Esercizio 1. Si dica per quali valori di $k \in \mathbb{R}$ il seguente sistema lineare ammette soluzioni:

$$\begin{cases} x_1 + 2x_2 + 5x_4 = 0 \\ 2x_1 + 5x_2 + 11x_4 = 4 \\ x_1 + x_2 + (k - k^2)x_3 + (5 - k^2)x_4 = -k^2 - 3 \\ -x_2 + 2(k^2 - k)x_3 + (3k^2 - 4)x_4 = 2(k^2 + k - 4) \end{cases}$$

Risolvere il sistema per i valori di k per i quali la soluzione non è unica.

Svolg:

$$A|b = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 2 & 5 & 0 & 11 & 4 \\ 1 & 1 & k-k^2 & 5-k^2 & -k^2-3 \\ 0 & -1 & 2(k^2-k) & 3k^2-4 & 2(k^2+k-4) \end{array}\right)$$

Riduciamo

$$\begin{array}{c} \mathrm{I}^\circ \\ \mathrm{II}^\circ - 2\mathrm{I}^\circ \\ \mathrm{III}^\circ - \mathrm{I}^\circ \\ \mathrm{IV}^\circ \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & -1 & k-k^2 & -k^2 & -k^2-3 \\ 0 & -1 & 2(k^2-k) & 3k^2-4 & 2(k^2+k-4) \end{array}\right)$$

$$\begin{array}{c} \\ \\ \mathrm{III}^\circ + \mathrm{II}^\circ \\ \mathrm{IV}^\circ + \mathrm{II}^\circ \end{array} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & k-k^2 & -k^2+1 & -k^2+1 \\ 0 & 0 & 2(k^2-k) & 3k^2-3 & 2k^2+2k-4 \end{array}\right) \quad \mathrm{IV}^\circ + 2\mathrm{III}^\circ$$

$\cancel{2k^2}+2k-4 \cancel{-2k^2}+2$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & \boxed{k-k^2} & 1-k^2 & 1-k^2 \\ 0 & 0 & 0 & \boxed{k^2-1} & 2k-2 \end{array}\right)$$

$rg(A) \geq 2$ $\quad k-k^2 \neq 0 \quad k(1-k) \neq 0 \quad k \neq 0 \quad k \neq 1$

$\quad k^2-1 \neq 0 \quad k \neq \pm 1$

Se $k \in \mathbb{R} \setminus \{-1, 0, 1\}$ $\quad rg\, A = 4 = rg(A|b)$ il sistema ha soluzione per R-C e le sol. dipendono da $n^\circ param = n^\circ incognite - rg(A) = 4 - 4 = 0 \Rightarrow$ la soluzione è unica.

Se $k=-1$ $\quad A|b = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$

$rg(A)=3 \neq rg(A|b)=4$ $\quad$ per $k=-1$ il sistema non ha soluzione

$Sol = \emptyset$

Se $k=0$ $\quad A|b = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right)$ $\quad$ IV° + III°

$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right)$ $\quad$ $rg A=3 \neq rg(A|b)=4$ non ha soluzioni.

Se $k=1$ $\quad A|b = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad$ $0=0$, $0=0$

$rg A = 2 = rg(A|b)$

$\begin{cases} x_2 + x_4 = 4 \\ x_1 + 2x_2 + 5x_4 = 0 \end{cases}$ $\quad$ $\begin{cases} x_2 = 4 - x_4 \\ x_1 = -2x_2 - 5x_4 = -8 + 2x_4 - 5x_4 = \\ \quad = -8 - 3x_4 \end{cases}$

$\left\{ \begin{pmatrix} -8-3x_4 \\ 4-x_4 \\ x_3 \\ x_4 \end{pmatrix} \;\middle|\; x_3, x_4 \in \mathbb{R} \right\} = \boxed{\begin{pmatrix} -8 \\ 4 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle}$

$x_3 = 0$

$x_4 = 0$

Esercizio 2. Data l'applicazione lineare $f_s : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ definita da:

$$f_s(x, y, z) = (4x + (s-4)y, 2x + (s-2)y, (s-2)x + (2-s)y + sz),$$

i) scrivere la matrice A_s associata ad f_s rispetto alla base canonica di $\mathbb{R}^3$;

ii) determinare nucleo ed immagine di f_s al variare del parametro reale s;

iii) determinare, al variare di $s \in \mathbb{R}$, un sottospazio T_s di $\mathbb{R}^3$ tale che $\ker f_s \oplus T_s = \mathbb{R}^3$;

iv) stabilire per quali valori del parametro reale s la matrice A_s è diagonalizzabile su $\mathbb{R}$;

v) per ognuno dei valori trovati in iv) determinare una base $\mathcal{B}_s$ di $\mathbb{R}^3$ costituita da autovettori di A_s;

vi) posto $s = 4$ si ortonormalizzi la base $\mathcal{B}_4$. Si ottiene in questo modo una base ortonormale di $\mathbb{R}^3$ costituita da autovettori di A_4?

