3$.

Let us classify the singularities. Since $p \equiv 1$ clearly $\operatorname{ord}(p, z_k) = 0$. Moreover $q'(z) = 4z^3$, therefore $q'(z_k) \neq 0$ for every k. Thus $\operatorname{ord}(q, z_k) = 1$, so each z_k is a first order pole for f.

Now, by residues theorem (7.9.2)

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\, dx = i2\pi\left(\operatorname{Res}(f, e^{i\frac{\pi}{4}}) + \operatorname{Res}(f, e^{i\frac{3\pi}{4}})\right).$$

To compute the residue at z_k we can apply the reduced formula (7.8.2):

$$\operatorname{res}(f, z_k) = \frac{p(z_k)}{q'(z_k)} = \frac{1}{4z_k^3} = \frac{1}{4}e^{-i\frac{3}{4}\pi}, \frac{1}{4}e^{-i\frac{\pi}{4}},\ k = 0, 1,$$

hence

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^4}\, dx = i2\pi\left(\frac{1}{4}e^{-i\frac{3}{4}\pi} + \frac{1}{4}e^{-i\frac{\pi}{4}}\right) = i\frac{\pi}{2}\left(\frac{-1-i}{\sqrt{2}} + \frac{1-i}{\sqrt{2}}\right) = -i\frac{\pi}{2}\frac{2}{\sqrt{2}}i = \frac{\pi}{\sqrt{2}}. \quad \square$$

Example 7.9.2: (∗∗)

Q. *Compute*

$$\int_0^{+\infty} \frac{x^2+3}{(x^2+1)(x^2+4)}\,dx.$$

A. Let $f(x) := \frac{x^2+3}{(x^2+1)(x^2+4)}$. Clearly $f \in \mathsf{C}(\mathbb{R})$, and $f(x) \sim_{\pm\infty} \frac{x^2}{x^4} = \frac{1}{x^2}$, f is absolutely integrable on $\mathbb{R}$. Moreover, since f is even,

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\,dx.$$

To compute the integral, we apply the residue theorem. The singular points of $f(z) = \frac{z^2+3}{(z^2+1)(z^2+4)}$ are $z = \pm i$ and $z = \pm 2i$. Writing

$$f(z) = \frac{(z-3i)(z+3i)}{(z-i)(z+i)(z-2i)(z+2i)},$$

we see immediately that all the singular points are first order poles. Therefore, by (7.9.2)

$$\int_{-\infty}^{+\infty} f(x)\,dx = i2\pi\left(\mathrm{Res}(f,i) + \mathrm{Res}(f,2i)\right).$$

To compute the residues, we apply formula (7.8.1).

$$\mathrm{Res}(f,i) = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{(z-3i)(z+3i)}{(z+i)(z-2i)(z+2i)} = \frac{(-2i)(4i)}{(2i)(-i)(3i)} = -\frac{4}{3}i,$$

and

$$\mathrm{Res}(f,2i) = \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{(z-3i)(z+3i)}{(z-i)(z+i)(z+2i)} = \frac{(-i)(5i)}{(i)(3i)(4i)} = \frac{5}{12}i.$$

Therefore

$$\int_{-\infty}^{+\infty} f(x)\,dx = i2\pi\left(-\frac{4}{3}i + \frac{5}{12}i\right) = \frac{11}{6}\pi. \quad \square$$

The next example is not an application of formula (7.9.2), nonetheless it involves the same order of ideas.

Example 7.9.3: (∗∗∗)

Q. *Compute*

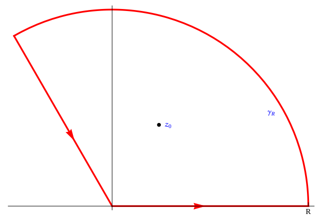
$$\int_0^{+\infty} \frac{1}{1+x^3}\,dx$$

A. Clearly the integral exists finite. Let $f(z) = \frac{1}{1+z^3}$, $f \in \mathsf{H}(\mathbb{C}\backslash\{e^{i\frac{\pi}{3}+ik\frac{2\pi}{3}} \;:\; k = 0,1,2\})$. It is easy to check that all the singularities of f are first order poles. So, if

$$\gamma_r := [0,r] + \sigma_r + \left[re^{i\frac{2\pi}{3}}, 0\right], \quad \text{where } \sigma_r(t) = re^{it},\ t \in \left[0, \frac{2\pi}{3}\right].$$

by the residues theorem we have,

$$\oint_{\gamma_r} f = i2\pi\,\mathrm{res}\left(f, e^{i\frac{\pi}{3}}\right) = i2\pi\frac{1}{3e^{i\frac{2\pi}{3}}}.$$



