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Evaluation of uncertainty

Lecture #6

Electronic measurements

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The concept of uncertainty

The concept of uncertainty as a quantifiable attribute is relatively new in the history of measurement, although error and error analysis have long been a part of the practice of measurement science or metrology. It is now widely recognized that, when all of the known or suspected components of error have been evaluated and the appropriate corrections have been applied, there still remains an uncertainty about the correctness of the stated result, that is, a doubt about how well the result of the measurement represents the value of the quantity being measured.

JCGM 100:2008 – Evaluation of measurement data – **Guide to the expression of uncertainty in measurement**

The concept of uncertainty

Measurement result:

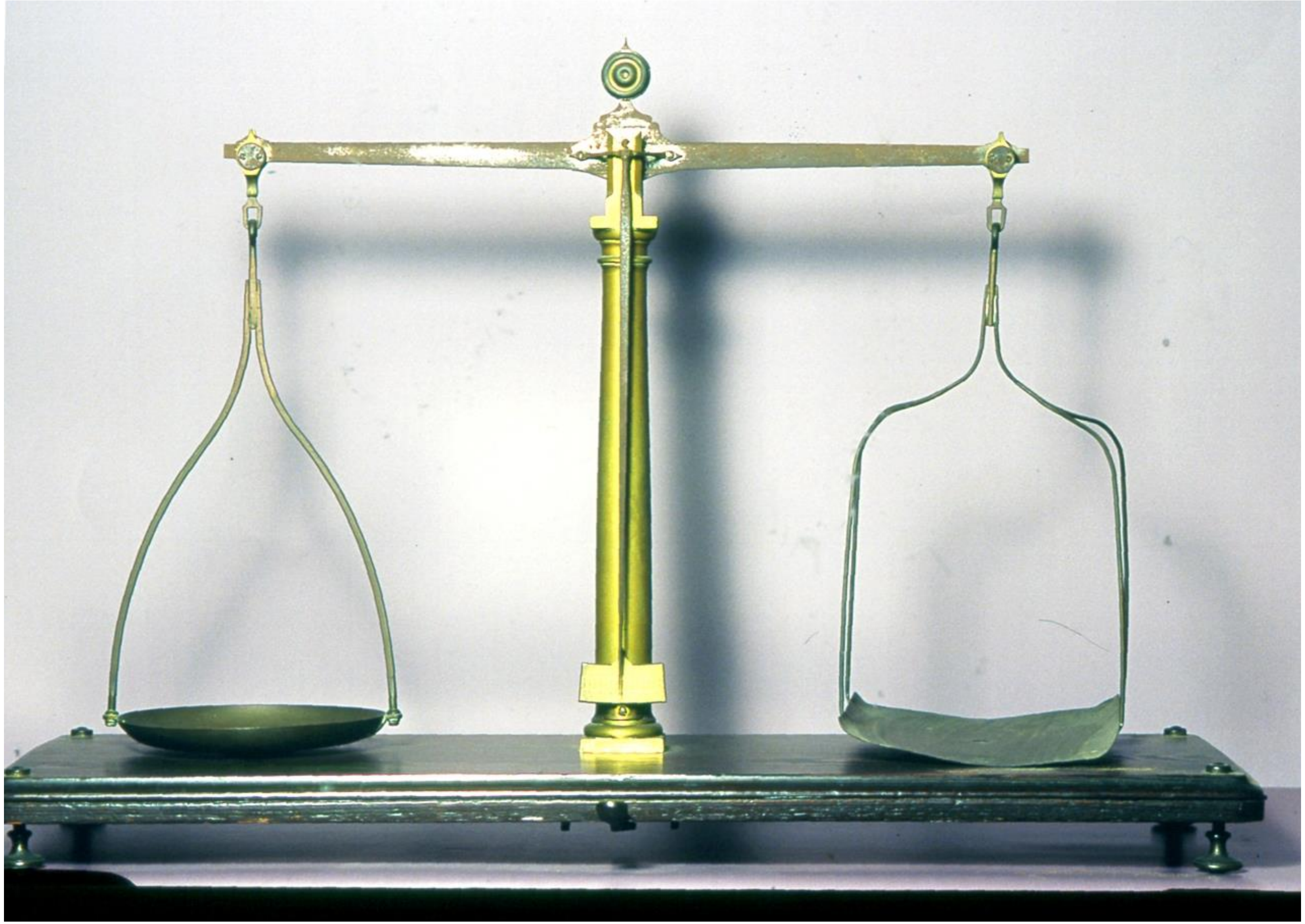
- The measured value
- The unit
- An evaluation of **measurement uncertainty**

