Università di Padova,
Laurea Magistrale in Scienze Statistiche
Corso: Teorie e modelli demografici

Population projections

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Objectives

At the end of this session, you should be able to:

- Define and distinguish between a population projection and a population forecast
- Apply simple mathematical models of growth to extrapolate total population
- Undertake a straightforward projection of population size by age and sex using the cohort-component method.

References

- Preston: Ch. 6

Population projections – Why to use them

- Government policymakers and planners around the world use population projections to gauge future demand for food, water, energy, and services, and to forecast future demographic characteristics.
- Population projections can alert policymakers to major trends that may affect economic development and help policymakers craft policies that can be adapted for various projection scenarios
- Users of population projections need to understand the reliability and the limitations of projection series. Awareness of how projections are prepared and the possible sources of uncertainty in the numbers can help policymakers more effectively incorporate projections in their planning process.

Population projections – How to use them

- The population of a geographic area grows or declines through the interaction of three factors: fertility, mortality, and migration.
- To project population size at a future date, demographers make assumptions about:
  - levels of fertility and mortality and
  - about how many people will move into or out of an area before that date.

The net population increase or decrease over the period is added to the “baseline” (beginning) population to project future population.

Population projections – How to use them

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- All of the major international agencies that project populations base their projections on current population estimates and assumptions about how fertility, mortality, and migration will change over time.
- Recent projection methodologies have focused on identifying uncertainty in projections — that is, on developing estimates of the probability that the future population size will fall within a certain range.

The specificity of human population

Human populations have two fundamental characteristics that reduce uncertainty about how they will develop in the future:

- First, a substantial overlap exists between the current population and the future population. For example, everyone who will be aged 25 or more years in 25 years time has already been born.
- Second, one fundamental aspect of the human condition is that every year that passes we all get exactly one year older until we eventually die.

Population projections & forecasts– Definitions

“**Population projections** are calculations which show the future development of a population when certain assumptions are made about the future course of fertility, mortality, and migration. They are in general **purely formal calculations**, developing the implications of the assumptions that are made.

A **population forecast** is a projection in which the assumptions are considered predictive to yield a **realistic picture** of the probable future development of a population”.(UnitedNations,1958:45)

DIFFERENT METHODS

TOTAL METHODS

COHORT COMPONENT METHODS

BAYESIAN METHODS

DIFFERENT METHODS

TOTAL METHODS

Total Methods

Total methods calculate **trends in the size of the population** as a whole using a mathematical model of population growth.

They may then distribute this total into sub-groups in ratio to the current structure of the population or an extrapolated forecast of its structure.

Therefore, such approaches are sometimes known instead as *ratio methods* of projection.

Total Methods I

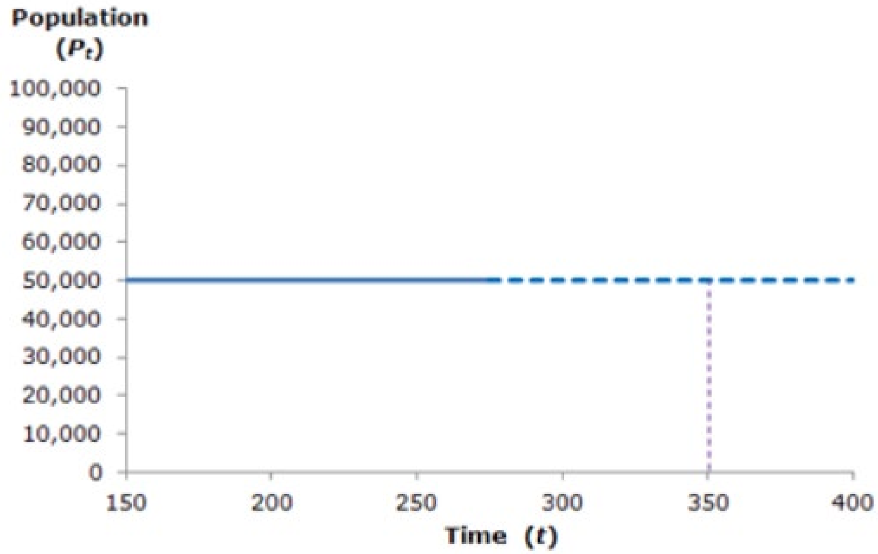
Total methods of projection involve fitting a mathematical model to data on past trends in the size of the population and using the fitted model to extrapolate the population forward (or on occasion backward into the more distant past).

The main steps involved in the procedure are to:

- select an appropriate model of the growth process
- estimate the parameters of the model from past estimates of the population
- extrapolate the fitted curves and read off the projected population.

Zero population growth

The simplest model of population growth is the zero population growth model, which assumes that the size of the population is unchanging. This assumption implies that, even if one only has a single existing estimate of the size of the population, one can project its size at other dates.