We have $\oint_{\gamma_r} = \int_{[0,r]} + \int_{\sigma_r} + \int_{[re^{i\frac{2\pi}{3}},0]}$. Notice that

$$\int_{\left[re^{i\frac{2\pi}{3}},0\right]} f = -\int_0^r f\left(te^{i\frac{2\pi}{3}}\right) e^{i\frac{2\pi}{3}}\, dt = -e^{i\frac{2\pi}{3}} \int_0^r \frac{1}{1+\left(te^{i\frac{2\pi}{3}}\right)^3}\, dt = -e^{i\frac{2\pi}{3}} \int_0^r \frac{1}{1+t^3}\, dt.$$

Moreover, by the triangular inequality

$$\left|\int_{\sigma_r} f\right| \leqslant \int_0^{\pi/3} |f(re^{it})||ire^{it}|\, dt = r\int_0^{\pi/3} \frac{1}{|r^3e^{i3t}+1|}\, dt.$$

Since $r \longrightarrow \infty$, $|r^3e^{i3t}+1| \geqslant |r^3e^{i3t}| - 1 = r^3 - 1$, so

$$\left|\int_{\sigma_r} f\right| \leqslant \frac{r}{r^3-1}\frac{\pi}{3} \longrightarrow 0.$$

Therefore

$$\left(1 - e^{i\frac{2\pi}{3}}\right)\int_0^{+\infty} \frac{1}{1+t^3}\, dt = i2\pi\frac{1}{3e^{i\frac{2\pi}{3}}},$$

that is

$$\int_0^{+\infty} \frac{1}{1+t^3}\, dt = \frac{i2\pi}{3e^{i\frac{2\pi}{3}}\left(1-e^{i\frac{2\pi}{3}}\right)}\frac{e^{-i\frac{\pi}{3}}}{e^{-i\frac{\pi}{3}}} = -\frac{\pi}{3e^{i\pi}\frac{e^{i\frac{\pi}{3}}-e^{-i\frac{\pi}{3}}}{2i}} = \frac{\frac{\pi}{3}}{\sin\frac{\pi}{3}}.$$

7.9.2. Fourier integrals. A *Fourier integral* is an integral of type

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx,\ \xi \in \mathbb{R}.$$

The integral is convergent if f is absolutely integrable because $|e^{i\xi x}f(x)| = |f(x)|$ for $x \in \mathbb{R}$. We assume that f is actually defined on $\mathbb{C}\backslash S$, where S is a finite set of poles. We can proceed as in the previous case writing

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx = \lim_{r\to+\infty}\int_{-r}^{r} e^{i\xi x} f(x)\, dx = \lim_{r\to+\infty}\int_{[-r,r]} e^{i\xi z} f(z)\, dz.$$

Notice that if $f \in H(\mathbb{C}\backslash S)$ then $e^{i\xi\sharp}f(\sharp) \in H(\mathbb{C}\backslash S)$ and since $e^{i\xi z} \neq 0$, if $z_j \in S$ is a pole for f, then it is also a pole for $e^{i\xi z}f(z)$ of the same order (exercise). Therefore, considering $\gamma_r := [-r,r] + \sigma_r$ where $\sigma_r(t) = re^{it}$, $t \in [0,\pi]$ we have, as r is big enough, by the residues theorem

$$\oint_{\gamma_r} e^{i\xi z} f(z)\, dz = i2\pi \sum_{z_j\in S_+} \mathrm{Res}\left(e^{i\xi z}f(z), z_j\right),$$

where $S_+ = S \cap \{\mathrm{Im}\, z > 0\}$. Because $\oint_{\gamma_r} = \int_{[-r,r]} + \int_{\sigma_r}$, the main problem is computing the limit

$$\lim_{r\to+\infty}\int_{\sigma_r} e^{i\xi z} f(z)\, dz.$$

Notice that

$$|e^{i\xi z} f(z)| = |f(z)|e^{-\xi\,\mathrm{Im}\, z}.$$

Hence, if $\xi \geqslant 0$, being $\sigma_r \subset \{\operatorname{Im} z > 0\}$, we have

$$\left|\int_{\sigma_r} e^{i\xi z} f(z)\, dz\right| \leqslant \int_0^{2\pi} |f(re^{it})||ire^{it}|\, dt = \int_0^{2\pi} |re^{it} f(re^{it})|\, dt.$$

