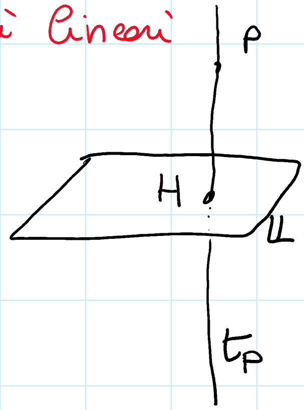
Proiezioni ortogonali su sottovarietà lineari



Definizione: $\mathbb{R}^3$

$\mathbb{L}: \ L + V_L$

Proiezione ortogonale del punto P su L è

$$\boxed{\{H\} = L \cap t \qquad \text{con} \qquad t = P + V_L^{\perp}}$$

Proiezione ortogonale su un piano:

$\mathbb{R}^3 \qquad \mathbb{L} = \pi \qquad P_\pi(P) = H = \pi \cap t_P = \pi \cap (P + V_\pi^{\perp})$

Esempio:

$\pi: 2x - y + 3z = 8 \qquad P = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}$ determinare la proiezione ortogonale di P su π.

Svolg:

$$t_P \ \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\rangle$$

$$P_\pi(P) = \pi \cap t_P \qquad \begin{cases} 2x - y + 3z = 8 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + a \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7+2a \\ 1-a \\ 3+3a \end{pmatrix} \end{cases}$$

$2(7+2a) - 1 + a + 3(3+3a) = 8$

$14 + 4a - 1 + a + 9 + 9a = 8 \qquad 14a + 14 = 0 \qquad a = -1 \qquad \boxed{P_\pi(P) = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}}$

Esercizio:

Determinare la proiezione ortogonale della retta

$$t: \overset{P}{\begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix}} + \left\langle \overset{v_t}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \right\rangle$$ sul piano $\pi: 2x - y + 3z = 8$

$\boxed{P_\pi(t) = \pi \cap \sigma}$ con $\boxed{\sigma : P + \langle v_t, v_\pi^\perp \rangle}$

$t \subseteq \sigma$ 2 generatori

$$\sigma : \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right\rangle$$ determiniamo l'eq. cartesiana di σ

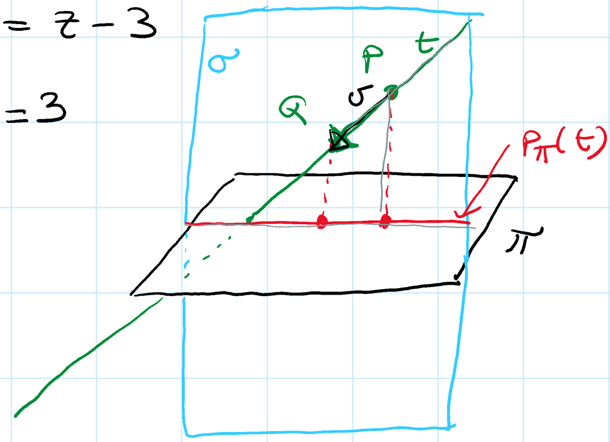
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 + a + 2b \\ 1 + a - b \\ 3 + 3b \end{pmatrix}$$

$$\begin{cases} x = 7 + a + 2b \\ \cancel{y = 1 + a - b} \\ z = 3 + 3b \end{cases} \qquad a = y - 1 + b \qquad \begin{cases} x = 7 + y - 1 + b + 2b \\ z = 3 + 3b \end{cases}$$

$$\begin{cases} 3b = x - 6 - y \\ 3b = z - 3 \end{cases} \qquad x - 6 - y = z - 3$$

$$\sigma : x - y - z = 3$$

$$P_\pi(t) = \begin{cases} 2x - y + 3z = 8 \\ x - y - z = 3 \end{cases}$$



2° modo: $t : P \vee Q$ $\qquad t : \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + \left\langle \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{v} \right\rangle \qquad P = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} \qquad Q = \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 3 \end{pmatrix}$

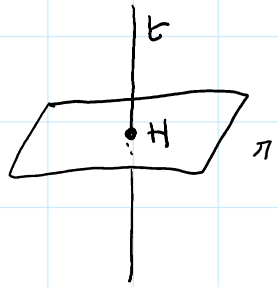
$$P_\pi(t) = P_\pi(P) \vee P_\pi(Q)$$

Per esercizio calcolare $P_\pi(Q)$ e verificare che $P_\pi(t) = P_\pi(P) \vee P_\pi(Q)$

Esercizio: determinare la proiezione ortogonale su $\pi : x + y + 8z = 1$ della retta $t = \overset{P}{\begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix}} + \left\langle \overset{v_t}{\begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}} \right\rangle$

Svolg:

$P_\pi(t) = \pi \cap \sigma = \{H\}$ $\quad \boxed{\sigma: P + \langle v_t, V_\pi^\perp \rangle} \quad \sigma: \begin{pmatrix}2\\1\\8\end{pmatrix} + \langle \begin{pmatrix}1\\1\\8\end{pmatrix}, \cancel{\begin{pmatrix}1\\1\\8\end{pmatrix}} \rangle$



$$P_\pi(t) = \begin{cases} x+y+8z = 1 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\1\\8\end{pmatrix} + a\begin{pmatrix}1\\1\\8\end{pmatrix} = \begin{pmatrix}2+a\\1+a\\8+8a\end{pmatrix} \end{cases}$$

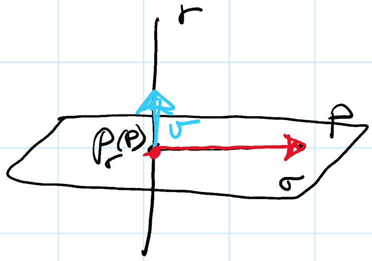
$2+a+\cancel{1}+a+64+64a=\cancel{1} \qquad 66a+66=0 \qquad \boxed{a=-1}$

$$P_\pi(t) = \left\{ \overset{H}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \right\}$$

Esempio: calcolare la proiezione ortogonale del punto $P=\begin{pmatrix}1\\3\\1\end{pmatrix}$ sulla retta $r: \begin{pmatrix}0\\0\\0\end{pmatrix} + \langle \underbrace{\begin{pmatrix}1\\0\\1\end{pmatrix}}_{v} \rangle$.

