# Problem sheet 1: Preliminaries

**Exercise 1.** Let $U \subseteq \mathbb{R}^n$ be an open bounded set. Let $C^{0,\alpha}(U)$ for $\alpha \in (0,1]$ be the space of Hölder continuous functions of exponent $\alpha$, so $u \in C(\bar{U})$ and there exists $C > 0$ such that $|u(x) - u(y)| \leq C|x-y|^\alpha$ for all $x, y \in U$. We define the norm $\|u\|_\alpha = \|u\|_\infty + \sup_{x \neq y \in U} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$.

(1) Show that if $u_n \in C^{0,\alpha}(U)$ is a sequence with $\|u_n\|_\alpha \leq C$, then up to a subsequence, $u_n \to u$ in $C(\overline{U}, \|\cdot\|_\infty)$ (that is the immersion $(C^{0,\alpha}(U), \|\cdot\|_\alpha) \to (C(\overline{U}), \|\cdot\|_\infty)$ is compact for every $\alpha \in (0,1]$).
(2) Show that $(C^{0,\alpha}(U), \|u\|_\alpha)$ is a Banach space.
(3) Show that if $u_n \in C^{0,\alpha}(U)$ is a sequence with $\|u_n\|_\alpha \leq C$, then up to a subsequence, $u_n \to u$ in $C^{0,\beta}(U)$, where $u \in C^{0,\alpha}(U)$ (that is the immersion $C^{0,\alpha}(U) \to C^{0,\beta}(U)$ is compact for every $\beta < \alpha$).

Hint: use the Ascoli-Arzelà compactness theorem.

**Exercise 2** (Characterization of exterior sphere condition)**.** Let $U \subseteq \mathbb{R}^n$ be a bounded open set. Show that the following two conditions are equivalent.

(1) **Exterior sphere condition**: there exists $r > 0$ such that $\forall x \in \partial U$, there exists $y_x \in \mathbb{R}^n \setminus U$ such that $B(y_x, r) \subseteq \mathbb{R}^n \setminus U$, $\overline{B(y_x, r)} \cap \overline{U} = \{x\}$.
(2) There exist $C, R > 0$, such that $\forall x \in \partial U$, there exists a vector $\nu(x) \in \mathbb{R}^n$, $|\nu(x)| = 1$ such that

$$(C) \qquad \nu(x) \cdot (y - x) \leq C|y-x|^2 \qquad \forall y \in \overline{B(x,R)} \cap \overline{U}.$$

Hint: to prove 1 implies 2 define $\nu(x) = \frac{y_x - x}{|y_x - x|}$. To prove that 2 implies 1 take $r < R/2, r < 1/(2C)$.

**Exercise 3** (Hardy's inequality)**.** Let $u \in C^1(\overline{B(0,r)})$ where $B(0,r) \subseteq \mathbb{R}^n$ the ball of center 0 and radius $r$. Assume that the dimension of the space is $n \geq 3$.

(1) Show that

$$\frac{1}{r} \int_{\partial B(0,r)} u^2 dS = \frac{1}{r^2} \int_{B(0,r)} (nu^2 + 2u\nabla u \cdot x) dx \leq \int_{B(0,r)} \left( \frac{n+1}{r^2} u^2 + |\nabla u|^2 \right) dx.$$

Hint: Apply divergence theorem to $xu^2$ and then Young inequality (that is $2ab \leq a^2/c + cb^2$, for every $c > 0$).

(2) Observing that $\operatorname{div} \frac{x}{|x|^2} = \frac{n-2}{|x|^2}$ and using divergence theorem, show that for $\varepsilon \in (0, r)$

$$\int_{B(0,r)\setminus B(0,\varepsilon)} (n-2)\frac{u^2}{|x|^2} dx = -\int_{B(0,r)\setminus B(0,\varepsilon)} 2u\nabla u \cdot \frac{x}{|x|^2} dx + \frac{1}{r}\int_{\partial B(0,r)} u^2 dS - \frac{1}{\varepsilon}\int_{\partial B(0,\varepsilon)} u^2 dS.$$

Conclude that, for $\delta \in (0, n-2)$

$$\int_{B(0,r)\setminus B(0,\varepsilon)} (n-2-\delta)\frac{u^2}{|x|^2} dx \leq \frac{1}{\delta}\int_{B(0,r)} |\nabla u|^2 dx + \frac{1}{r}\int_{\partial B(0,r)} u^2 dS - \frac{1}{\varepsilon}\int_{\partial B(0,\varepsilon)} u^2 dS.$$

Hint: recall Young inequality.

(3) Using 1, and 2, prove that $\frac{u(x)}{|x|} \in L^2(B(0,r))$ and there exists $C = C(n)$ such that

$$\int_{B(0,r)} \frac{u^2(x)}{|x|^2} dx \leq C \int_{B(0,r)} \left( \frac{u^2(x)}{r^2} + |\nabla u|^2 \right) dx.$$

(4) Show that if $n = 1, 2$ and $u(0) \neq 0$ then $\frac{u(x)}{|x|} \notin L^2(B(0,1))$.
Finally show that if $u \in C^1(\mathbb{R}^n)$, $n \geq 3$ with $u, |\nabla u| \in L^2(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

Hint: use 2, with $\delta = \frac{n-2}{2}$.