Additional information:

- Measurement conditions
- Followed procedure to obtain the results

- **Measurement** → Process aimed at **removing uncertainty**, as far as possible, by producing empirical evidence
- **Measurement Uncertainty** → **Residual** uncertainty

Equal-arm balance



Equal-arm balance

- Gravity acceleration is uniform
- The two mass quantities are equal when the two arms are level

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Principle of equality of mechanical torques

Balance of effects on an auxiliary quantity

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Measurement by DIRECT comparison

- **Discrimination threshold**: minimum displacement from the null position

Equal-arm balance

- Possible situations at the null condition:
  - Mass A is still **slightly larger** than mass B (or vice versa);
  - The balance arm holding mass A is found to be **slightly longer** than the arm holding mass B (or viceversa);
  - A very **small vibration** effect prevents reaching a static balance condition.

There will be a whole set of mass quantities for which comparison with the measurand yields a null result

Uncertainty: *non-negative parameter characterizing the dispersion of the quantity values being attributed to a measurand, based on the information used*

Uncertainty

- Different parameters can be representative of dispersion:
  - A standard deviation, called **standard measurement uncertainty**, that will be indicated by the symbol *u* (or a specified multiple of it, $k \cdot u$, with $k > 1$)
  - The half-width of an interval delimiting the **range of dispersion**, indicated by the symbol *U*

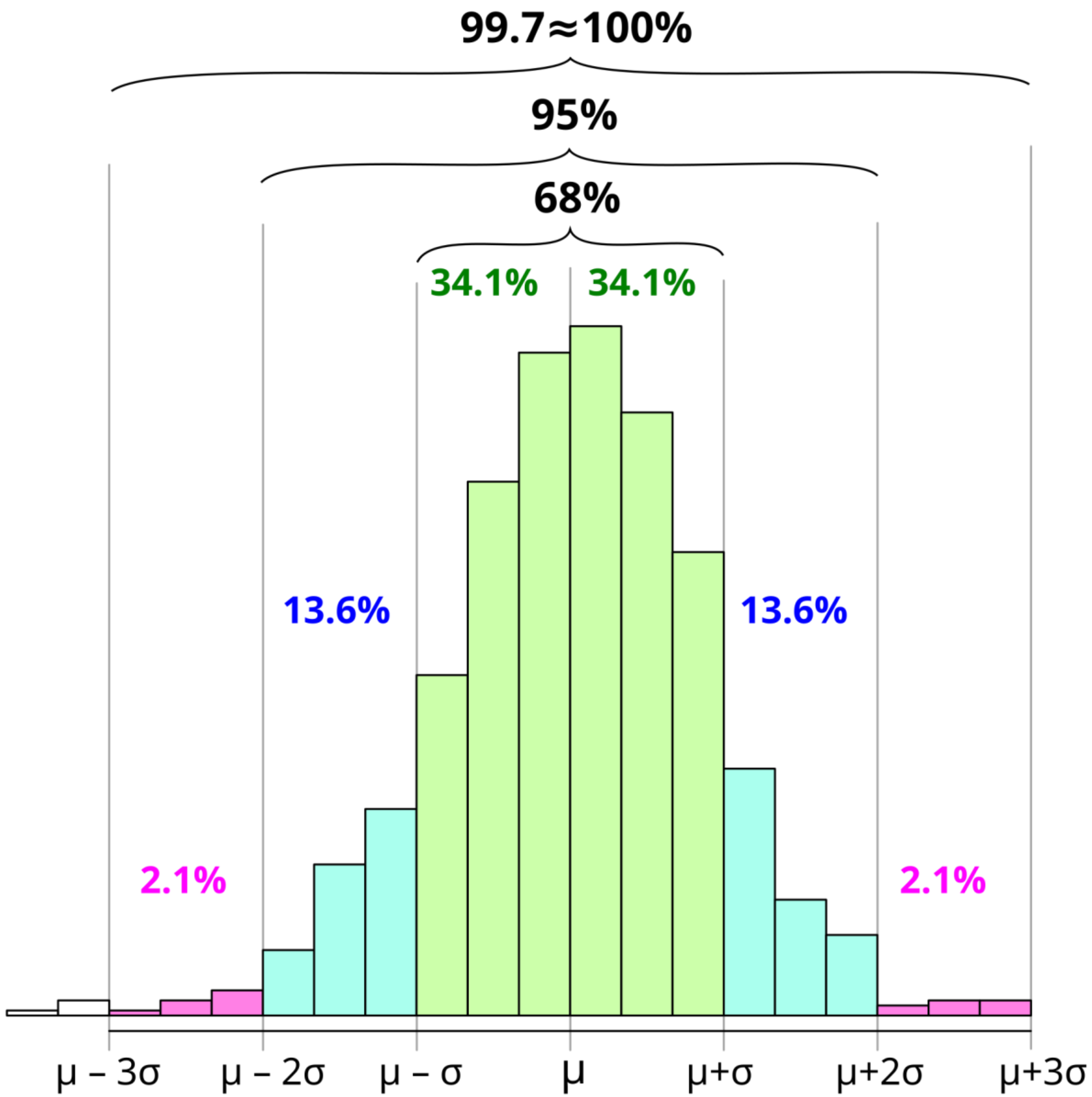
- *X* measured value for the quantity x
- u_x measurement uncertainty