vii) Determinare la matrice associata ad f_s rispetto alla base canonica nel dominio e alla base $\mathcal{B} = \{(1,1,1), (1,1,0), (1,0,0)\}$ nel codominio.

$$f_s\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4x + (s-4)y \\ 2x + (s-2)y \\ (s-2)x + (2-s)y + sz \end{pmatrix}$$

i) $$A_s = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ s-2 & 2-s & s \end{pmatrix}$$

$$\det A_s = s(4s - \cancel{8} - 2s + \cancel{8}) = 2s^2$$

Se $s \in \mathbb{R} \setminus \{0\}$ $\quad f_s$ è iniettiva $\quad \operatorname{Ker} f_s = \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ $\quad B_{\operatorname{Ker} f_s} = \emptyset$

f_s è suriettiva $\quad \operatorname{Im} f_s = \mathbb{R}^3 \quad B_{\operatorname{Im} f_s} = \{e_1, e_2, e_3\}$

$\dim \operatorname{Ker} f_s = 0 \quad \dim \operatorname{Im} f_s = 3 \quad 3 = 0 + 3$

Se $s = 0$ $\quad A_0 = \begin{pmatrix} 4 & -4 & 0 \\ 2 & -2 & 0 \\ -2 & 2 & 0 \end{pmatrix}$ $\quad \dim \operatorname{Im} f_0 = \operatorname{rg} A_0 = 1$

$\operatorname{Im} f_0 = \left\langle \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle \quad B_{\operatorname{Im} f_0} = \left\{\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$

$\operatorname{Ker} f_0 \quad A_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{cases} 4x - 4y = 0 \\ 2x - 2y = 0 \\ -2x + 2y = 0 \end{cases} \quad x = y \quad \begin{pmatrix} y \\ y \\ z \end{pmatrix}$

$\operatorname{Ker} f_0 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \quad \dim \operatorname{Ker} f_0 = 2 \quad B_{\operatorname{Ker} f_0} = \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$3 = 2+1$

iii) Determinare T_s : $\underset{0}{\operatorname{Ker} f_s} \oplus \underset{3}{T_s} = \underset{3}{\mathbb{R}^3}$

Se $s \in \mathbb{R} \setminus \{0\}$ $\quad T_s = \mathbb{R}^3 \quad \left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\} \oplus \mathbb{R}^3 = \mathbb{R}^3$

Se $s = 0$ $\quad \operatorname{Ker} f_0 = \left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\rangle \qquad T_0 = \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\operatorname{Ker} f_0 : \ x - y = 0 \qquad T_0' = \left\langle \begin{pmatrix}1\\-1\\0\end{pmatrix} \right\rangle = \operatorname{Ker} f_0^{\perp}$

iv) $A_s = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ \underline{s-2} & 2-s & s \end{pmatrix}$

$$P_{A_s}(x) = \det(A_s - x\,\mathbb{I}_3) = \det \begin{pmatrix} 4-x & s-4 & 0 \\ 2 & s-2-x & 0 \\ s-2 & 2-s & s-x \end{pmatrix} =$$

$$= (s-x)\left(4s \cancel{-8} \overbrace{-4x - sx + 2x}^{-2x-sx} + \underline{x^2} - 2s \cancel{+8}\right) =$$

$$= (s-x)\left(x^2 - (s+2)x + 2s\right)$$

$x = s \qquad x^2 - (s+2)x + 2s = 0 \qquad x_{1,2} = \dfrac{s+2 \pm \sqrt{s^2+4s+4-8s}}{2} =$

$$= \frac{s+2 \pm \sqrt{s^2-4s+4}}{2} = \frac{s+2 \pm \sqrt{(s-2)^2}}{2} = \frac{s+2 \pm (s-2)}{2} =$$

$$= \begin{cases} \dfrac{s\cancel{+2}+s\cancel{-2}}{2} = s \\[2ex] \dfrac{\cancel{s}+2-\cancel{s}+2}{2} = 2 \end{cases} \qquad P_{A_s}(0) = \det A_s$$

$s \qquad s \qquad 2$

Se $s \in \mathbb{R} \setminus \{2\}$ ci sono due autovalori $\quad s \neq 2$

$\left.\begin{array}{ll} s & m_a(s) = 2 = m_g(s) \\ 2 & m_a(2) = 1 = m_g(2) \end{array}\right\} \Rightarrow A_s$ è diagonalizzabile perché ha tutti aut. reali e $m_a = m_g$ $\forall$ autovalore.