Assume now that

$$\boxed{\lim_{z\to\infty} z f(z) = 0.}$$

This assumption is pretty natural, it says that f decays faster than $\frac{1}{z}$ at infinity, a "natural" requirement to have integrability on $]-\infty, +\infty[$. Then, for $|z| \geqslant r$, $|zf(z)| \leqslant \varepsilon$, hence

$$\left|\int_{\sigma_r} e^{i\xi z} f(z)\, dz\right| \leqslant 2\pi\varepsilon,$$

that is

$$\lim_{r\to+\infty} \int_{\sigma_r} e^{i\xi z} f(z)\, dz = 0.$$

Thus, we can conclude that

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx = i2\pi \sum_{z_j \in S_+} \operatorname{Res}\left(e^{i\xi \sharp} f(\sharp), z_j\right),\ \forall \xi \geqslant 0. \tag{7.9.3}$$

If $\xi \leqslant 0$ this argument fails because $e^{-\xi \operatorname{Im} z}$ is unbounded. However, this can be easily fixed by changing γ_r with $\widetilde{\gamma}_r := [-r, r] + (-\widetilde{\sigma}_r)$ where $\widetilde{\sigma}_r(t) = re^{it}$, $t \in [\pi, 2\pi]$. For r is big enough, the residue theorem gives

$$\oint_{\widetilde{\gamma}_r} e^{i\xi z} f(z)\, dz = -i2\pi \sum_{z_j \in S_-} \operatorname{Res}\left(e^{i\xi \sharp} f(\sharp), z_j\right),$$

where $S_- = S \cap \{\operatorname{Im} z < 0\}$ and there's the $-$ because the path is clockwise oriented.

Example 7.9.4: $(**)$

Q. *Compute*

$$\int_{-\infty}^{+\infty} \frac{e^{i\xi x}}{x^2 - 2x + 2}\, dx,\ \xi \in \mathbb{R}.$$

A. It is easy to check that $f(x) := \frac{1}{x^2-2x+2}$ is integrable. Indeed: $f \in \mathsf{C}(\mathbb{R})$ and $|f(x)| \sim_{\pm\infty} \frac{1}{x^2}$. As a function of complex variable, f is defined on $\mathbb{C}\backslash\{z^2 - 2z + 2 = 0\}$, that is on $\mathbb{C}\backslash\{z_1, z_2\}$ where

$$z_{1,2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i2}{2} = 1 \pm i.$$

These are poles of first order (check). Moreover

$$\lim_{z\to\infty} z f(z) = 0,$$

therefore

$$\int_{-\infty}^{+\infty} e^{i\xi x} f(x)\, dx = \begin{cases} i2\pi \operatorname{Res}\left(e^{i\xi\sharp} f(\sharp), 1 + i\right), & \xi \geqslant 0, \\ \\ -i2\pi \operatorname{Res}\left(e^{i\xi\sharp} f(\sharp), 1 - i\right), & \xi < 0. \end{cases}$$

Easily we get

$$\operatorname{Res}\left(e^{i\xi\sharp} f(\sharp), 1 + i\right) = \left.\frac{e^{i\xi z}}{2z - 2}\right|_{z=1+i} = \frac{e^{i\xi - \xi}}{2i},\ \ \operatorname{Res}\left(e^{i\xi\sharp} f(\sharp), 1 - i\right) = -\frac{e^{i\xi + \xi}}{2i},$$

that is
$$\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{x^2-2x+2}\,dx = \pi e^{-|\xi|}(\cos\xi + i\sin\xi).$$

