Problem sheet 5: Functions of bounded variation

Exercise 1. Let $u \in BV(\mathbb{R}^n)$ with compact support. Let μ_i be the signed valued Radon measure associated to $\frac{\partial}{\partial x_i} T_u$. Show that $\mu_i(\mathbb{R}^n) = 0$ (for every i).

Exercise 2 (Subaddictivity of the perimeter)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be an open set and E, F measurable sets with $\mathrm{Per}(E,\Omega), \mathrm{Per}(F,\Omega) < +\infty$. Show that

$$\mathrm{Per}(E \cup F, \Omega) + \mathrm{Per}(E \cap F, \Omega) \leq \mathrm{Per}(E,\Omega) + \mathrm{Per}(F,\Omega).$$

Hint: Proceed by smooth approximation. Let $f, g \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$ with $0 \leq f, g, \leq 1$ and check (pointwise) that

$$|\nabla(f + g - fg)| + |\nabla(fg)| \leq |\nabla f| + |\nabla g|.$$

Exercise 3 (Isoperimetric problem with confinement)**.** Let $m > 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ be a positive continuous function with $\lim_{|x| \to +\infty} g(x) = +\infty$. Show that the energy

$$\mathcal{F}(E) = \mathrm{Per}(E) + \int_E g(x)dx$$

admits a minimizer among sets $E \subseteq \mathbb{R}^n$ of finite perimeter such that $|E| = m$.

Hint:use the Kolmogorov criterium for compactness in $L^1(\mathbb{R}^n)$

Exercise 4 (Plateau type problem)**.** Let $F \subset \mathbb{R}^n$ be a set of finite perimeter. Let U be a open bounded set of class C^1. Show that there exists a solution to the minimization problem

$$\inf\{\mathrm{Per}(G), \mid G \subseteq \mathbb{R}^n, \text{measurable}, G \setminus U = F \setminus U\}.$$