Monday, 29 September 2025 10:09

# Abstract Measures

What is an abs. meas?

It is an ext. of area / volume:

$X$ generic set $(\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^m)$

$$\underset{\substack{\rotatebox{-90}{$\in$} \\ [0,+\infty]}}{\mu(E)} = \text{meas of } E \subset X$$

$$E \longmapsto \mu(E) \in [0,+\infty]$$

$\mu$ is a funct of sets

$$\mu : \boxed{\mathcal{F}} \subset \underbrace{\mathcal{P}(X)}_{\text{parts of } X} \longrightarrow [0,+\infty]$$

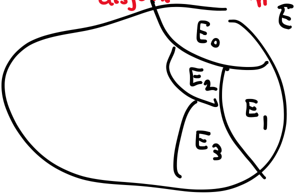
parts of $X$ (class of all the subsets of $X$)

s.t.

1. $\mu(\emptyset) = 0$

2. $\mu(\bigcup E_n) = \sum_n \mu(E_n)$

disjoint $E_n \underset{E}{\cap} E_m = \emptyset \quad n \neq m$



**Def:** (σ-algebra of sets)

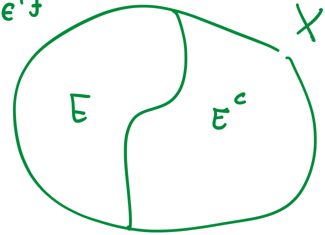
Let $X$ be a set. A class $\mathcal{F} \subset \mathcal{P}(X)$ is called σ-alg if

i) $\emptyset, X \in \mathcal{F}$

ii) if $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$

iii) if $(E_n)_{n \in \mathbb{N}} \in \mathcal{F}$

$\Rightarrow \bigcup_n E_n \in \mathcal{F}$



**Examples** (*) $X$ any set

. $\mathcal{F} = \{\emptyset, X\}$ (trivial σ-alg)

. $\mathcal{F} = \mathcal{P}(X)$

. $\mathcal{F} = \{\emptyset, X, A, A^c\}$

$\emptyset \subsetneq A \subsetneq X$

Ex (**)

$X$ any set

$\mathcal{F} = \{E \subset X : \text{at least one of } E \text{ or } E^c \text{ is countable}\}$

$\updownarrow$

$\mathbb{N}$

$\mathcal{F}$ is a $\sigma$-alg.

Sol: We check the axioms of a $\sigma$-alg:

i) $\phi, X \in \mathcal{F}$ ?

$\phi \in \mathcal{F} \Leftrightarrow \boxed{\phi}$ or $\phi^c = X$ is count

$X \in \mathcal{F} \Leftrightarrow X$ or $X^c = \boxed{\phi}$ is count

ii) $E \in \mathcal{F} \Leftrightarrow E$ or $E^c$ is count (Hp)

$\Updownarrow$

Th: $E^c \in \mathcal{F} \Leftrightarrow E^c$ or $(E^c)^c = E$ is count

iii) $(E_n) \in \mathcal{F} \Leftrightarrow \boxed{\forall n \quad E_n \text{ or } E_n^c \text{ is count}}$

Th: $\bigcup_n E_n \in \mathcal{F} \Leftrightarrow \bigcup E_n$ or $\left(\bigcup_n E_n\right)^c$ is count

$$\left(\bigcup_n E_n\right)^c = \bigcap_n E_n^c$$

$\Leftrightarrow \bigcup E_n$ or $\boxed{\bigcap_n E_n^c}$ is count.

either $E_n$ is count $\forall n \Rightarrow \bigcup_n E_n$ is count

or $\exists N : E_N^c$ is count $\Rightarrow \bigcap_n E_n^c \subset E_N^c$ count

$\Downarrow$

$\bigcap_n E_n^c$ count ■

Examples of families which are not $\sigma$-alg.