Example 7.9.5: $(***)$

Q. *Compute*
$$\int_0^{+\infty} \frac{\cos(\xi x)}{1+x^4}\,dx,\ \xi\in\mathbb{R}.$$

A. Clearly $g(x)=\frac{\cos(\xi x)}{1+x^4}$ is continuous on $\mathbb{R}$, hence locally integrable on $[0,+\infty[$ and since $|g(x)|\leqslant \frac{1}{1+x^4}\sim_{+\infty}\frac{1}{x^4}$, g is absolutely integrable at $+\infty$. So, g is integrable on $[0,+\infty[$. To compute the integral, notice first that being g even we have
$$\int_0^{+\infty} \frac{\cos(\xi x)}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(\xi x)}{1+x^4}\,dx = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{+\infty} \frac{e^{i\xi x}}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{e^{i\xi x}}{1+x^4}\,dx,$$
being $\int_{-\infty}^{+\infty}\frac{\sin(\xi x)}{1+x^4}\,dx=0$ (integral of an odd function over a symmetric interval). Therefore, if $f(z)=\frac{1}{1+z^4}$, $f\in \mathsf{H}(\mathbb{C}\setminus\{z^4+1=0\})$. We have $z^4+1=0$ iff $z=z_k:=e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)}$, $k=0,1,2,3$. It is also clear that each of these points is a first order pole for f. Finally, notice that $\lim_{z\to\infty_{\mathbb{C}}} zf(z)=0$. Hence (7.9.3) gives immediately
$$\int_{-\infty}^{+\infty} \frac{e^{i\xi x}}{1+x^4}\,dx = i2\pi\left(\mathrm{Res}\left(e^{i\xi\sharp}f, e^{i\frac{\pi}{4}}\right)+\mathrm{Res}\left(e^{i\xi\sharp}f, e^{i\frac{3}{4}\pi}\right)\right),\quad \forall\xi\geqslant 0.$$
Applying the reduced formula $\mathrm{Res}(e^{i\xi\sharp}f,z)=\frac{e^{i\xi z}}{4z^3}$ we have
$$\mathrm{Res}(e^{i\xi\sharp}f,z_0)=\frac{e^{i\xi e^{i\frac{\pi}{4}}}}{4e^{i\frac{3}{4}\pi}}=\frac{1}{4}e^{-\frac{\xi}{\sqrt{2}}}e^{i\left(\frac{\xi}{\sqrt{2}}-\frac{3}{4}\pi\right)},\quad \mathrm{Res}(e^{i\xi\sharp}f,z_1)=\frac{e^{i\xi e^{i\frac{3}{4}\pi}}}{4e^{i\frac{\pi}{4}}}=\frac{1}{4}e^{-\frac{\xi}{\sqrt{2}}}e^{i\left(-\frac{\xi}{\sqrt{2}}-\frac{\pi}{4}\right)},$$
so
$$\int_{-\infty}^{+\infty} \frac{e^{i\xi x}}{1+x^4}\,dx = i\frac{\pi}{2}e^{-\frac{\xi}{\sqrt{2}}}\left(e^{i\left(\frac{\xi}{\sqrt{2}}-\frac{3}{4}\pi\right)}+e^{i\left(-\frac{\xi}{\sqrt{2}}-\frac{\pi}{4}\right)}\right).$$
Taking the real part we finally deduce
$$\int_0^{+\infty} \frac{\cos(\xi x)}{1+x^4}\,dx = \frac{\sqrt{2}\pi}{4}e^{-\frac{\xi}{\sqrt{2}}}\left(\cos\frac{\xi}{\sqrt{2}}+\sin\frac{\xi}{\sqrt{2}}\right),\ \forall\xi\geqslant 0.$$
If $\xi<0$ we can avoid the computation and argue by symmetry. Indeed: the integral is an even function of ξ clearly. We deduce
$$\int_0^{+\infty} \frac{\cos(\xi x)}{1+x^4}\,dx = \frac{\sqrt{2}\pi}{4}e^{-\frac{|\xi|}{\sqrt{2}}}\left(\cos\frac{|\xi|}{\sqrt{2}}+\sin\frac{|\xi|}{\sqrt{2}}\right),\ \forall\xi\in\mathbb{R}.\quad\square$$

7.9.3. Integrals involving exponentials. In this subsection, we apply the method to the problem of computing a generalized integral of a function based on e^z.

Example 7.9.6: $(***)$

Q. *Discuss convergence and compute*
$$\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\,dx.$$

where $\alpha \in \mathbb{R}$.

A. Let $f(x) = \frac{x^\alpha}{1+x^2}$. Clearly $f \in \mathsf{C}(]0,+\infty[)$, therefore f is integrable on every interval $[a,b] \subset]0,+\infty[$. Moreover

$$f(x) \sim_{0+} x^\alpha, \quad f(x) \sim_{+\infty} \frac{x^\alpha}{x^2} = \frac{1}{x^{2-\alpha}},$$

hence, by asymptotic comparison, f is integrable at $0+$ iff $\alpha > -1$, while it is integrable at $+\infty$ iff $2-\alpha > 1$, that is iff $\alpha < 1$. Therefore, the proposed integral converges iff $-1 < \alpha < 1$.
To compute the integral, we change variable setting $x = e^t$. We have

