Sketch of solutions, sheet 2 (Measure theory)

Es 1

Observe that if $x \in N$, every A open containing x satisfies $\mu(A) > 0$

Let f such that $f(\bar{x}) > 0 \to \exists A$ open containing $\bar{x}$ such that $f(x) \geq \frac{f(\bar{x})}{2} > 0 \quad \forall x \in A$

If $x \notin \operatorname{supp} \mu$, $\exists A$ open s. that $\mu(A) = 0$ and $x \in A$. Let $r > 0$ $\overline{B(x,r)} \subseteq A$. Construct f s. that $f = 1$ on $\overline{B(x,r)}$, $f = 0$ in $\mathbb{R}^n \setminus A$, $f(x) \in [0,1] \quad \forall x$

then $\int_{\mathbb{R}^n} f(x)\, d\mu = \int_A f(x)\, d\mu = 0$

Ex 2 Let $s > c_i \ \forall i \Rightarrow \mathcal{H}^s(A_i) = 0 \Rightarrow$

$$\mathcal{H}^s(\cup_i A_i) \leq \sum_i \mathcal{H}^s(A_i) = 0 \Rightarrow \forall s > \sup_i c_i$$

$$\mathcal{H}^s(\cup_i A_i) = 0$$

Let $s < \sup_i c_i \Rightarrow \exists k \ \ s < c_k \Rightarrow \mathcal{H}^s(A_k) = +\infty \Rightarrow$

$$\mathcal{H}^s(\cup_i A_i) \geq \mathcal{H}^s(A_k) = +\infty \Rightarrow \forall s < \sup_i s_i$$

$$\mathcal{H}^s(\cup_i A_i) = +\infty$$

$\sup c_i = \dim_{\mathcal{H}}(\cup_i A_i)$: $\square$.

If $c_i < n \Rightarrow \mathcal{H}^n(A_i) = 0 \Rightarrow \mathcal{H}^n(A) = 0$

Ex 3 let $A \subseteq \mathbb{R}^n$ take a cover of A

$A \subseteq \bigcup_i C_i$ diam $C_i \leq \delta$

$f(A) \subseteq \bigcup_i f(C_i)$ diam $f(C_i) \leq C\delta$

$$\mathcal{H}^s_{C\delta}(f(A)) \leq \frac{\omega_s}{2^s} \sum_i \left(\text{diam } f(C_i)\right)^s \leq$$

$$\leq \frac{\omega_s}{2^s} \sum_i C^s \left(\text{diam } C_i\right)^s = \frac{\omega_s}{2^s} C^s \sum_i \left(\text{diam } C_i\right)^s$$

Taking the infimum on (C_i) $\Rightarrow$

$$\mathcal{H}^s_{C\delta}(f(A)) \leq \frac{\omega_s}{2^s} C^s \mathcal{H}^s_\delta(A)$$

$$\downarrow \delta \to 0$$

$$\mathcal{H}^s(f(A)) \leq \frac{\omega_s}{2^s} C^s \mathcal{H}^s(A).$$