- $X = \mathbb{R}$, $\mathcal{F} = \{ I : I \subset \mathbb{R} \text{ interval} \}$

$$I = \overset{(a,b)}{]a,b[}, \; ]a,b], \; [a,b[, \; [a,b],$$

$$[a,+\infty[, \; ]a,+\infty[$$

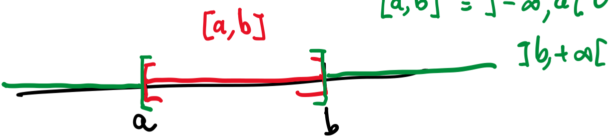
$$]-\infty,b[, \; ]-\infty,b]$$

$$]-\infty,+\infty[ \; \equiv \mathbb{R}.$$

$\mathcal{F}$ is not a $\sigma$-alg.

Sol: i) $\emptyset = ]0,0[$, $X = \mathbb{R} = ]-\infty,+\infty[$.

ii)



$$[a,b]^c = ]-\infty,a[ \, \cup \, ]b,+\infty[$$

- $X = \mathbb{R}$ $\quad \mathcal{F} = \{ \bigcup_n I_n : (I_n)_{n \in \mathbb{N}} \text{ intervals} \}$

Is $\mathcal{F}$ a $\sigma$-alg ? (***)

- $X = \mathbb{R}^m$ $\quad \mathcal{F} = \{ O \subset \mathbb{R}^m : O \text{ is open} \}$

Here $\emptyset, X \in \mathcal{F}$.

But if $O \in \mathcal{F} \Rightarrow O^c$ is open ?

($\Leftrightarrow$ $\mathcal{O} = \emptyset, X$)

$X = \mathbb{R}^1$ $\qquad \mathcal{O} = ]a,b[$

$\mathcal{O}^c = ]-\infty, a] \cup [b, +\infty[$

↑ is not open

· also $\mathcal{F} = \{E : E \text{ is open or } E \text{ closed}\}$ is not a $\sigma$-alg.

**Prop:** Let $X$ be a set, $\mathcal{G} \subset \mathcal{P}(X)$ a family of sub sets of $X$ (not nec. $\sigma$-alg) The $\sigma$-alg gen. by $\mathcal{G}$ (not. $\sigma(\mathcal{G})$) is the smallest $\sigma$-alg containing $\mathcal{G}$

$$\sigma(\mathcal{G}) = \bigcap_{\mathcal{F}\ \sigma\text{-alg},\ \mathcal{F} \supset \mathcal{G}} \mathcal{F}$$

This $\sigma(\mathcal{S})$ is well-defd and it is a $\sigma$-alg that $\supset \mathcal{S}$.

Moreover, if $\tilde{\mathcal{F}}$ is $\sigma$-alg $\supset \mathcal{S} \Rightarrow \tilde{\mathcal{F}} \supset \sigma(\mathcal{S})$

**Proof:** $\sigma(\mathcal{S})$ is well-defd. We have to check that $\bigcap_{\substack{\mathcal{F}\ \sigma\text{-alg} \\ \mathcal{F} \supset \mathcal{S}}} \mathcal{F}$ makes sense

$\Leftrightarrow$ $\exists$ at least one $\mathcal{F}$ $\sigma$-alg s.t. $\mathcal{F} \supset \mathcal{S}$.

$\mathcal{F} = \mathcal{P}(X)$ is a $\sigma$-alg and $\mathcal{S} \subset \mathcal{P}(X) = \mathcal{F}$.

$\sigma(\mathcal{S})$ is a $\sigma$-alg:

i) $\emptyset, X \in \sigma(\mathcal{S})$ : $\forall \mathcal{F} \supset \mathcal{S}$, $\mathcal{F}$ $\sigma$-alg

$$\Downarrow$$

$$\emptyset, X \in \mathcal{F}$$

$$\Downarrow$$

$$\emptyset, X \in \bigcap_{\substack{\mathcal{F}\ \sigma\text{-alg} \\ \mathcal{F} \supset \mathcal{S}}} \mathcal{F} = \sigma(\mathcal{S})$$