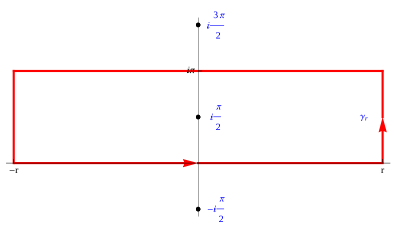
$$\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\,dx = \int_{-\infty}^{+\infty} \frac{e^{\alpha t}}{1+e^{2t}} e^t\,dt = \int_{-\infty}^{+\infty} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt = \lim_{r\to+\infty} \int_{-r}^{r} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt = \lim_{r\to+\infty} \int_{[-r,r]} g,$$

where of course $g(z) := \frac{e^{(\alpha+1)z}}{1+e^{2z}}$, $z \in \mathbb{C}$. Singular points of g are z such that $1+e^{2z} = 0$, that is

$$e^{2z} = -1, \iff 2z = \log 1 + i(\arg(-1) + k2\pi) = i\pi + i2\pi k,\ k \in \mathbb{Z}, \iff z = i\frac{\pi}{2} + i\pi k,\ k \in \mathbb{Z}.$$

To complete $[-r,r]$ obtaining a circuit, we consider the rectangle

$$\gamma_r := [-r,r] + [r, r+i\pi] - [-r+i\pi, r+i\pi] - [-r, -r+i\pi].$$



This path includes only one singular point, $z = i\frac{\pi}{2}$. By the residues theorem then,

$$\oint_{\gamma_r} g = i2\pi \mathrm{Res}\left(g, i\frac{\pi}{2}\right).$$

Let us compute the residue. Being $g = \frac{N}{D}$, $N \neq 0$ e $D'(z) = 2e^{2z} \neq 0$ it follows that $i\frac{\pi}{2}$ is a first order pole and the residue can be computed by the reduced formula

$$\mathrm{Res}\left(g, i\frac{\pi}{2}\right) = \frac{N\left(i\frac{\pi}{2}\right)}{D'\left(i\frac{\pi}{2}\right)} = \frac{e^{i(\alpha+1)\frac{\pi}{2}}}{2e^{i\pi}} = -\frac{1}{2}e^{i(\alpha+1)\frac{\pi}{2}}.$$

On the other side,

$$\oint_{\gamma_r} = \int_{[-r,r]} + \int_{[r,r+i\pi]} - \int_{[-r+i\pi,r+i\pi]} - \int_{[-r,-r+i\pi]}.$$

Notice that

$$\int_{[-r+i\pi,r+i\pi]} g = \int_{-r}^{r} \frac{e^{(\alpha+1)(t+i\pi)}}{1+e^{2(t+i\pi)}}\,dt = e^{i(\alpha+1)\pi} \int_{-r}^{r} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\,dt = e^{i(\alpha+1)\pi} \int_{[-r,r]} g.$$

Now: we claim that the two "vertical" integrals vanish for $r \longrightarrow +\infty$. To show this, notice that, according to the triangular inequality,

$$\left|\int_{[r,r+i\pi]} g\right| = \left|\int_0^\pi \frac{e^{(\alpha+1)(r+it)}}{1+e^{2(r+it)}}\,dt\right| \leqslant \int_0^\pi \left|\frac{e^{(\alpha+1)(r+it)}}{1+e^{2(r+it)}}\right|\,dt = \int_0^\pi \frac{e^{(\alpha+1)r}}{|1+e^{2r}e^{i2t}|}\,dt.$$

Now, since $r > 0$ we have that $|e^{2r}e^{i2t}| = e^{2r} > 1$ therefore, by the triangular inequality $|z+w| \geqslant |z| - |w|$, we get $|1+e^{2r}e^{i2t}|\ e^{2r} - 1 > 0$. Inserting this into the previous estimate we obtain

$$\left| \int_{[r,r+i\pi]} g \right| \leqslant \int_0^\pi \frac{e^{(\alpha+1)r}}{e^{2r}-1}\, dt = \pi \frac{e^{(\alpha+1)r}}{e^{2r}-1} \longrightarrow 0, \quad r \longrightarrow +\infty,$$

being $\alpha + 1 < 2$ (recall that $-1 < \alpha < 1$). Similarly,

$$\left| \int_{[-r,-r+i\pi]} g \right| \leqslant \int_0^\pi \left| \frac{e^{(\alpha+1)(-r+it)}}{1+e^{2(-r+it)}} \right| dt = \int_0^\pi \frac{e^{-(\alpha+1)r}}{|1+e^{-2r}e^{i2t}|}\, dt.$$