Arithmetic growth

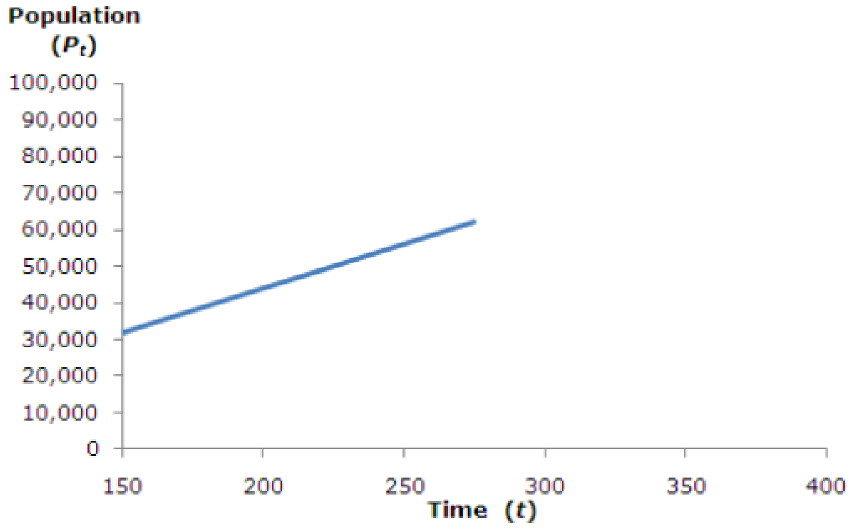
The model assumes that a constant numeric change occurs in the size of the population in every period of the same length.
A minimum of two estimates of the population for different dates are needed to estimate the annual increment in the population and project its size at other dates. The model can be fitted to a longer series of estimates of the population by means of a simple linear regression of population size on time.

If P(t) refers to the population at time t and P(t+n) refers to the population n years later:

$$P(t+n) = P(t) + a \times n$$

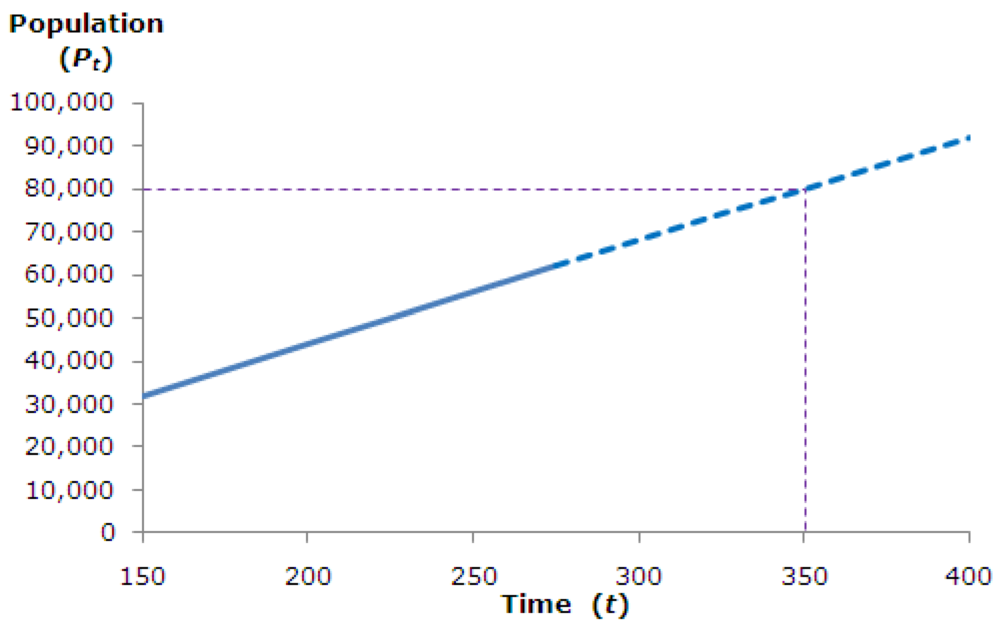
where a is the constant annual increase in the population

$$a = (P(t+n) - P(t)) / n$$



Arithmetic growth - Example

If the population is 62,000 at time 275 and was 50,000 at time 225, what will be the population at time 350?



a = (62,000 − 50,000) ⁄ 50 = **240**

P(350)= 62,000 + 240 × 75 = **80,000**

Exponential growth

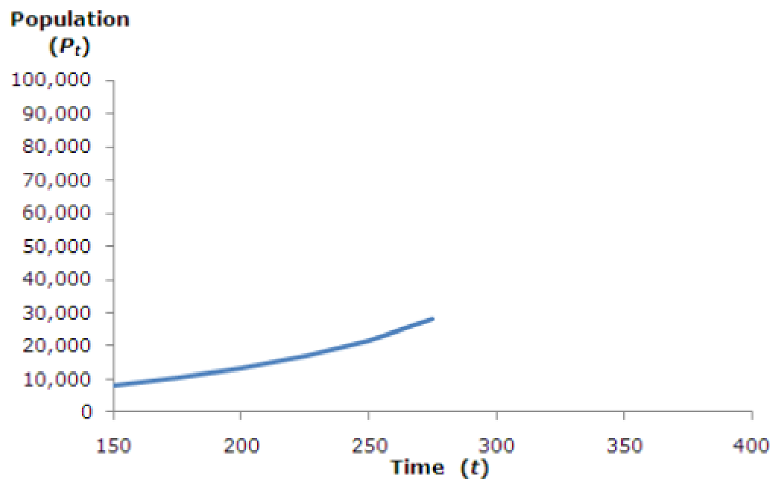
- Instead of assuming that the population is growing by a constant amount, the exponential model assumes that the population is growing at a constant rate.
- In order to project the population forward or backward, one requires an estimate its growth rate. A minimum of two estimates of the population for different dates are needed to calculate this. The model can be fitted to longer time series of estimates of the population by means of a linear regression of the log of population size on time.

