Sketch of solutions (Sheet 1)

EXERCISE 1

1) $\|u_n\|_\alpha \leq C \Rightarrow \underbrace{\|u_n\|_\infty \leq C}_{\text{EQUIBDD}}$ and $\underbrace{|u_n(x) - u_n(y)| \leq C|x-y|^\alpha}_{\text{EQUICONTINUITY}}$

$\Rightarrow$ we may apply ASCOLI-ARZELÀ thm $\Rightarrow \exists\, u_{n_k}$ such that $u_{n_k} \to u$ uniformly in $\bar{U}$

$u \in \mathcal{C}(\bar{U})$ $\quad \|u_{n_k} - u\|_\infty \longrightarrow 0 \quad n_k \to +\infty$

2) u_n Cauchy in $(\mathcal{C}^{0,\alpha}, \|\cdot\|_\alpha) \Rightarrow \left[\begin{array}{l}\|u_n\|_\alpha \leq C \text{ and} \\ u_n \text{ is Cauchy in } (\mathcal{C}(\bar{U}), \|\cdot\|_\infty)\end{array}\right.$

since $\|f\|_\infty \leq \|f\|_\alpha$. Since $(\mathcal{C}(\bar{U}), \|\cdot\|_\infty)$ is Banach, then $u_n \longrightarrow u$ in $(\mathcal{C}(\bar{U}), \|\cdot\|_\infty) \Rightarrow$ also pointwise $\Rightarrow |u(x) - u(y)| \leq C|x-y|^\alpha$

$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\alpha} \leq \lim_k \frac{|(u_k-u_n)(x) - (u_k-u_n)(y)|}{|x-y|^\alpha} \longrightarrow 0 \text{ as } k, n \to +\infty$$

since u_n is Cauchy in $\mathcal{C}^{0,\alpha}(U)$

$\Rightarrow \|u_n - u\|_\alpha \to 0 \Rightarrow u_n \to u$ in $(\mathcal{C}^{0,\alpha}, \|\cdot\|_\alpha)$

(3) First of all we get that $\|u_n\|_\infty \leq C$ and moreover

$|u_n(x) - u_n(y)| \leq C|x-y|^\alpha \quad \forall x, y \in \bar{U} \Rightarrow$ by A.A. up to subsequences $u_n \to u$ in $\mathcal{C}(\bar{U})$. By the same argument as in (2), $u \in \mathcal{C}^{0,\alpha}(\bar{U})$. $\|u\|_{\mathcal{C}^{0,\alpha}} \leq C$.

Let $\beta < \alpha$. Fix $\delta > 0$ and consider

① $|x-y| \leq \delta$
$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} = |x-y|^{\alpha-\beta} 2\|u_n - u\|_{\mathcal{C}^{0,\alpha}(\bar{U})}$$
$$\leq \delta^{\alpha-\beta} 4C$$

② $|x-y| \geq \delta$
$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} \leq \delta^{-\beta} 2\|u - u_n\|_\infty$$

So for each $\delta > 0$ consider $n \geq n(\delta)$ $\|u - u_n\|_\infty \leq \delta^\alpha$

and we get
$$\frac{|(u-u_n)(x) - (u-u_n)(y)|}{|x-y|^\beta} \leq \tilde{C}\delta^{\alpha-\beta} \xrightarrow[\delta \to 0]{} 0.$$

EXERCISE 2.

$1) \Longrightarrow 2)$ $\quad \exists\, y_x \in \mathbb{R}^n \setminus U \quad \overline{B(y_x, r)} \cap \overline{U} = \{x\}$

$\cdot\ y_x = x + \nu(x)\, r \qquad \nu(x) = \dfrac{y_x - x}{|y_x - x|}$

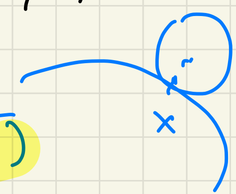
Let $R>0$ $\ y \in \overline{B(x,R)} \cap \overline{U}$. then $y \notin \overline{B(y_x, r)}$

$\Longrightarrow \cancel{r^2} < |y - y_x|^2 = |y - x - \nu(x) r|^2 = |y-x|^2 + \cancel{r^2} - 2\,\nu(x)\cdot(y-x)\cdot r$

$\Longrightarrow \quad \nu(x)\cdot(y-x) < \dfrac{1}{2r}|y-x|^2$ $\quad$ cond (C) with $C = \dfrac{1}{2r}$