Now: if $r > 0$ we have $|e^{-2r}e^{i2t}| = e^{-2r} < 1$ therefore, always by triangular inequality $|z+w| \geqslant |z| - |w|$, we get $|1+e^{-2r}e^{i2t}| \geqslant 1 - e^{2r} > 0$, hence

$$\left| \int_{[-r,-r+i\pi]} g \right| \leqslant \int_0^\pi \frac{e^{-(\alpha+1)r}}{1-e^{-2r}}\, dt = \pi \frac{e^{-(\alpha+1)r}}{1-e^{-2r}} \longrightarrow 0, \quad r \longrightarrow +\infty,$$

being $\alpha + 1 > 0$. Finally,

$$-i\pi e^{i(\alpha+1)\frac{\pi}{2}} = \lim_{r\to+\infty} \left(\int_{-r}^r \frac{e^{(\alpha+1)t}}{1+e^{2t}}\, dt - e^{i(\alpha+1)\pi} \int_{-r}^r \frac{e^{(\alpha+1)t}}{1+e^{2t}}\, dt \right) = \left(1 - e^{i(\alpha+1)\pi}\right) \int_{-\infty}^{+\infty} \frac{e^{(\alpha+1)t}}{1+e^{2t}}\, dt,$$

from which

$$\int_0^{+\infty} \frac{x^\alpha}{1+x^2}\, dx = \frac{1}{2} \frac{e^{i(\alpha+1)\frac{\pi}{2}}}{e^{i(\alpha+1)\pi} - 1} \frac{e^{-i(\alpha+1)\frac{\pi}{2}}}{e^{-i(\alpha+1)\frac{\pi}{2}}} = \frac{\pi}{2} \frac{1}{\sin\left((\alpha+1)\frac{\pi}{2}\right)}. \quad \square$$

Example 7.9.7: $(***)$

Q. *Discuss convergence and compute the*

$$\int_0^{+\infty} \frac{(\log x)^2}{1+x^4}\, dx.$$

A. Clearly the integrand $f \in \mathscr{C}(]0,+\infty[)$, thus f is integrable on every $[a,b] \subset]0,+\infty[$. For the integrability we check the behaviour of f at $x = 0, +\infty$. We have, $f(x) \sim_{0+} (\log x)^2$ which is easy to check that it is integrable. For $x \longrightarrow +\infty$, noticed that $\log x = o(x)$ we have that $f(x) = o\left(\frac{x}{1+x^4}\right) = o\left(\frac{1}{x^3}\right)$, clearly integrable. We conclude that f is integrable in generalized sense on $[0,+\infty[$.
To compute the integral, we start with a change of variable $t = \log x$. We get

$$\int_0^{+\infty} \frac{(\log x)^2}{1+x^4}\, dx = \int_{-\infty}^{+\infty} \frac{t^2}{1+e^{4t}} e^t\, dt = \lim_{r\to+\infty} \int_{[-r,r]} g,$$

where $g(z) := \frac{z^2 e^z}{1+e^{4z}}$. Clearly $g \in H(\mathbb{C}\backslash\{e^{4z} = -1\})$. Now

$$e^{4z} = -1, \iff 4z = i\pi + i2\pi k,\ k \in \mathbb{Z}, \iff z = i\frac{\pi}{4} + ik\frac{\pi}{2},\ k \in \mathbb{Z}.$$

Now e^{4z} does not change if we replace z with $z + i\frac{\pi}{2}$. It is therefore natural to consider the path

$$\gamma_r = [-r,r] + \left[r, r+i\frac{\pi}{2}\right] - \left[-r+i\frac{\pi}{2}, r+i\frac{\pi}{2}\right] - \left[-r, -r+i\frac{\pi}{2}\right].$$

