PLAN of the COURSE

1) Preliminaries (calculus + measure theory)
   Radon measures (~'20), Hausdorff measures
2) theory of distributions (Schwartz '50)
3) Sobolev space (Sobolev '30)
4) geometric measure theory, BV functions

REF
- EVANS, PDE for preliminaries & Sobolev spaces
- FOLLAND Real Analysis for distributions / Hausdorff
- EVANS - GARIEPY Fine properties ..

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All the information at the MOODLE PAGE !

Feel free to ask questions (directly, by email, or to ask for one office - hour).

PRELIMINARIES. SPACE of TEST FUNCTIONS

$U \subseteq \mathbb{R}^n$ $\quad U$ open set (it could also be $\mathbb{R}^n$)

$V \subset\subset U$ if $\overline{V}$ is compact and $\overline{V} \subset U$.

$\mathcal{C}(U) = \{ f : U \to \mathbb{R} \text{ continuous} \}$

$\mathcal{C}^m(U) = \{ f : U \to \mathbb{R} \quad D^\alpha f \in \mathcal{C}(U) \quad \forall \alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in \mathbb{N}^n$
$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \leq m \}$

$\mathcal{C}^{0,1}(U) = \{ f \quad \exists C_f \quad |f(x) - f(y)| \leq C_f |x-y| \quad \forall x,y \in U \}$ LIPSCHITZ FUNCTIONS

$\mathcal{C}^{0,\alpha}(U) = \{ f \quad \exists C_f \quad |f(x) - f(y)| \leq C_f |x-y|^\alpha \quad \forall x,y \in U \}$
$0 < \alpha < 1$ HÖLDER CONTINUOUS

$\mathcal{C}^{0,1}_{loc}(U) = \{ f \quad \forall K \subset U \text{ compact} \quad \exists C_{K,f}$
$|f(x) - f(y)| \leq C_{K,f} |x-y| \quad x,y \in K)$

Def $f \in C^{0,1}(U)$ $\quad C_f =$ Lipschitz constant of $f := \sup\limits_{\substack{x,y \in U \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|}$

$f \in C^{0,\alpha}(U)$ $\quad C =$ Hölder constant $=$

$$= \sup_{\substack{x \neq y \\ x,y \in U}} \frac{|f(x)-f(y)|}{|x-y|^\alpha} .$$

Obs $f \in C^{0,1}(U)$ $\quad \exists\, \bar{f} \in C^{0,1}(\mathbb{R}^n)$ LIPSCHITZ EXTENSION of f

such that $\bar{f} = f$ on U

$\downarrow$

several Lipschitz extension $\quad$ Let C_f the Lipschitz constant of f

$$\bar{f}_1(x) = \min_{y \in U} \{ f(y) + C_f |y-x| \}$$ is the smallest extension

$$\bar{f}_2(x) = \max_{y \in U} \{ f(y) - C|y-x| \}$$

Also true for Hölder (by subadditivity of concave functions)

ASCOLI-ARZELÀ THEOREM (compactness theorem)

K compact $\quad f_n \in \mathcal{C}(K) \quad (f_n)_n$ sequence

1) $\|f_n\|_\infty = \sup_{x \in K} |f_n(x)| \le C$
2) $\forall \varepsilon \ \exists \delta \quad |x-y| \le \delta \Rightarrow |f_n(x) - f_n(y)| \le \varepsilon$

true $\forall n$ with the same $C>0$ and $\delta = \delta(\varepsilon)$.

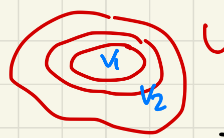
$\exists f_{n_k}$ such that $f_{n_k} \to f$ uniformly $\quad f \in \mathcal{C}(K)$

that is $\|f_{n_k} - f\|_\infty = \sup_{x \in K} |f_{n_k}(x) - f(x)| \to 0$

EXTENSION

Let U be σ-compact that is $\quad U = \bigcup_{k=1}^{\infty} V_k \quad$ • $V_k \subset V_{k+1}$ • $V_k \subset\subset U$

$\left(ex \quad V_k = \{x \in U \quad d(x, \mathbb{R}^n \setminus U) > \frac{1}{k} \quad |x| < k\}\right)$



If $f_n \in \mathcal{C}(U)$ • $\exists C>0 \quad \sup_{x \in K} |f_n(x)| \le C \ \forall n$

• $\forall \varepsilon \ \exists \delta = \delta(\varepsilon) \quad |x-y| \le \delta \Rightarrow |f_n(x) - f_n(y)| \le \varepsilon \qquad \Longrightarrow$

$\Longrightarrow \exists f_{n_k}$ and $f \in \mathcal{C}(U)$ such that

$f_{n_k} \to f$ locally uniformly, that is

$\forall K \subset\subset U$ compact $\|f_{n_k} - f\|_\infty \to 0$.

