Problem sheet 4: Sobolev spaces

Exercise 1. Let $p \in [1, +\infty)$. Show that if $f \in W^{1,p}(\mathbb{R})$ then

$$\|f\|_\infty^p \leq 2\|f\|_p \|f'\|_p^{p-1}.$$

Hint: proceed as in the proof of the fact that $W^{1,p}(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$, showing that for all $L > 0$ and all $x \in R$, $|f(x)| \leq L^{-1/p}\|f\|_p + L^{1-1/p}\|f'\|_p$.

Exercise 2 (Poincaré on strips)**.** Let $n > 1$ and consider for $-\infty < a < b < +\infty$,

$$\Omega = \{(x', x_n)\ x' \in \mathbb{R}^{n-1}, a < x_n < b\}.$$

Show that for every $u \in W_0^{1,p}(\Omega)$ for $p \in [1, +\infty)$ there holds

$$\|u\|_p \leq |b - a| \|Du\|_p.$$

Hint: reduce to $u \in C_c^\infty(\Omega)$ and write it as $u(x', x_n) = \int_a^{x_n} \frac{\partial}{\partial x_n} u(x', t) dt$. Then use Hölder.

Exercise 3. Let U be an open bounded set with C^1 boundary in $\mathbb{R}^n$. Show that for all $u \in W_0^{1,2}(U) \cap W^{2,2}(U)$ there holds

$$\|\nabla u\|_2^2 \leq \|u\|_2 \|\Delta u\|_2$$

where $\Delta u = \mathrm{div} \nabla u$ (in weak sense).

Hint: Recall the density result $\overline{C^\infty(\bar{U})}^{\|\cdot\|_{k,p}} = W^{k,p}(U)$ and the definition of $W_0^{1,p}$. Integrate by parts.

Exercise 4. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open connected set of class C^1, let $g \in L^2(\Omega)$ with $\int_U g(x)dx = 0$ and consider the energy

$$E(u) = \int_\Omega |\nabla u|^2 + u(x)g(x)dx \qquad u \in W^{1,2}(\Omega).$$

Note that $E(u + k) = E(u)$ for every constant k. Let $C = \{u \in W^{1,2}(\Omega), \int_\Omega u(x)dx = 0\}$. Show that there exists a minimizer of $E(u)$ in C.

Show that the minimizer is unique and compute the (Euler-Lagrange) equation solved by the minimizer.

Hint: use Hölder inequality and then Poincaré inequality to show that every minimizing sequence is bounded. Then proceed by direct methods.

Exercise 5. Let $U \subseteq \mathbb{R}^n$ be a bounded open set of class C^1 and $p \in [1, +\infty]$. Consider the closed ball

$$B = \{u \in W^{1,p}(U), \|u\|_{W^{1,p}} \leq 1\}.$$

In which functional spaces is this set compact?

Let $p = 1$. Is B closed in $L^1(U)$?