By the residues theorem
$$\oint_{\gamma_r} g = i2\pi \operatorname{Res}\left(g, i\frac{\pi}{4}\right).$$
Clearly $z = i\frac{\pi}{4}$ is a pole of first order ($g = \frac{N}{D}$ with $D'(z) = 4e^{4z} \neq 0$ for any $z \in \mathbb{C}$ and $N(i\frac{\pi}{4}) \neq 0$), therefore,
$$\operatorname{Res}\left(g, i\frac{\pi}{4}\right) = \frac{N\left(i\frac{\pi}{4}\right)}{D'\left(i\frac{\pi}{4}\right)} = \frac{-\frac{\pi^2}{16}e^{i\frac{\pi}{4}}}{4(-1)} = \frac{\pi^2}{64}e^{i\frac{\pi}{4}}.$$
Now,
$$\int_{[-r+i\pi, r+i\pi]} g = \int_{-r}^{r} \left(t + i\frac{\pi}{2}\right)^2 e^{i\frac{\pi}{2}} \frac{e^t}{1+e^{2t}}\, dt = i\left(\oint_{-r}^{r} \frac{t^2e^t}{1+e^{2t}}\, dt + i\pi \int_{-r}^{r} \frac{te^t}{1+e^{2t}}\, dt - \frac{\pi^2}{4}\int_{-r}^{r} \frac{e^t}{1+e^{2t}}\, dt\right).$$
Notice that the second and third integrals are easily computed. The second is the integral of an odd function over a symmetric interval (so it vanishes), because
$$\frac{-te^{-t}}{1+e^{-2t}} = -\frac{te^{-t}}{\frac{e^{2t}+1}{e^{2t}}} = -\frac{te^t}{e^{2t}+1}.$$
About the third integral, letting $r \longrightarrow +\infty$, it converges to
$$\int_{-\infty}^{+\infty} \frac{e^t}{1+e^{2t}}\, dt \overset{x=e^t}{=} \int_0^{+\infty} \frac{x}{1+x^2}\frac{1}{x}\, dx = \int_0^{+\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{4}.$$
Therefore
$$\int_{[-r,r]} - \int_{[-r+i\pi, r+i\pi]} = (1-i)\int_{-r}^{r} \frac{t^2e^t}{1+e^{2t}}\, dt + i\frac{\pi^2}{4}\int_{-r}^{r} \frac{e^t}{1+e^{2t}}\, dt \longrightarrow (1-i)\int_{-\infty}^{+\infty} \frac{t^2e^t}{1+e^{2t}}\, dt + i\frac{\pi^3}{16}.$$
Now, we claim that the two vertical integrals vanish for $r \longrightarrow +\infty$. Indeed:
$$\left|\int_{[r, r+i\frac{\pi}{2}]} g\right| = \left|\int_0^{\pi/2} g(r+it)i\, dt\right| \leqslant \int_0^{\pi/2} \frac{|r+it|^2e^r}{|1+e^{2r}e^{i2t}|}\, dt \leqslant \frac{\pi}{2}\frac{(r+\pi)^2e^r}{e^{2r}-1} \longrightarrow 0, \quad r \longrightarrow +\infty.$$
Similarly,
$$\left|\int_{[-r, -r+i\frac{\pi}{2}]} g\right| = \left|\int_0^{\pi/2} g(-r+it)i\, dt\right| \leqslant \int_0^{\pi/2} \frac{|-r+it|^2e^{-r}}{|1+e^{-2r}e^{i2t}|}\, dt \leqslant \frac{\pi}{2}\frac{(r+\pi)^2e^{-r}}{1-e^{-2r}} \longrightarrow 0, \quad r \longrightarrow +\infty.$$
Therefore
$$\frac{\pi^2}{64}e^{i\frac{\pi}{4}} = \lim_{r\to+\infty} \oint_{\gamma_r} g = (1-i)\int_{-\infty}^{+\infty} \frac{t^2e^t}{1+e^{2t}}\, dt + i\frac{\pi^3}{16}.$$
Noticed that $e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, and taking the real parts in both members we finally obtain,
$$\int_0^{+\infty} \frac{(\log x)^2}{1+x^4}\, dx = \int_{-\infty}^{+\infty} \frac{t^2e^t}{1+e^{2t}}\, dt = \frac{\pi^2}{64}\frac{\sqrt{2}}{2}. \quad \square$$

7.10. Exercises

Exercise 7.10.1 (∗). *For each of the following power series, determine their radius of convergence:*

1. $\sum_{n=1}^{\infty} \frac{z^n}{n}$. 2. $\sum_{n=1}^{\infty} nz^n$. 3. $\sum_{n=0}^{\infty} 2^n z^n$. 4. $\sum_{n=0}^{\infty} \frac{10^n z^n}{n!}$. 5. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$.

6. $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$. 7. $\sum_{n=0}^{\infty} n! z^n$.

Exercise 7.10.2 (∗∗). *Solve the following equations in the unknown* $z \in \mathbb{C}$*:*

$$1.\ \cosh^2 z + 1 = 0. \quad 2.\ \cosh(2z) + 1 = 0. \quad 3.\ e^{iz} = 1. \quad 4.\ \sinh(iz+1) = 0.$$

$$5.\ \cos z = i. \quad 6.\ e^{z^2} = 1. \quad 7.\ e^{iz^2} = i. \quad 8.$$

Exercise 7.10.3 (∗∗). *Let* log *be the principal logarithm. Show, with an example, that* $\log(zw)$ *can be different from* $\log z + \log w$.