In this model:

$$P(t+n) = P(t) \times e^{rn}$$

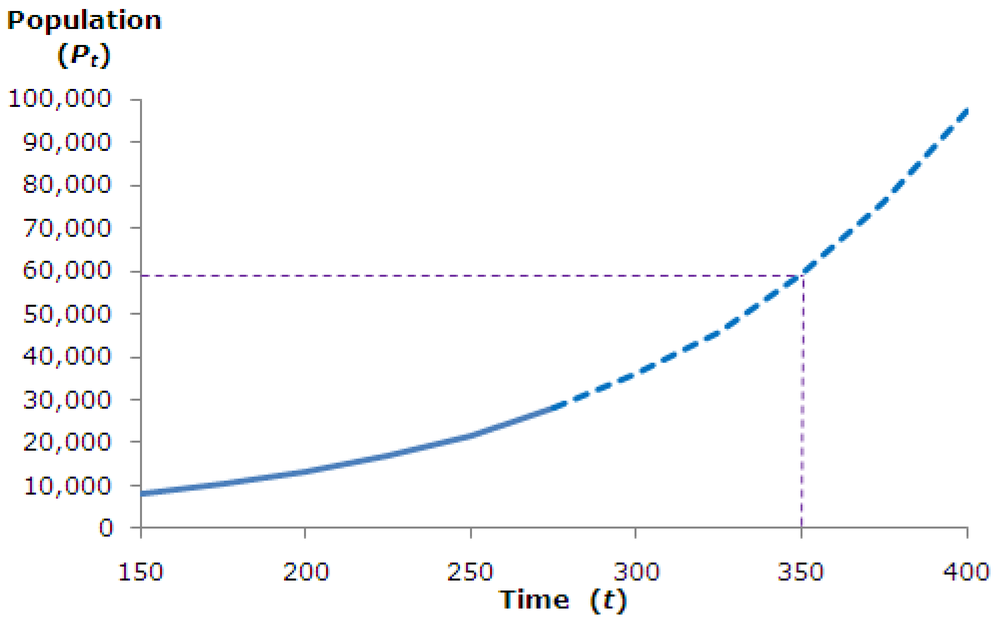
where r is the constant annual growth rate:

$$r = \log_e(P(t+n)/P(t))/n$$



Exponential growth - Example

If the population is 27,923 at time 275 and was 16,936 at time 225, what is *r* and what will be the population at time 350?



Exponential growth - Example

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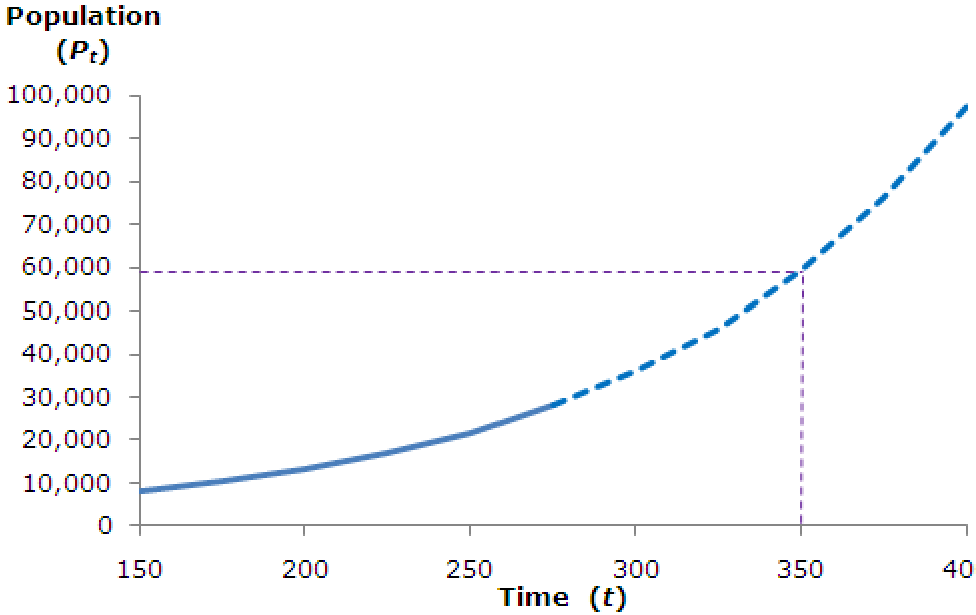


$r = \log_e(27{,}923\,/\,16{,}936)\,/\,(275-225)$
$= 0.5\,/\,50 =$ **0.01 per year**

$P(350) = P(275) \times e^{0.01\times(350-275)}$
$= 27{,}923 \times e^{0.75} =$ **59,113**

Logistic growth

The model assumes that the growth rate slows over time, eventually dropping to zero at which point the population size stabilises.
It assumes that limits exist to population growth.

To fit the model and project the population's size at other dates requires either three or more estimates of the population for different dates or two estimates and a separate assumption about the final size of the population when it ceases growing.