$2) \Longrightarrow 1)$ $\quad r < R/2 \quad r < \dfrac{1}{2C}$ $\quad$ by contradiction $\exists\, y \neq x$

$y \in \overline{B(x + \nu(x) r, r)} \cap \overline{U}$



$\Longrightarrow y \in \overline{U} \cap \overline{B(x, R)}$ $\quad$ since $\overline{B(x,R)} \supseteq \overline{B(x + \nu(x) r, r)}$

$r^2 \geq |y - x - \nu(x) r|^2 = |y-x|^2 + r^2 - 2r\,\nu(x)\cdot(y-x) \geq (C) \geq$

$\geq |y-x|^2 + r^2 - 2rC\,|y-x|^2 \Longrightarrow \quad 0 \geq (1 - 2rC)\,|y-x|^2 > 0$

ABSURD!

$\boxed{\text{Ex } 3}$ (1) Note that $\operatorname{div}(x u^2) = n u^2 + 2u\, x\cdot\nabla u$

$$(\ast)\quad \int_{B(0,r)} n u^2 + 2u\, x\cdot \nabla u = \int_{\partial B(0,r)} u^2 x\cdot \frac{x}{|x|}\, dS = r \int_{\partial B(0,r)} u^2\, dS$$

since $\nu(x) = \frac{x}{|x|}$ on $\partial B(0,r)$

Note that

$$|2u\, x\cdot\nabla u| \le 2|u|\, |x|\cdot|\nabla u| \le 2|u| r\cdot |\nabla u| \le |u|^2 + r^2 |\nabla u|^2$$

substitute in $(\ast)$ and divide by r^2.

(2) first equality is just divergence theorem. Observe that

$$\left|2u \frac{x}{|x|^2}\cdot\nabla u\right| \le 2\frac{|u|}{|x|}|\nabla u| \le \frac{1}{\delta}|\nabla u|^2 + \delta \frac{u^2}{|x|^2}.$$ for every $\delta > 0$.

(3) By (1)

$$\int_{B_r\setminus B_\varepsilon} (n-2-\delta)\frac{u^2}{|x|^2} \le \frac{1}{\delta}\int_{B_r} |\nabla u|^2 + \int_{B_r} u^2 \frac{(n+1)}{r^2} + \int_{B_r} |\nabla u|^2\, dx \underbrace{- \frac{1}{\varepsilon}\int_{\partial B_\varepsilon} u^2\, dS}_{\le \|u\|_\infty^2 \cdot \frac{|\partial B_\varepsilon|}{\varepsilon} = \|u\|_\infty^2 \cdot C\varepsilon^{n-2} \to 0} \quad \forall \varepsilon < r$$

So $\lim_{\varepsilon\to 0^+} \int_{B_r\setminus B_\varepsilon} (n-2-\delta)\frac{u^2}{|x|^2}\, dx$ exists finite.

(4) $u(0) \neq 0 \Rightarrow$ assume without loss of generality $u(0) = \alpha > 0$.

then $\exists\, \varepsilon'$ $\quad u(x) \geq \frac{\alpha}{2} \quad \forall\, x \in B_{\varepsilon'}$

by integration on spheres (co-area f.)

$$\int_{B_r} \frac{u^2(x)}{|x|^2}\,dx \geq \int_{B_{\varepsilon'}} \frac{u^2(x)}{|x|^2}\,dx \geq \frac{\alpha^2}{4} \int_{B_{\varepsilon'}} \frac{1}{|x|^2}\,dx = \frac{\alpha^2}{4} \int_0^{\varepsilon'} \frac{n\omega_n r^{n-1}}{r^2} = \frac{\alpha^2}{4}\, n\omega_n \underbrace{\int_0^{\varepsilon'} r^{n-3}\,dr}_{+\infty}$$

if $n=1,\ n=2$

If $u, |\nabla u| \in L^2(\mathbb{R}^n)$ choose $\delta = \frac{n-2}{2}$ in (3) and

we get

$$\frac{n-2}{2} \int_{B_r} \frac{u^2(x)}{|x|^2}\,dx \leq \frac{2}{n-2} \int_{B_r} |\nabla u|^2 + \underbrace{\frac{1}{r} \int_{\partial B_r} u^2\,dS}_{\to\, 0 \text{ as } r \to +\infty}$$

sending $r \to +\infty$ we conclude.

$$\int_{\mathbb{R}^n} u^2 = \int_0^{+\infty} \int_{\partial B_r} u^2\,dS < +\infty$$

so $\liminf_{r \to +\infty} \int_{\partial B_r} u^2\,dS = 0$