(by A.A using a diagonalization argument)

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$(\mathcal{C}(K), \underbrace{\|\cdot\|_\infty}_{\text{NORM}})$ is a Banach space (by Ascoli-Arzelà)

$\|f\|_\infty = \sup_{x \in K} |f(x)|$

$(\mathcal{C}(K), \|\cdot\|_\infty) \hookrightarrow (L^\infty(K), \|\cdot\|_\infty)$ measurable functions which are essentially bounded.

closed subspace of

$(\mathcal{C}^m(K), \|\cdot\|_{\infty, m})$ is a Banach space (by Ascoli-Arzelà)

$$\|f\|_{\infty, m} = \sup_{x \in K} \sup_{|\alpha| \leq m} \|D^\alpha f\|_\infty .$$

$\mathcal{C}^\infty(K) = \bigcap_{m \geq 0} \mathcal{C}^m(K)$ define a seminorm $\|f\|_m = \sup_{|\alpha| = m} \|D^\alpha f\|_\infty$

$d(f,g) := \sum_{m=0}^{+\infty} 2^{-m} \frac{\|f-g\|_m}{1+\|f-g\|_m}$

$(\mathcal{C}^\infty(K), d_K)$ is COMPLETE METRIC SPACE (NO NORM! NOT A BANACH SPACE)

FRÉCHET SPACE

$f_n \to f$ in $\mathcal{C}^\infty(K) \iff d_K(f_n, f) \to 0 \iff \|D^\alpha f_n - D^\alpha f\|_\infty \to 0 \quad \forall \alpha \in \mathbb{N}^n$

Obs: BOUNDED SETS ARE PRECOMPACT

(NEVER TRUE for INFINITE DIMENSIONAL BANACH SPACES!)

in a Banach space of INFINITE DIMENSION, CLOSED BALLS ARE NEVER COMPACT.

Let $f_n \in \mathcal{C}^\infty(K)$ s.that $\exists\, C > 0$ $d(f_n, 0) \leq C \implies$ $\exists\, f_{n_k}$ subsequence and $f \in \mathcal{C}^\infty(K)$ such that $d_K(f_{n_k}, f) \to 0$

(CONSEQUENCE of ASCOLI-ARZELÀ THM + DIAGONALIZATION ARGUMENT)

$U \subseteq \mathbb{R}^n$ open $\quad f : U \to \mathbb{R}$

$\operatorname{supp} f$ is a (relatively) closed set in U (support of f)

If f is continuous $\operatorname{supp} f = \overline{\{x \in U \quad f(x) \neq 0\}}$ closure

More generally: f measurable

$\operatorname{supp} f = U \setminus A \qquad A =$ union of open sets $V \subseteq U$ such that $f \equiv 0$ on V.

$\mathcal{C}_c(U) = \{ f \in \mathcal{C}(U)$ with $\operatorname{supp} f$ compact $\}$

$(\mathcal{C}_c(U), \|\cdot\|_\infty)$ is not a Banach space!

$\overline{\mathcal{C}_c(U), \|\cdot\|_\infty}^{\|\cdot\|_\infty} = (\mathcal{C}_0(U), \|\cdot\|_\infty) = \{ f : \overline{U} \to \mathbb{R} \quad f = 0 \text{ on } \partial U \}$

$= \{ f \in \mathcal{C}(U) \mid \forall \varepsilon \ \{x \in U \ |f(x)| > \varepsilon\} \subset\subset U \}$

indeed if $f_n \in C_c(U)$ and $f_n \to f$ uniformly in U (that is $\|f_n - f\|_\infty = \sup_{x \in U} |f_n(x) - f(x)| \to 0$)

then $f \in C(U)$ and $f = 0$ on ∂U but in general f has no compact support!

Ex:

$$f_n(x) = \begin{cases} e^{\frac{n-1}{n|x|^2 - (n-1)}} & \frac{\sqrt{n}|x|}{\sqrt{n-1}} < 1 \\ 0 & \frac{\sqrt{n}|x|}{\sqrt{n-1}} \geq 1 \end{cases}$$

$f_n(x) \in C_c(B(0,1))$

$B(0,1) = \{x \ |x| \leq 1\}$

$f_n \to f = e^{\frac{1}{|x|-1}}$ in $B(0,1)$ uniformly.

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"DUAL OF THE SPACE OF DISTRIBUTIONS"

$$C_c^\infty(U) \; \left(= \mathcal{D}(U)\right) = \{ f \in C^\infty(U) \;\; \text{supp } f \;\; \text{compact} \}$$

$$C^\infty(U) = \bigcap_{m \geq 0} C^m(U)$$

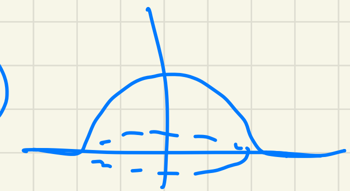
SPACE of TEST FUNCTIONS

Obs

in $C_c^\infty(\mathbb{R}^n)$ there are NO ANALYTIC FUNCTIONS!

ex

$$f(x) = \begin{cases} e^{\frac{1}{|x|^2 - 1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \qquad f \in C_c^\infty(\mathbb{R}^n)$$



$C_c^\infty(U)$ is not a Fréchet space! it is an INDUCTIVE LIMIT of Fréchet spaces

NON METRIZABLE

We put on $C_c^\infty(U)$ a topology τ' such that $C_c^\infty(U)$ is a HAUSDORFF LOCALLY COMPACT TOPOLOGICAL SPACE

$f_n \to f$ in $C_c^\infty(U)$ $\iff$ $\exists K$ COMPACT $\subseteq U$ such that $\text{supp } f_n, \text{supp } f \subseteq K$ AND $\|D^\alpha f_n - D^\alpha f\|_\infty \to 0 \;\; \forall \alpha$