Logistic growth I

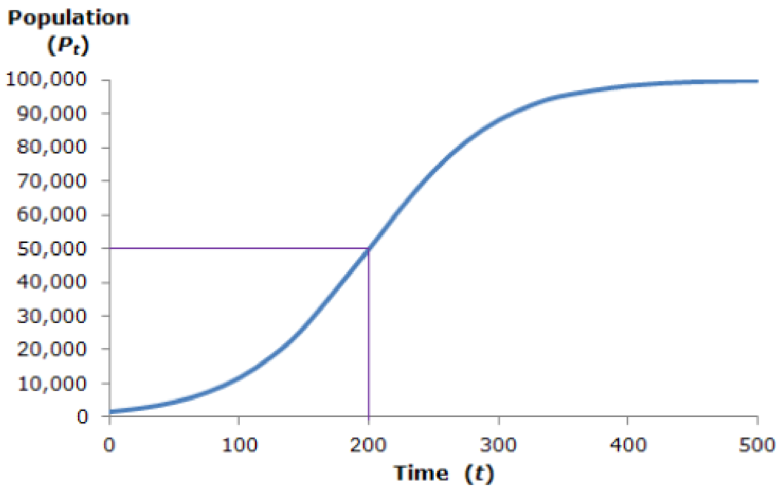
In the logistic growth model:

$$P(t) = P(\infty)/(1 + e^{-s(t-h)})$$

where $P(\infty)$ represents the final size of the population and time is measured relative to point *h*, which is the date at which the population reaches half its final size.

The *s* parameter determines the growth rate, *r*, at each time and therefore how rapidly the population reaches its final size:

$$r = s(1 - P(t)/P(\infty))$$



In a table

t	$P(t)$	r
$-\infty$	0	s
h	$P(\infty)/2$	$s/2$
$+\infty$	$P(\infty)$	0

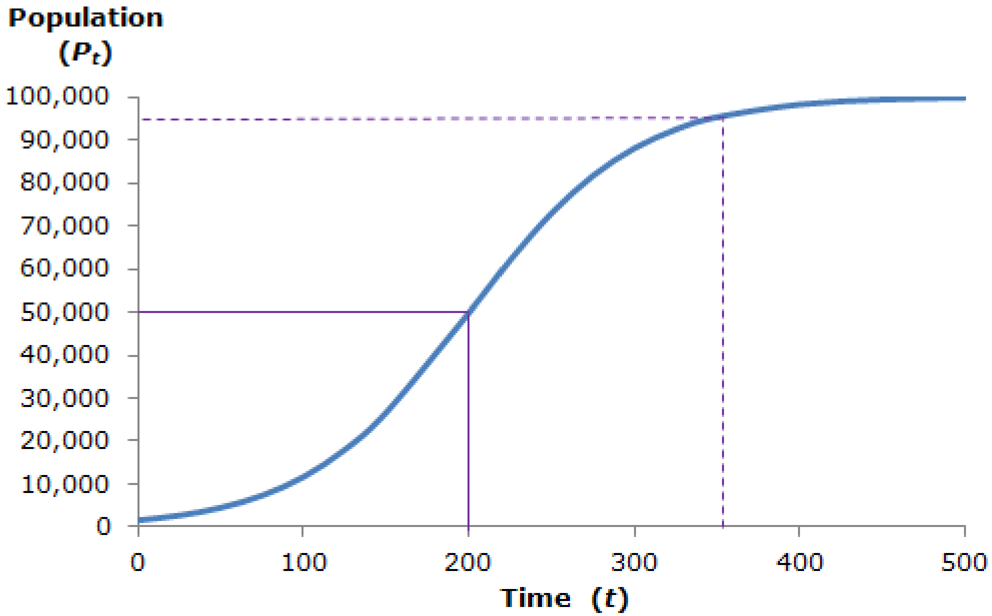
Logistic growth - Example

At time 500 the population is close to achieving its final size of 100,000, having reached half that size at time 200.
If s = 0.02, what is the projected size of the population at time 350?

Logistic growth - Example

At time 500 the population is close to achieving its final size of 100,000, having reached half that size at time 200.