Coverage probability

$$P[X - k \cdot u_x \leq x \leq X + k \cdot u_x | x = X]$$

- k **Coverage Factor**
- $k \cdot u_x$ **Expanded Uncertainty**

Coverage factor



Uncertainty

- Uncertainty is in general split into two contributions that must be evaluated separately:
  - **Type A uncertainty**: is the part of the uncertainty that can be estimated with **statistical methods**, i.e., through the analysis of the results of repeated tests (estimation of the standard deviation of the accidental error)
  - **Type B uncertainty**: is the part of the uncertainty that cannot be evaluated through the statistical analysis of the results of repeated tests, but which must be estimated through **information known a priori**, on the instruments and on the measurement process. Its determination is based on appropriate assumptions on the probability density of systematic errors. These contributions cannot in any way be eliminated or compensated by the operator, who can only take them into account

Type A uncertainty

- Its estimation is made through the statistical analysis of the results obtained with a finite number ***N*** **of independent and repeated observations** q_k of the measurand X, with $q = 1, \ldots, N$.
- **Experimental mean**:

$$\bar{q} = \frac{1}{N}\sum_{k=1}^{N} q_k$$

- **Experimental variance**:

$$s^2(q_k) = \frac{1}{N-1}\sum_{k=1}^{N} (q_k - \bar{q})^2$$

Type A uncertainty

- **Standard deviation:**

$$s(q) = \sqrt{\frac{1}{N-1} \sum_{k=1}^{N} (q_k - \bar{q})^2}$$

- **Type A uncertainty**:

$$u(q) = \frac{s(q)}{\sqrt{N}} = \sqrt{\frac{1}{N} \cdot \frac{1}{N-1} \sum_{k=1}^{N} (q_k - \bar{q})^2}$$

Type B uncertainty

- Its estimation takes place starting from **known information on the uncertainty** of the measurement system already available to the user.
- This information may be of a different nature, such as:
  - data acquired in **previous measurements**
  - the **metrological characteristics** of the instrument specified by the manufacturer
  - the **known properties of some materials**
  - information drawn from the **user's previous experience**
- Among these, the most widespread and frequently used information is that reported with the instrument, in **manuals or technical sheets**

