# Problem sheet 2: Measure theory

**Exercise 1.**
Let $\mu$ be a Radon measure on $\mathbb{R}^n$. We define the support of $\mu$ as

$$\text{supp } \mu = \mathbb{R}^n \setminus N \qquad \text{where } N = \cup\{A \subseteq \mathbb{R}^n, A \text{ open set}, \mu(A) = 0\}.$$

Prove that $\bar{x} \in \text{supp } \mu$ iff $\int_{\mathbb{R}^n} f(x) d\mu(x) > 0$ for all $f \in C_c(\mathbb{R}^n, [0,1])$ with $f(\bar{x}) > 0$.

**Exercise 2.** Let $A_i \subseteq \mathbb{R}^n$ with $dim_{\mathcal{H}} A_i = c_i$. Show that $dim_{\mathcal{H}}(\cup_i A_i) = \sup_i c_i$.

Deduce that if $c_i < n$, and $\sup_i c_i = n$, then $A = \cup_i A_i$ satisfies $dim_{\mathcal{H}}(A) = n$ and $|A| = 0$.

**Exercise 3.** Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a Lipschitz function (that is $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^n$). Show that for all $s > 0$ and all $A \subseteq \mathbb{R}^n$, there holds

$$\mathcal{H}^s(f(A)) \leq C^s \mathcal{H}^s(A)$$

where $C$ is the Lipschitz constant of $f$ ($C = \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$)