$2 \quad m_a(2) = 1 = m_g(2)$) perché ha tutti aut. reali e $m_a = m_g$ $\forall$ autovalore.

$m_g(s) = \dim V_s = \dim \operatorname{Ker}(A_s - s\mathbb{I}_3) =$

$= 3 - rg \begin{pmatrix} 4-s & s-4 & 0 \\ 2 & -2 & 0 \\ s-2 & 2-s & 0 \end{pmatrix} = 3-1 = 2$

Se $s=2$ $\quad \begin{matrix} s & s & 2 \\ 2 & 2 & 2 \end{matrix}$ vi è un solo autovalore

$2 \quad m_a(2) = 3 \neq m_g(2) = 2 \qquad s=2 \quad A_2 = \begin{pmatrix} 4 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$m_g(2) = \dim V_2 = \dim \operatorname{Ker}(A_2 - 2\mathbb{I}_3) =$

$= 3 - rg \begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 3-1 = 2$

$\Rightarrow$ A_2 non è diagonalizzabile perché $m_a(2) \neq m_g(2)$.

v) Cerchiamo una base B_s di autovettori per $s \in \mathbb{R} \setminus \{2\}$

$m_a(s) = 2 = m_g(s)$

$m_a(2) = 1 = m_g(2)$

$V_s = \operatorname{Ker}(A_s - s\mathbb{I}_3) =$

$= \operatorname{Ker} \begin{pmatrix} 4-s & s-4 & 0 \\ 2 & -2 & 0 \\ s-2 & 2-s & 0 \end{pmatrix} \quad \begin{cases} (4-s)x + (s-4)y = 0 \\ 2x - 2y = 0 \\ (s-2)x + (2-s)y = 0 \end{cases} \qquad \boxed{x - y = 0}$

$x = y$

$V_s = \left\{ \begin{pmatrix} y \\ y \\ z \end{pmatrix} \middle| \; y, z \in \mathbb{R} \right\} = \boxed{\left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle}$

$V_2 = \operatorname{Ker}(A_s - 2\mathbb{I}_3) \qquad A_s = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ \underline{s-2} & 2-s & s \end{pmatrix}$

$V_2 = \operatorname{Ker} \begin{pmatrix} 2 & s-4 & 0 \\ 2 & s-4 & 0 \\ s-2 & 2-s & s-2 \end{pmatrix}$

$\frac{1}{s-2}$ $\qquad \underline{s \neq 2} \qquad s - 2 \neq 0$

$$\begin{array}{c} I^\circ \\ II^\circ - I^\circ \\ \frac{III^\circ}{s-2} \end{array}\begin{pmatrix} 2 & s-4 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad \begin{cases} x-y+z=0 \\ 2x+(s-4)y=0 \end{cases}$$

$$\begin{cases} x = \frac{4-s}{2}y \\ z = y-x = y - \frac{4-s}{2}y = \frac{2-4+s}{2}y = \frac{s-2}{2}y \end{cases}$$

$$\begin{pmatrix} \frac{4-s}{2}y \\ y \\ \frac{s-2}{2}y \end{pmatrix} \qquad y=2 \qquad \begin{pmatrix} 4-s \\ 2 \\ s-2 \end{pmatrix} \qquad \boxed{V_2 = \left\langle \begin{pmatrix} 4-s \\ 2 \\ s-2 \end{pmatrix} \right\rangle}$$

$$B_s = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \underbrace{\begin{pmatrix} 4-s \\ 2 \\ s-2 \end{pmatrix}}_{} \right\} \quad V_s$$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

$$r1) \quad B_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \right\} \qquad GS$$

$$u_1 = \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \qquad u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$w_2 = v_2 - \underbrace{(v_2 \cdot u_1)}_{0} u_1 = v_2 \qquad u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_3 \qquad w_3 = v_3 - (v_3 \cdot u_1)u_1 - (v_3 \cdot u_2)u_2 =$$

$$= \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} - \frac{2}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-1-0 \\ 2-1-0 \\ 2-0-2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad u_3 = \frac{w_3}{\|w_3\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

Non è autovettore

$$\mathcal{C} = \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$$ è base di autovettori per A_4?

$$A_4 = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & -2 & 4 \end{pmatrix} \qquad A_s = \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ \underline{s-2} & 2-s & s \end{pmatrix}$$

$A_4 \neq A_4^t \Rightarrow$ NON è ortogonalmente digonalizzabile quindi $\mathcal{E}$ NON è base di autovettori.