Global uncertainty

- **Global uncertainty** takes into account the contributes of both types of uncertainty:

$$u_c = \sqrt{u_a^2 + u_{1b}^2 + u_{2b}^2 + \cdots + u_{Nb}^2}$$

Uncertainty as a design parameter

- **Ideally**, uncertainty should be minimized
- **In practice**, optimal uncertainty is a trade-off among contrasting requirements

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- Product specifications → **TOLERANCE**: acceptable range of variation for parameter values

- **Conformity assessment**: test to check that tolerance bounds are met

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Adequacy of measuring systems accuracy

Uncertainty as a design parameter

- **Measurement capability index** (relative uncertainty required in conformance testing):

$$C_m = \frac{T}{k \cdot u}$$

where T is the tolerance range

- In general $1 < T \leq 10$

- Example: resistor
    - Tolerance $\pm 10\%$
    - Required relative uncertainty $1 - 2\%$

Sources of uncertainty

- **Definitional uncertainty**, or intrinsic uncertainty, that results from the finite amount of detail in the definition of a measurand. Is the practical minimum achievable measurement uncertainty
- **Interactions between the measuring system and the measurand**. In several cases interaction effects are recognizable and can be either compensated or corrected for
- Inadequate knowledge of **influence quantities**
- **Measuring system uncertainty**

Evaluation of uncertainty

- **Measurement model:** mathematical relation among all quantities known to be involved in a measurement
- Measurement function:

$$y = f(x_1, x_2, \dots, x_n)$$

- y measurand
- $x_1, x_2, \dots, x_n$ input quantities measured or determined somehow

- $u_1, u_2, \dots, u_n$ uncertainties related to the measured values $X_1, X_2, \dots, X_n$
- u_y uncertainty associated to the measured value $Y = f(X_1, X_2, \dots, X_n)$

COMBINED STANDARD UNCERTAINTY

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Evaluation of uncertainty

- **Preliminary assumptions:**
  - Uncertainties $u_1, u_2, \ldots, u_n$ are small enough compared to the corresponding measured values
  - The measurement function $f(\cdot)$ is invertible in $\underline{X} = (X_1, X_2, \ldots, X_n)$ and its gradient in $x = X$ is non-zero

- **First order Taylor approximation**

$$Y + \delta_Y = f(X_1, X_2, \ldots, X_n) + \sum_{i=1}^{N} \left.\frac{\partial f}{\partial x_i}\right|_{x_i = X_i} \delta_i$$

where δ_Y and δ_i with $i = 1, 2, \ldots, n$ are random variables representing dispersion of the corresponding quantities

Evaluation of uncertainty

- δ_Y linear combination of variables δ_i with coefficients $\frac{df}{dx_i}$

$$\delta_Y = \sum_{i=1}^{N} \left.\frac{\partial f}{\partial x_i}\right|_{x_i=X_i} \delta_i$$

- In many practical cases input quantities $x_1, x_2, \dots, x_n$ are assumed to be independent
- **Combined standard uncertainty**:

$$u_y = \sqrt{\sum_{i=1}^{N} \left(\left.\frac{\partial f}{\partial x_i}\right|_{x_i=X_i} \right)^2 \cdot u_i^2}$$