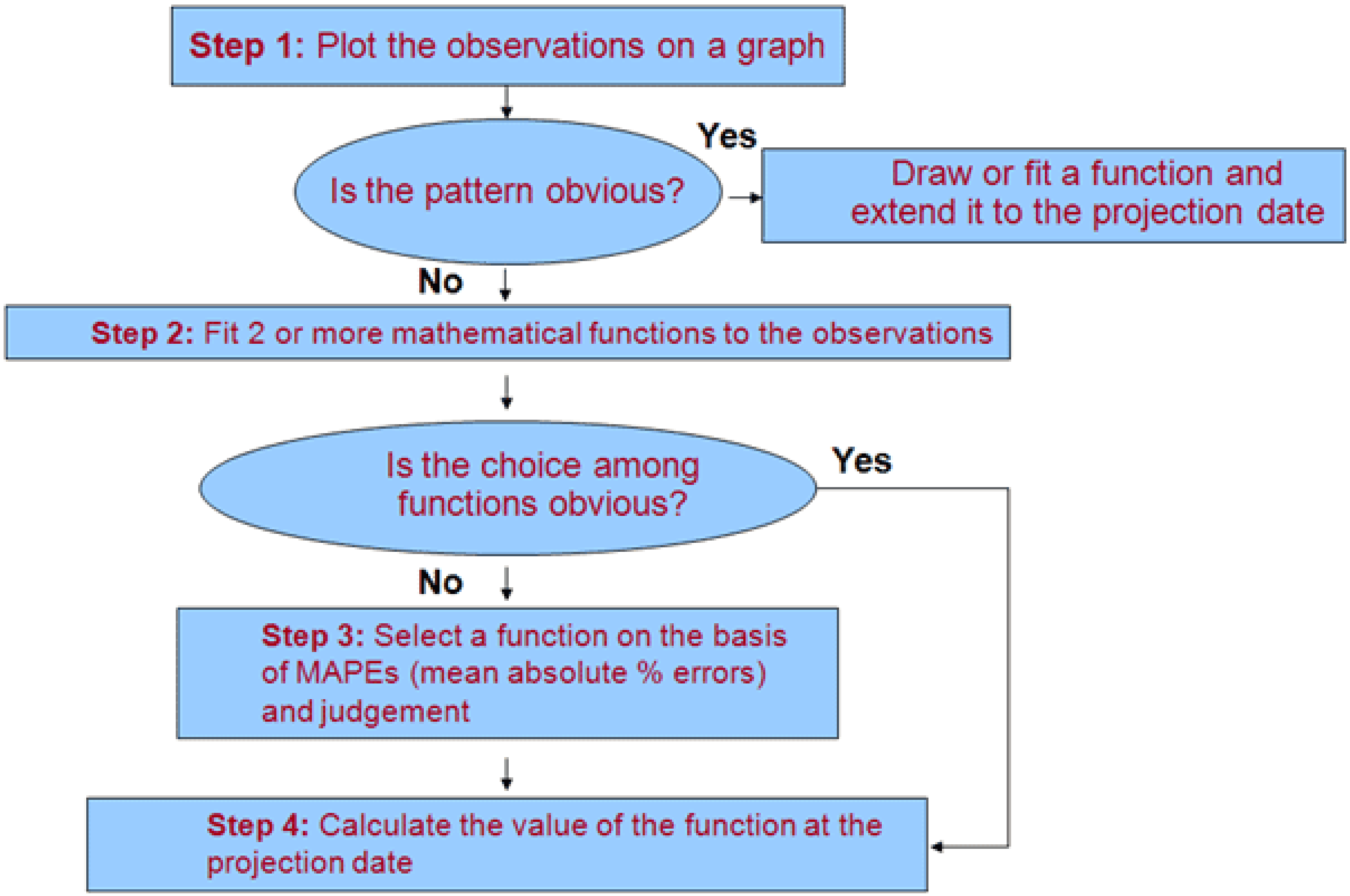
If *s* = 0.02, what is the projected size of the population at time 350?



$P(350) = 100{,}000 / (1+e-0.02(350-200))$

$= 100{,}000 / (1+e-3) =$ **95,257**

To RE-CAP


Step 1: Plot the observations on a graph
Is the pattern obvious?
Yes
Draw or fit a function and extend it to the projection date
No
Step 2: Fit 2 or more mathematical functions to the observations
Is the choice among functions obvious?
Yes
No
Step 3: Select a function on the basis of MAPEs (mean absolute % errors) and judgement
Step 4: Calculate the value of the function at the projection date

DIFFERENT METHODS

COHORT COMPONENT METHODS

Cohort component method

- It models the age-sex structure of populations and not just their size by taking into account the components of demographic change: fertility, mortality, and migration - and not just population growth.

- The procedure for making cohort-component population projections was developed by Whelpton in the 1930s.

Overview

The steps of a cohort-component projection are:

- to project every age cohort for one projection interval at a time
- to calculate the births during this interval and add in the newly-born children
- to adjust for migration
- before moving on to repeat the procedure to project the population to the end of the next interval.

Cohort component method

- It can be thought of as an elaboration of the ideas encapsulated in the demographic balancing equation:

$$P(t+n) = P(t) + B(t) - D(t) + I(t) - E(t)$$

extended to individual age cohorts.

Cohort component method

- They make use of the fact that every year of time that passes, every member of a population becomes a year older.

Es. Thus, after 5 years the survivors of the cohort aged 0-4 years at some baseline date will be aged 5-9 years, 5 years after that they will aged 10-14 years, and so on.

Data required

Detailed assumptions have to be made about each of the components of population growth throughout the period covered by the projection:

- Base year *population* subdivided by age and sex
- Sex-specific life tables for each projection interval in the projection period (*mortality*)
- Age-specific *fertility* rates for each projection interval in the projection period
- Age- and sex-specific net *migration* for each interval in the projection period (unless one is assuming that the population is closed to migration).

Cohort component method I

- The approach consists of segmenting the population into different subgroups differentially exposed to the "risks" of fertility, mortality, and migration and separately computing the changes over time in each group.
- In any population, exposure varies by age and sex so at a minimum the method segregates the population by age and sex.
- Other differentiation may recognize race, nationality, location (region, urban/rural), educational attainment, or religion.

Cohort component method II

- Is a discrete-time model of population dynamics. Population characteristics are only calculated at certain moments of time separated by lengthy time intervals.

- The projection period is usually divided into time intervals of the same length as the age intervals that are employed. Projection is then carried out one projection interval at a time.

Cohort component method: Steps

For each projection interval, the method basically consists of three steps:

1) Project forward the population in each subgroup at the beginning of the time interval in order to estimate the number still alive at the beginning of the next interval;

2) Compute the number of births for each subgroup over the time interval, add them across groups, and compute the number of those births who survive to the beginning of the next interval;

3) Add immigrants and subtract emigrants in each subgroup during the interval; compute the number of births to these migrants during the interval, and project forward the number of migrants and the number of their births that will survive to the beginning of the next interval.

Cohort component method: Steps I

1) If the population is only segregated by age and sex, the first step is technically straightforward: use a single decrement lifetable for each sex to compute the population alive at the baseline.