vii) $A_{f_s, \mathcal{E}, \mathcal{B}}$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{tikzcd}
(\mathbb{R}^3, \mathcal{E}) \arrow[r, "A_s"] & (\mathbb{R}^3, \mathcal{E}) \arrow[d, "T_{\mathcal{E}}^{\mathcal{B}}"] \\
(\mathbb{R}^3, \mathcal{E}) \arrow[u, "I_3 = T_{\mathcal{E}}^{\mathcal{E}}"] \arrow[r, "M"] & (\mathbb{R}^3, \mathcal{B})
\end{tikzcd}$$

$$T_{\mathcal{E}}^{\mathcal{B}} \cdot A_s \cdot I_3 = M$$

$$T_{\mathcal{E}}^{\mathcal{B}} A_s = M \qquad T_{\mathcal{E}}^{\mathcal{B}} = \left(T_{\mathcal{B}}^{\mathcal{E}}\right)^{-1}$$

$$(H \mid I_3) \qquad (I_3 \mid H^{-1})$$

$$H = T_{\mathcal{B}}^{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} III^\circ \\ II^\circ - III^\circ \\ I^\circ - II^\circ \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) \quad \text{(right block: } H^{-1}\text{)} \qquad T_{\mathcal{E}}^{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & s-4 & 0 \\ 2 & s-2 & 0 \\ \underline{s-2} & 2-s & s \end{pmatrix}$$

Esercizio 3. Nello spazio euclideo di dimensione tre si considerino i punti $A = (1,1,1)$, $B = (2,1,0)$, $C = (2,1,2)$.

a) Mostrare che i punti A, B, C non sono allineati e determinare il piano π che li contiene;

b) determinare, se possibile, un punto D tale che il quadrilatero di vertici A, B, C, D sia un

a) Mostrare che i punti A, B, C non sono allineati e determinare il piano π che li contiene;

b) determinare, se possibile, un punto D tale che il quadrilatero di vertici A, B, C, D sia un quadrato;

c) determinare la retta r ortogonale al piano π e passante per B;

d) calcolare la distanza della retta r dalla retta s passante per $(0,0,0)$ e parallela al vettore $v = (1,2,-1)$;

e) determinare il luogo dei punti equidistanti dai punti A, B, C.

a) $A = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad B = \begin{pmatrix}2\\1\\0\end{pmatrix}$ $\quad C = \begin{pmatrix}2\\1\\2\end{pmatrix}$

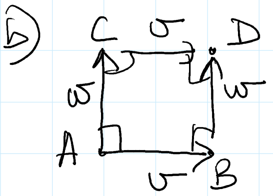
Non allineati $\Leftrightarrow$ $\dim\langle B-A, C-A\rangle = 2$

$\pi: A + \langle B-A, C-A\rangle$

$$\boxed{\begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \overset{a}{\begin{pmatrix}1\\0\\-1\end{pmatrix}}, \overset{b}{\begin{pmatrix}1\\0\\1\end{pmatrix}} \right\rangle}$$

$y=1$ $\quad \begin{cases} x = 1+a+b \\ \boxed{y=1} \\ z = -a+b \end{cases}$

b)



$v = B - A = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ $\quad \|B-A\| = \sqrt{2}$

$w = C - A = \begin{pmatrix}1\\0\\1\end{pmatrix}$ $\quad \|C-A\| = \sqrt{2}$

$\Rightarrow v \cdot w = 0$

$D = B + w = \begin{pmatrix}2\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$

$= C + v = \begin{pmatrix}2\\1\\2\end{pmatrix} + \begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$

$D - C = v = B - A$
$C - A = w = D - B$
è un quadrato

c) $\pi: \boxed{y} = 1$ $\quad z = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$r: B + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle = \begin{pmatrix}2\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $\quad r \perp \pi \quad B \in r$

d) $s: \underbrace{\begin{pmatrix}0\\0\\0\end{pmatrix}}_{S} + \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\rangle$ $\quad (S + \langle v_s \rangle)$

$r: \underbrace{\begin{pmatrix}2\\1\\0\end{pmatrix}}_{R} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $\quad (R + \langle v_r \rangle)$

s e r sono complanari $\Leftrightarrow$

$\dim(\langle S-R, v_s, v_r\rangle) \le 2$

$\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix} = 2 \neq 0$

sono rette sghembe

$d(r,s) = d(R, \sigma)$ con σ piano contenente s e parallelo a r;

$d(r,s) = d(R, \sigma)$ con σ piano contenente s e parallelo a r:

$\sigma: \begin{pmatrix}0\\0\\0\end{pmatrix} + \left\langle \begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$ $\qquad R = \begin{pmatrix}2\\1\\0\end{pmatrix}$

$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + a\begin{pmatrix}1\\2\\-1\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}a\\2a+b\\-a\end{pmatrix}$ $\qquad \begin{cases} \cancel{x=a} \\ y=2a+b \\ z=-a \end{cases}$ $\qquad a=x$

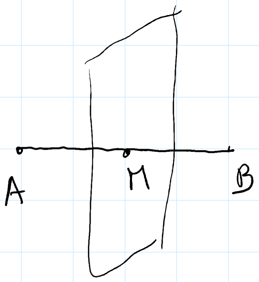
$\begin{cases} \cancel{y=2x+b} \\ z=-x \end{cases}$ $\qquad b=y-2x$

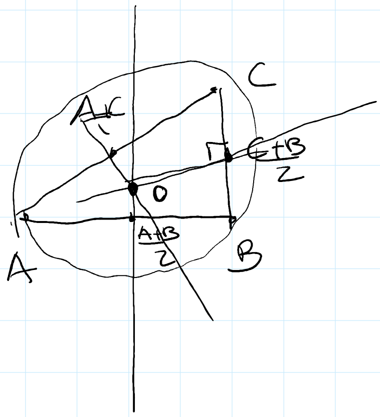
$\sigma: x+z=0$

$d(r,s) = d(R,\sigma) = \dfrac{|2|}{\sqrt{1^2+0^2+1^2}} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

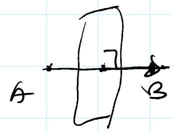
e)

Punti equidistanti da A, B, C





Punti eq. da A e B $\qquad \dfrac{A+B}{2} + \langle B-A \rangle^{\perp}$



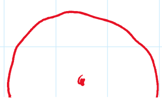
$A = \begin{pmatrix}1\\1\\1\end{pmatrix} \quad B = \begin{pmatrix}2\\1\\0\end{pmatrix} \qquad \dfrac{A+B}{2} = \begin{pmatrix}3/2\\1\\1/2\end{pmatrix} \qquad B-A = \begin{pmatrix}1\\0\\-1\end{pmatrix}$

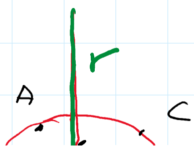
$x - z = \dfrac{3}{2} - \dfrac{1}{2} = 1 \qquad\qquad x - z = 1$

$B = \begin{pmatrix}2\\1\\0\end{pmatrix} \quad C = \begin{pmatrix}2\\1\\2\end{pmatrix} \qquad \dfrac{B+C}{2} = \begin{pmatrix}2\\1\\1\end{pmatrix} \qquad C-B = \begin{pmatrix}0\\0\\2\end{pmatrix}$

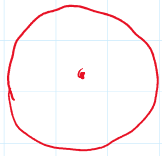
$\dfrac{B+C}{2} + \langle C-B \rangle^{\perp} \qquad 2z = 2 \qquad z = 1$

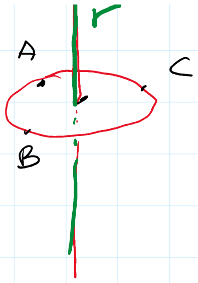
$r \quad \boxed{\begin{cases} x-z=1 \\ z=1 \end{cases}}$





$r \quad \boxed{\begin{cases} x - z = 1 \\ z = 1 \end{cases}}$





$$\frac{A+C}{2} + \langle C-A \rangle^{\perp}$$

Esercizio 4. Determinare tutti i sottospazi U di $\mathbb{R}^3$ tali che la proiezione ortogonale di $(1,2,0)$ su U sia $(1,1,1)$.

$U \leq \mathbb{R}^3 \qquad P_U \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

$P_U(v) = v_{//} \qquad \boxed{v = v_{//} + v_{\perp}} \qquad \boxed{\begin{matrix} v_{//} \in U \\ v_{\perp} \in U^{\perp} \end{matrix}}$

$v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad v_{//} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_{\perp} = v - v_{//} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\dim U \geq 1$
$\dim U^{\perp} \geq 1$

U	$\boxed{1}$	2
$U^{\perp}$	$\boxed{2}$	1

$$\begin{cases} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in U \\ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \in U^{\perp} \end{cases}$$

Se U ha dim 1 $\Rightarrow U = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

Se U ha dim 2 $\Rightarrow U^{\perp}$ ha dim 1 $\Rightarrow U^{\perp} = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle$

$$\boxed{\begin{matrix} U = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \\ U = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle^{\perp} \end{matrix}}$$