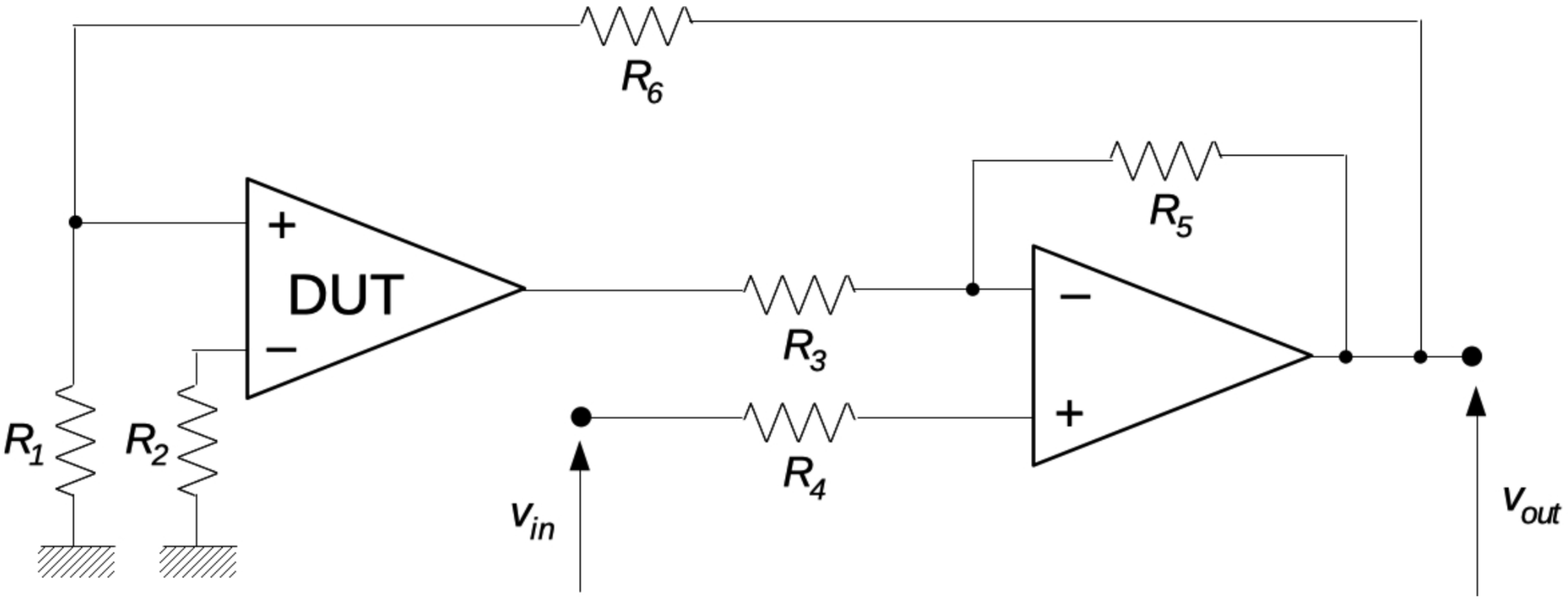
- When input quantities are **not independent**, cross-correlations must be accounted for

Evaluation of uncertainty

Combined standard uncertainty for some elementary measurement functions

Y	u_Y	$\frac{u_Y}{Y}$
$X_1 \pm X_2$	$\sqrt{u_1^2 + u_2^2}$	$\frac{\sqrt{u_1^2+u_2^2}}{X_1 \pm X_2}$
$X_1 \cdot X_2$	$\sqrt{X_2^2 u_1^2 + X_1^2 u_2^2}$	$\sqrt{\frac{u_1^2}{X_1^2} + \frac{u_2^2}{X_2^2}}$
$\frac{X_1}{X_2}$	$\sqrt{\frac{1}{X_2^2} u_1^2 + \frac{X_1^2}{X_2^4} u_2^2}$	$\sqrt{\frac{u_1^2}{X_1^2} + \frac{u_2^2}{X_2^2}}$
X^α	$\sqrt{\alpha^2 \left[X^{(\alpha-1)}\right]^2 u_X^2}$	$\alpha \cdot \frac{u_X}{X}$

OpAmp test circuit



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OpAmp test circuit

- **Open-loop gain**

$$G_{ol} = \frac{v_{in}}{v_{out}} \frac{R_1 + R_6}{R_1} = \frac{v_{in}}{v_{out}} \left(1 + \frac{R_6}{R_1}\right)$$

- v_{in} and v_{out} **measured RMS values**
- R_1 and R_6 resistors characterized by a **nominal value and a tolerance (or measured)**

- Propagation of uncertainty

$$\delta_{gain} = \left(\frac{\delta_{in}}{V_{in}} + \frac{\delta_{our}}{V_{out}}\right) \cdot \mathrm{gain} + \left(\frac{\delta_1}{R_1} + \frac{\delta_6}{R_6}\right) \cdot \frac{R_6}{R_1} \frac{V_{in}}{V_{out}}$$

