Analitical and Stochastic Mathematical Methods for Engineering (ASMME)

(AY 2025/26 course presentation)

Paolo Guiotto

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- to offer an integrated approach combining analytical and probabilistic methods

Typical PDE $A[u] = f$ Workflow

- *Abstract problem:* find u such that $\boxed{A[u] = f}$ with $f \in Y$ (known) $u \in X$ (unknown).

  Iconic examples:
  - $\Delta u = f$ (Laplace)
  - $\partial_t u = \Delta u + f$ (heat equation)
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- *Limit passage:*

$$\boxed{A[u_n] \to A[u] \;\; \text{in } Y} \;\; \Rightarrow \;\; \boxed{A[u] = f},$$

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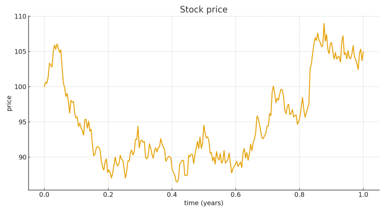
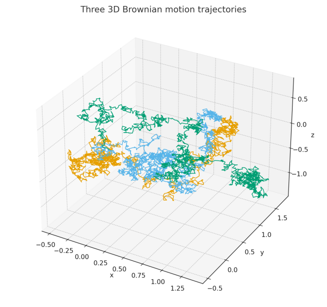
$\Longrightarrow$ **need of measure and integral theory** (Lebesgue 1904)

MOTIVATIONS (PROBABILITY)





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- Events: subsets of Ω (an event is a set of paths)

motivation (probability)

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- Coordinate process: $W_t(\omega) := \omega(t)$, $t \geqslant 0$
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- $W_0 = 0$ (starts in the origin)
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Connections :

$$\begin{cases} u(t,x) = \frac{1}{2}\Delta u(t,x), & \text{(heat equation)} \\ u(0,x) = f(x), \end{cases} \quad \Longrightarrow \quad u(t,x) = \mathbb{E}_{\mathbb{W}}[f(x+B_t)]$$

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In fact, modern Probability arose as a branch of Measure Theory

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- Normed spaces, Banach spaces, Space of functions
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Orthogonal Projection $\Pi_U f$	Conditional Expectation $\mathbb{E}[X \mid \mathscr{F}]$

From **Analysis**:

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- Basic discrete probability

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- operating with Wiener measure and carrying out calculations involving Brownian motion.

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The assessment process is divided into two levels:

- the **mandatory final written exam**
- the **optional in-course assessment**.

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The written exam is based on

- **problem-solving**
- **writing statements** and (if required) **doing proofs** of fundamental results of the theory.

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Along the course, students are given a series of homework assignments. For each homework, students participate as

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At the end of the course, every participant in the optional in-course assessment receives an overall evaluation, which is added as a bonus to the average score of the mandatory final exam.

EXAM RULES

1. **Exam Structure and Scheduling:**
    - **two separate parts** (Analysis and Probability).
    - 4 calls per each part (Jan*, Feb, Jun/Jul, Sept)
    - the two parts are scheduled on different dates
    - each part can be taken in any order and on any of the available dates.
2. **Passing Criteria:**
    - minimum of 16/30 in each part (Analysis and Probability)
    - overall rounded average of at least 18/30

   Participation in at least 75% of the optional in-course assessment activity earns a bonus of +15% on the rounded overall average.
3. **Retakes:** Each part can be retaken (in case, any previous score for that part is forfeited).
4. **Exam Validation:** if eligible, register on the official validation list in Uniweb
5. **Oral Exam:** in exceptional circumstances, where further assessment is deemed necessary, an oral integration exam may be requested.