ii) $E \in \sigma(\mathcal{S}) \overset{?}{\Longrightarrow} E^c \in \sigma(\mathcal{S})$

$$\Downarrow$$

$E \in \mathcal{F}$ $\forall \mathcal{F}$ $\sigma$-alg $\mathcal{F} \supset \mathcal{S}$

$$\Downarrow$$

$E^c \in \mathcal{F}$ 〃 〃

$\Downarrow$

$$E \in \bigcap_{\substack{\mathcal{F}\ \sigma\text{-alg}\\ \mathcal{F}\supset\mathcal{S}}} \mathcal{F} = \sigma(\mathcal{S})$$

iii) $(E_n) \subset \sigma(\mathcal{S}) \overset{?}{\Rightarrow} \bigcup E_n \in \sigma(\mathcal{S})$

$\Downarrow$

$E_n \in \mathcal{F} \quad \forall \mathcal{F}\ \sigma\text{-alg} \quad \mathcal{F}\supset\mathcal{S}$

$\Downarrow$

⋮

(you finish). and also [redacted] ∎

**Def:** (abstract meas.)

Let $X$ be a set, $\mathcal{F}$ be a $\sigma$-alg of sets of $X$. A function $\mu : \mathcal{F} \longrightarrow [0,+\infty]$ is called **measure** if

i) $\mu(\emptyset) = 0$.

ii) (countable additivity) $\forall (E_n)_n \subset \mathcal{F}$ s.t. $E_n \cap E_m = \emptyset \quad n \neq m \quad \Rightarrow$

$$\mu\Big(\bigcup_n E_n\Big) = \sum_n \mu(E_n)$$

(notation : $\bigsqcup_n E_n := \bigcup_n E_n$ when $E_n \cap E_m = \emptyset$, $n \neq m$.)

$$\mu\Big(\bigsqcup_n E_n\Big) = \sum_n \mu(E_n).$$

Rmk: $\sum_n \mu(E_n)$ is the sum of a series with positive terms, and by def

$$\sum_{n=0}^{\infty} \underset{\substack{\rotatebox{90}{$\leq$} \\ 0}}{\mu(E_n)} := \lim_{N \to +\infty} \sum_{n=0}^{N} \mu(E_n)$$

with the agreement that

$$a + (+\infty) = +\infty.$$

$a \geq 0$ $\quad a \in \mathbb{R}$

Moreover, since $\mu(E_n) \geq 0$ $\forall n$, series

$\sum_n \mu(E_n)$ is always conv or div. ■

Example (*) (Dirac $\delta$ - meas)

$X$ any set, $\mathcal{F}$ $\sigma$- alg, $x \in X$ fixed

$$\begin{tikzcd} \delta_x : \mathcal{F} \arrow[r] \arrow[d] & {[0,+\infty]} \\ {\delta_x(E) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}} \end{tikzcd}$$

$\delta_x$ is a meas.

Sol: Let's check the axioms:

i) $\delta_x(\phi) = 0$

ii) $\delta_x\left(\bigsqcup_n E_n\right) \overset{?}{=} \sum_n \delta_x(E_n)$

$$\begin{aligned} \delta_x\left(\bigsqcup E_n\right) = 0 &\iff x \notin \bigsqcup E_n \\ &\iff x \notin E_n \quad \forall n \\ &\iff \delta_x(E_n) = 0 \quad \forall n \\ &\iff \sum \delta_x(E_n) = 0 \end{aligned}$$

$$\begin{aligned} \delta_x\left(\bigsqcup E_n\right) = 1 &\iff x \in \bigsqcup E_n \\ &\iff \exists! N : x \in E_N \end{aligned}$$

$\underbrace{\exists!}$ ≡ exists unique

$$\begin{aligned} &\iff \delta_x(E_N) = 1, \quad \delta_x(E_n) = 0 \quad \forall n \neq N \\ &\iff \sum \delta_x(E_n) = 1 \end{aligned}$$

∎

Do ex. 1.4.1/2/3/4