OpAmp test circuit

- Assuming $R_6 \gg R_1$

$$\delta_{gain} \cong \left(\frac{\delta_{in}}{V_{in}} + \frac{\delta_{our}}{V_{out}} + \frac{\delta_1}{R_1} + \frac{\delta_6}{R_6}\right) \cdot \mathrm{gain}$$

- **Relative combined standard uncertainty**

$$\frac{u_{gain}}{\mathrm{gain}} \cong \sqrt{\left(\frac{u_{in}}{V_{in}}\right)^2 + \left(\frac{u_{out}}{V_{out}}\right)^2 + \left(\frac{u_1}{R_1}\right)^2 + \left(\frac{u_6}{R_6}\right)^2}$$

- RMS combination of relative standard uncertainties

OpAmp test circuit

- Uncertainty can be associated to different factors
  - u_{in} and u_{out} are measurement uncertainties

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  - Evaluated by reference to instrument specifications

  - u_1 and u_6 are measured or **nominal values**

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    Determined by the specified tolerances

- u_{gain} determined by the choices made for the evaluation of these uncertainties

OpAmp test circuit

- A very simple alternative is to consider instead the half-widths U_i of the intervals that delimit the ranges of dispersion of each quantity

$$U_Y = \sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i} \right|_{x_i = X_i} U_i$$

- Worst-case evaluation of uncertainty → the bounds of individual intervals are all added up

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Exceedingly pessimistic assessments

Principle of maximum entropy

- Measurement given as a standard uncertainty u $\rightarrow$ Half-width can be determimed: $U = k \cdot u$
- Deterministic bounds are known $\rightarrow u$ needs to be assumed

Uncertainty is characterized only by means of bounds or by specified properties

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Principle of maximum entropy

Pdf describing the maximum dispersion

Indirect measurement of resistance

- Resistance is determined as the ratio of measured values of voltage across a resistor, and of current through it

- Example:
  - $V_m = 4.38562\ \mathrm{V}$
  - $I_m = 0.2867\ \mathrm{A}$

- **Voltmeter accuracy**: ±0.01% of measured value, +0.001% of full-scale value – uncertainty interval corresponds to a 99% coverage probability, full-scale value is 10 V
- **Amperometer accuracy**: ±0.2% of measured value, +0.01% of full-scale value, that is 0.3 A

Indirect measurement of resistance

- $U_V = 0.54$ mV
- $U_I = 0.6$ mA

$$R = \frac{V}{I}$$

- **Worst-case rule**

$$U_R = \left|\frac{\partial R}{\partial V}\right|_{V=V_m} \cdot U_V + \left|\frac{\partial R}{\partial I}\right|_{I=I_m} \cdot U_I = \frac{1}{I_m} \cdot U_V + \frac{V_m}{I_m^2} \cdot U_I$$

- **Propagation rule**

$$u_R = \sqrt{\left(\left.\frac{\partial R}{\partial V}\right|_{V=V_m}\right)^2 \cdot u_V^2 + \left(\left.\frac{\partial R}{\partial I}\right|_{I=I_m}\right)^2 \cdot u_I^2}$$

Indirect measurement of resistance

- Conversion of interval half-widths U into standard uncertainties u
  - **Voltage uncertainty** is given as an interval with an associated 99% coverage probability. This is typically done with an assumed Gaussian distribution, therefore $U_V = k \cdot u_V$ where, in this case: k = 2.6. It follows that: u_V = 0.21 mV
  - **Current uncertainty** is given as an interval, with no additional information. In this case the maximum entropy pdf is rectangular, therefore $u_I = U_I/\sqrt{3}$ = 0.36 mA

$$\frac{u_R}{R_m} = \sqrt{\left(\frac{u_V}{V_m}\right)^2 + \left(\frac{u_i}{I_m}\right)^2} = \sqrt{(0.5 \times 10^{-3})^2 + (12 \times 10^{-3})^2}$$

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