2) The second step is more complicated since every birth is produced by two individuals. In practice, the normal strategy is to pretend that births are produced by women only. The number of births can then be estimated by applying fertility rates to women only ("female-dominant" model)

3) The third step adds some practical difficulty to the projection as one needs to project not only the total number of migrants in each projection interval but also the timing of migration within the interval, since exposure to birth and death depends on when migrants enter or leave the population.

Projection of a closed female population

First step

$$ {}_5N_x^F(t+5) = {}_5N_{x-5}^F(t) \cdot \frac{{}_5L_x}{{}_5L_{x-5}} $$

Proportion of the person aged x-5 to x that will be alive 5 years later in a stationary population subject to the appropriate lifetable.

This survivorship ratio would be exactly correct if the age distribution of the population within the interval x-5 to x were the same as the age distribution within that age interval in the stationary population subject to the same lifetable.

Open-ended age group

Slight complication: open-ended age group (85+).

This group consists of two subgroups, those 80-84 at baseline who would then be 85-89, and those 85+ at baseline who would be 90+ five years later.

$$_{\infty}N_x^F(t+5) = \left({}_5N_{x-5}^F(t) \cdot \frac{{}_5L_x}{{}_5L_{x-5}} \right) + \left({}_{\infty}N_x^F(t) \cdot \frac{T_{x+5}}{T_x} \right)$$

However, we do not have T_{90+}.

Open-ended age group I

It is customary to combine the last two age groups at baseline and project them together (so 80-84 and 85+ are treated as 80+, who then become 85+) using the survival ratio

T_{85}/T_{80}, where $T_{80} = {}_5L_{80} + T_{85}$,

so

$$ {}_\infty P_{85}^{t+5} = \left({}_5P_{80}^{t} + {}_\infty P_{85}^{t} \right) \frac{T_{85}}{{}_5L_{80} + T_{85}} $$

First age group

Second step

- The population under age 5 at t+5 consists of births during the period from t to t+5 who survive to the end of the projection period.
- Female births, in turn, are obtained by applying the fertility rates to the women exposed to the risk of having a child between t and t+5.
- The number of girls under 5 at the end of the projection period depends on the initial population of women in the reproductive ages, their survival probabilities, the fertility rates, and the survival probabilities for female births.

Average # of women

Birth in each age group

$$ {}_5F_x \cdot 5 \cdot \left[\frac{{}_5N_x^F(t) + {}_5N_x^F(t+5)}{2} \right] = {}_5F_x \cdot 5 \cdot \left[\frac{{}_5N_x^F(t) + {}_5N_{x-5}^F(t) \cdot \frac{{}_5L_x}{{}_5L_{x-5}}}{2} \right] $$

First age group I

The total number of births will be:

$$B[t, t+5] = \sum_{x=\alpha}^{\beta-5} \frac{5}{2} \cdot {}_5F_x \cdot \left({}_5N_x^F(t) + {}_5N_{x-5}^F(t) \cdot \frac{{}_5L_x}{{}_5L_{x-5}} \right)$$

Where α and β are the lower and upper bounds of the childbearing ages.

The number of female births is then normally obtained by applying the ratio of male to female births (SRB):

$$B^F[t, t+5] = \frac{1}{1+SRB} \cdot B[t, t+5]$$

First age group II

- Finally, the number of females aged 0 to 4 at the end of the projection interval is obtained by surviving female births through time t to t+5.
- If births are assumed to be distributed evenly during the period t to t+5, then the relations of a stationary population can be invoked.
- In a stationary population, the ratio of the number of persons aged 0-4 to the number of births in the preceding 5-year period is ${}_5L_0/(5*l_0)$.

Thus

$$ {}_5N_0(t+5) = \frac{B[t,t+5]\cdot {}_5L_0}{5\cdot l_0} $$

$$ {}_5N_0^F(t+5) = B^F[t,t+5]\cdot \frac{{}_5L_0}{5\cdot l_0} $$

$$ = \frac{{}_5L_0}{2\cdot l_0}\cdot\frac{1}{1+SRB}\cdot\sum_{x=\alpha}^{\beta-5} {}_5F_x\cdot\left({}_5N_x^F(t) + {}_5N_{x-5}^F(t)\cdot\frac{{}_5L_x}{{}_5L_{x-5}}\right) $$

Example: Sweden, baseline 1993 (females); $l_o = 100{,}000$							
Age x	${}_5N_x^F$ (1993.0)	${}_5L_x^F$	${}_5F_x$	${}_5N_x^F$ (1998.0)	${}_5B_x$ [1993.0, 1998.0]	${}_5N_x^F$ (2003.0)	${}_5B_x$ [1998.0, 2003.0]
0	293,395	497,487					
5	248,369	497,138					
10	240,012	496,901					
15	261,346	496,531	0.0120				
20	285,209	495,902	0.0908				
25	314,388	495,168	0.1499				
30	281,290	494,213	0.1125				
35	286,923	492,760	0.0441				
40	304,108	490,447	0.0074				
45	324,946	486,613	0.0003				
50	247,613	480,665					
55	211,351	471,786					
60	215,140	457,852					
65	221,764	436,153					
70	223,506	402,775					
75	183,654	350,358					
80	141,990	271,512					
85+	112,424	291,707					
Sum	4,397,428						

Projection of a closed two sex population

The procedure described in the previous slides can be used for the male population projection.
Caution has to be used when computing the total number of births.