Exercise 7.10.4 (∗). *Check, with the definition, that* $\mathrm{Im}\, z$*,* $|z|$ *and* $\bar{z}$ *are not differentiable at every point* $z \in \mathbb{C}$*. Repeat the same check by using the CR conditions.*

Exercise 7.10.5 (∗). *For each of the following* $u = u(x, y)$ *determine* $v = v(x, y)$ *in such a way* $f = u + iv$ *be holomorphic on* $\mathbb{C}$*, determining also* f.

i) $u(x, y) = x$.
ii) $u(x, y) = y$.
iii) $u(x, y) = x^2$.
iv) $u(x, y) = x^2 - y^2$.
v) $u(x, y) = x^2 + y^2$.

Exercise 7.10.6 (∗∗). *Let*

$$f(z) := \begin{cases} \frac{z^3}{|z|^2}, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

Check that i) f *verifies the CR equations at* $z = 0$*, but ii)* f *is not* $\mathbb{C}$*−differentiable at* $z = 0$.

Exercise 7.10.7 (∗∗∗). *Let* $f = f(z) \in H(\mathbb{C})$*. Define*

$$g(z) := \overline{f(\bar{z})}.$$

Use CR conditions to check that $g \in H(\mathbb{C})$*. Can you use the chain rule in this example? Justify your answer.*

Exercise 7.10.8 (∗∗). *For each of the following functions, classify their singularities:*

i) $f(z) := \frac{z^2+2z+5}{(z+2)(z^2+2z+1)}$.
ii) $f(z) = \exp\left(\frac{z}{z+1}\right)$.
iii) $f(z) = \frac{e^z}{z+1}$.
iv) $f(z) = z^2 e^{\frac{1}{z-1}}$.
v) $f(z) = \frac{(z-1)^2(z+3)}{1-\sin\left(\frac{\pi z}{2}\right)}$.

Exercise 7.10.9 (∗∗). *Classify the singularity of* $f(z) = (z-2)\sin(\frac{1}{z+2})$ *at* $z = -2$*; i.e., determine whether it is removable, a pole, or an essential singularity.*

Exercise 7.10.10 (∗∗). *Compute the following integrals:*

$$1.\ \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2}\, dx. \quad 2.\ \int_0^{+\infty} \frac{1+x^2}{1+x^4}\, dx. \quad 3.\ \int_0^{+\infty} \frac{x^2}{(x^2+1)(x^4+1)}\, dx. \quad 4.\ \int_0^{+\infty} \frac{\cos x}{x^2+4}\, dx.$$

Exercise 7.10.11 (∗∗). *Compute the following Fourier integrals:*

$$1.\ \int_{-\infty}^{+\infty} \frac{1}{1+x^2} e^{i\xi x}\, dx. \quad 2.\ \int_{-\infty}^{+\infty} \frac{1}{1+x^4} e^{i\xi x}\, dx. \quad 3.\ \int_{-\infty}^{+\infty} \frac{1}{\cosh x} e^{i\xi x}\, dx.$$

Exercise 7.10.12 (∗∗). *Use contour integration to verify that for* $b > 0$,

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+b^2}\, dx = \pi \frac{e^{-b}}{b},\ \forall b > 0.$$

Exercise 7.10.13 $(**+)$**.** *Compute the following integrals:*

$$1.\ \int_0^{+\infty} \frac{x^{1/3}}{x^2+9x+8}\,dx. \quad 2.\ \int_0^{+\infty} \frac{\log x}{1+x^2}\,dx.$$

Exercise 7.10.14 $(**+)$**.** *Determine for which values of $a \in \mathbb{R}$ the integral*

$$I(a) := \int_{-\infty}^{+\infty} \frac{e^{ax}}{e^x+1}\,dx$$

exists, hence compute it by using complex integration.

Exercise 7.10.15 $(**+)$**.** *Determine for which values of $\alpha \in \mathbb{R}$ the integral*

$$I(\alpha) := \int_0^{+\infty} \frac{1}{(1+x)x^{\alpha}}\,dx,$$

converges, hence compute it by using comples integration.

Exercise 7.10.16 $(***)$**.** *Let $f \in \mathsf{H}(\mathbb{C})$ be such that*

$$\exists a, b > 0,\ :\ |f(z)| \leqslant a|z|^n + b,\ \forall z \in \mathbb{C}.$$

Show that, necessarily, there exists $c \in \mathbb{C}$ such that $f(z) = cz^n$. (hint: use Liouville's theorem).

Exercise 7.10.17 $(***)$**.** *Let $f \in \mathsf{H}(\mathbb{C})$. Show that there exists $g \in \mathsf{H}(\mathbb{C})$ such that $g' = f$ on $\mathbb{C}$.*