Derive the total number of births (male and female) from female fertility rates and derive the number of male births by applying a sex ratio at birth to the total number of female births.

$$B^M[t, t+5] = \frac{SRB}{1+SRB} \cdot B[t, t+5]$$

Example: Sweden, baseline 1993 (males)

Age x	${}_5N_x^M$ (1993.0)	${}_5L_x^M$	${}_5N_x^M$ (1998.0)	${}_5N_x^M$ (2003.0)
0	310,189	496,754		
5	261,963	496,297		
10	252,046	495,989		
15	274,711	495,113		
20	296,679	493,460		
25	333,726	491,475		
30	296,774	489,325		
35	299,391	486,487		
40	314,295	482,392		
45	338,709	476,532		
50	256,066	467,568		
55	208,841	452,941		
60	199,996	428,556		
65	197,282	390,707		
70	184,234	336,027		
75	133,856	261,507		
80	86,732	172,333		
85+	49,095	128,631		
Sum	4,294,585			
Total population size	8,692,013			

Projection of an open population - Emigration

- Easy to adapt the projection methodology to take into account emigration.
- Since emigration is a decrement from the population of interest, compute emigration rates by age and sex and then derive a two-decrement life table combining the risks of death and emigration.
- Use the corresponding survivorship ratios from the multiple decrement lifetable for the projection.

This method is entirely appropriate for populations in which the dominant migration flow is emigration.

Projection of an open population - Immigration

Difficulties:

- People already in the population are not at risk of immigrating into it and relating immigration flows to the population by age and sex does not provide the same advantages as it does for mortality or fertility.
- The formal difficulty of integrating migration in projection is that migration continuously affects the population at risk both of dying and of giving birth.
- If migration were taking place by discrete leaps exactly at the end of each projection interval, we would only need to add or subtract migrants at the right ages. But some migrants will not survive until the end of the interval and some may bear children who will survive until the end of the interval.

Projection of an open population – Immigration I

Solution:

Divide the number of migrants during the interval into two discrete quantities, and to assume that:

- half of the migrants moved exactly at the beginning of the projection interval
- the other half moved exactly at the end of the interval.

Projection of an open population – Immigration II

Third step

Let's denote as ${}_5Ix^F$ [t,t+5] the net flow of immigrants during the projection period in the age interval x to x+5

There are two additional terms in the number of survivors at the end of any projection interval for the age group x to x+5:

- half of the increments between the age x-5 and x are added at the beginning of the interval and survived to age x to x+5.
- half of the increments between the age x and x+5 are added directly at the end of the interval;

$$ {}_5N_x^F(t+5) = \left[\left({}_5N_{x-5}^F(t) + \frac{{}_5I_{x-5}^F[t,t+5]}{2}\right) \cdot \frac{{}_5L_x}{{}_5L_{x-5}}\right] + \frac{{}_5I_x^F[t,t+5]}{2} $$

Projection of an open population – Immigration III

The number of births in the period must also be adjusted.

- Increments at the end of the interval do not contribute to the number of births in the population during the interval.
- Increments at the beginning of the interval are usually assumed to bear children at the same rate as the population they are joining

$$\Delta B[t, t+5] = \sum_{x=\alpha}^{\beta-5} \frac{5}{4} \cdot {}_5F_x \cdot \left({}_5I_x^F(t) + {}_5I_{x-5}^F(t) \cdot \frac{{}_5L_x}{{}_5L_{x-5}} \right)$$

Negative values can be observed, which reflect the number of births that would have occurred but which were lost to the population through the emigration of potential mothers.

Projection of an open population – Immigration IV

Last step:

- Births are divided by sex based on a sex ratio at birth and survived forward using the lifetable to obtain the migration “correction” to the 0-4 age group.
- Since half of the migration at 0-4 is also to be added at the end of the interval, the equation for the first age group becomes:

$$ {}_5N_0^F(t+5) = B^F[t, t+5] \cdot \frac{{}_5L_0}{5 \cdot l_0} + \frac{{}_5I_0^F[t, t+5]}{2} $$

Example: Sweden, females, baseline 1993 (see Box 6.1)

Age x	${}_5N_x^F$ (1993.0)	${}_5L_x^F$	${}_5F_x$	${}_5I_x^F$ [1993.0, 1998.0]	${}_5N_x^F$ (1998.0)	${}_5B_x$ [1993.0, 1998.0]
0	293,395	497,487		6,840		
5	248,369	497,138		4,150		
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30	281,290	494,213	0.1125	5,470		
35	286,923	492,760	0.0441	3,155		
40	304,108	490,447	0.0074	1,770		
45	324,946	486,613	0.0003	1,115		
50	247,613	480,665		1,075		
55	211,351	471,786		845		
60	215,140	457,852		645		
65	221,764	436,153		530		
70	223,506	402,775		465		
75	183,654	350,358		300		
80	141,990	271,512		250		
85+	112,424	291,707		175		
Sum	4,397,428					

STEP 1

Calculate how many members of each living age cohort will survive the current projection interval.

STEP 2

Either add the immigrants to each age cohort and subtract the emigrants or, more simply, add net migrants to each age cohort.

STEP 3

Calculate how many births will occur during current projection interval and divide them into boys and girls.

STEP 4

Calculate how many of these births of each sex will remain alive at the end of the projection interval and adjust for net migration into the youngest age group.

STEP 5

Repeat the calculations for the next projection interval.

Accuracy of a forecast

- Difference between the forecasted size and the actual one, or the difference as a proportion of the projected size. This relative difference misses the temporal dimension of a forecast; a 10 percent error indicates different forecast qualities when it refers to a long-term projection as opposed to a short-term one.

- Measure of relative error: $E = \frac{N(T) - N^P(T)}{N(T) - N(0)}$

where N^P (T) is the predicted population size at time T, while N(0) and N(T) are the actual population size at time 0 and T. Time 0 can be chosen as the latest population estimate that was available when the forecast was prepared.

The measure does not directly control for the length of the forecast but rather for the actual population change

Accuracy of a forecast I

- Comparing the actual growth rate to the projected growth rate

$$\varepsilon = \bar{r}^{P}[0, T] - \bar{r}[0, T]$$

$$= \frac{\ln\left(\frac{N^{P}(T)}{N(T)}\right)}{T}$$

when T is measured in years, this ratio is annualized measure of the forecast error.

Matrix notation

To show how to summarize the cohort component projection methodology using matrix notation let's hypothesize to divide the population into five age groups only (0-14, 15-29, 30-44, 45-59, 60+) and refer to those as groups 1 to 5 (6 inT_6 will refer to age 75+). For a closed female-only population (denoted W), surviving the population at time t fifteen years forward implies

$$W_i(t+15) = W_{i-1}(t) \cdot \frac{L_i}{L_{i-1}}$$ for *i* = 2,3,4

$$W_5(t+15) = \left(W_4(t) \cdot \frac{L_5}{L_4}\right) + \left(W_5(t) \cdot \frac{T_6}{T_5}\right)$$ for *i* = 5

If we assume that fertility is limited to ages 15 to 45, the number of births will be:

$$B[t, t+15] = \frac{15}{2} \cdot F_2 \cdot \left(W_2(t) + W_1(t) \cdot \frac{L_2}{L_1} \right) + \frac{15}{2} \cdot F_3 \cdot \left(W_3(t) + W_2(t) \cdot \frac{L_3}{L_2} \right)$$

And the first age group fifteen years forward will be:

$$W_1(t+15) = B[t, t+15] \cdot \frac{1}{1+SRB} \cdot \frac{L_1}{15 \cdot l_0} = k \cdot \left(F_2 \cdot \frac{L_2}{L_1} \cdot W_1(t) + \left[F_2 + F_3 \cdot \frac{L_3}{L_2} \right] \cdot W_2(t) + F_3 \cdot W_3(t) \right)$$

with $k = (1/(1+SRB))(L_1/(2 \cdot l_0))$.

$$\begin{pmatrix} W_1(t+15) \\ W_2(t+15) \\ W_3(t+15) \\ W_4(t+15) \\ W_5(t+15) \end{pmatrix} = \begin{pmatrix} k \cdot F_2 \cdot \frac{L_2}{L_1} & k \cdot \left[F_2 + F_3 \cdot \frac{L_3}{L_2} \right] & k \cdot F_3 & 0 & 0 \\ \frac{L_2}{L_1} & 0 & 0 & 0 & 0 \\ 0 & \frac{L_3}{L_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{L_4}{L_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{L_5}{L_4} & \frac{T_6}{T_5} \end{pmatrix} \cdot \begin{pmatrix} W_1(t) \\ W_2(t) \\ W_3(t) \\ W_4(t) \\ W_5(t) \end{pmatrix}$$

The **Leslie matrix** (also called the Leslie model) is one of the most well known ways to describe the growth of populations (and their projected age distribution), in which a population is closed to migration, growing in an unlimited environment, and where only one sex, usually the female, is considered.

A Leslie Matrix contains:

- age-specific fertilities along the first row
- age-specific survival probabilities along the subdiagonal
- Zeros everywhere else

With age-structure, the only transitions that can happen are from one age to the next and from adult ages back to the first age class

$$\mathbf{W}(t+15) = \mathbf{L}[t, t+15] \cdot \mathbf{W}(t)$$

If we assume that the same projection matrix can be applied to a projection period of n successive 15-year intervals, then:

$$\mathbf{W}(t + 15 \cdot n) = \mathbf{L}^{\mathbf{n}} \cdot \mathbf{W}(t)$$

When the matrix L is raised to a high enough power n, the population age structure of W(t+15n) becomes constant and the population growth rate during each projection interval becomes constant. This result is related to the stable population theorem.

Matrix algebra offers a way to derive the constant age distribution and the constant growth rate.
When a population has reached the stable state, it must satisfy for any subsequent projection interval:

$$\mathbf{W}^{\mathbf{S}}(t + 15) = \mathbf{L} \cdot \mathbf{W}^{\mathbf{S}}(t) = \lambda \cdot \mathbf{W}^{\mathbf{S}}(t)$$

The dominant eigenvalue of L, gives the population's asymptotic growth rate (growth rate at the stable age distribution). The corresponding eigenvector

$$(\mathbf{L} - \lambda \cdot \mathbf{I}) \cdot \mathbf{W}^{\mathbf{S}}(t) = 0$$

provides the stable age distribution, the proportion of individuals of each age within the population, which remains constant at this point of asymptotic growth barring changes to vital rates. Once the stable age distribution has been reached, a population undergoes exponential growth at rate λ.