Sol:

$P_r(P) = r \cap \sigma \qquad \sigma: P + V_r^\perp$



$\sigma: \begin{pmatrix}1\\3\\1\end{pmatrix} + V_r^\perp \qquad \sigma_k: x+z=k \qquad$ cerchiamo $k \in \mathbb{R}$ tale che $P \in \sigma_k$

$1+1=k \qquad k=2$

$$P_r(P) = \begin{cases} x+z=2 \\ \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} + a\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}a\\0\\a\end{pmatrix} \end{cases} \qquad a+a=2 \quad 2a=2 \quad a=1$$

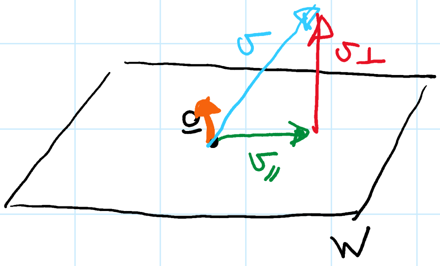
$P_r(P) = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad P = \begin{pmatrix}1\\3\\1\end{pmatrix} \qquad \boxed{P - P_r(P)} = \begin{pmatrix}0\\3\\0\end{pmatrix} \perp v = \begin{pmatrix}1\\0\\1\end{pmatrix}$

Relazione tra proiez. ort. su sottosp. e su sott. lin.

$\mathbb{R}^3$

$W: 2x+y+z=0 \qquad v=\begin{pmatrix}1\\3\\0\end{pmatrix}$

$P_W(v)$



$P_W(v)=v_{\parallel}$

$v=v_{\parallel}+v_{\perp}$ con $v_{\parallel}\in W$

con $v_{\perp}\in W^{\perp}$

$W^{\perp}=\left\langle\begin{pmatrix}2\\1\\1\end{pmatrix}\right\rangle \qquad \begin{pmatrix}2/\sqrt6\\1/\sqrt6\\1/\sqrt6\end{pmatrix}=u$

$\uparrow \sqrt6$

$v_{\perp}=(v\cdot u)u=\left[\begin{pmatrix}1\\3\\0\end{pmatrix}\cdot\begin{pmatrix}2/\sqrt6\\1/\sqrt6\\1/\sqrt6\end{pmatrix}\right]\begin{pmatrix}2/\sqrt6\\1/\sqrt6\\1/\sqrt6\end{pmatrix}$

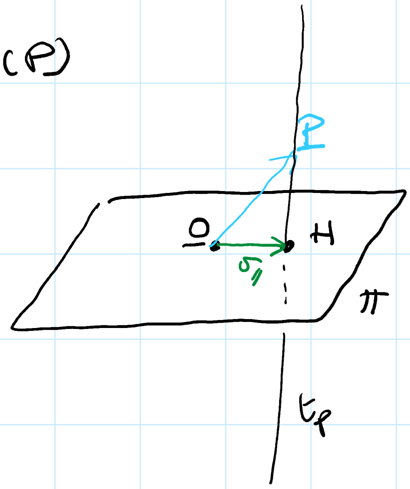
$=\frac{5}{6}\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}5/3\\5/6\\5/6\end{pmatrix}$

$v_{\parallel}=v-v_{\perp}=\begin{pmatrix}1\\3\\0\end{pmatrix}-\begin{pmatrix}5/3\\5/6\\5/6\end{pmatrix}=$

$=\begin{pmatrix}-2/3\\13/6\\-5/6\end{pmatrix}$

$\pi: 2x+y+z=0 \qquad P=\begin{pmatrix}1\\3\\0\end{pmatrix}=\underline{O}+v$

$P_{\pi}(P)$



$H=\underline{O}+v_{\parallel}$

$H=\pi\cap t_P \qquad t_P=\begin{pmatrix}1\\3\\0\end{pmatrix}+\left\langle\begin{pmatrix}2\\1\\1\end{pmatrix}\right\rangle$

$\begin{cases}2x+y+z=0\\ \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\0\end{pmatrix}+a\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}1+2a\\3+a\\a\end{pmatrix}\end{cases}$

$2+4a+3+a+a=0$

$6a=-5 \qquad a=-5/6$

$H=\begin{pmatrix}1-5/3\\3-5/6\\-5/6\end{pmatrix}=\begin{pmatrix}-2/3\\13/6\\-5/6\end{pmatrix}$

Esercizio: $r: \begin{pmatrix}2\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

1) Determinare, se esiste, l'equazione cartesiana di un piano π contenente r (1) ed ortogonale al vettore (2) $v = \begin{pmatrix}1\\0\\1\end{pmatrix}$.

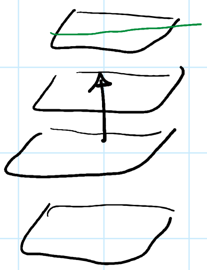
2) Verificare che $r \subseteq \sigma$ con $\sigma: x - z = 2$ e determinare le equazioni cartesiane delle rette s_1 e s_2 contenute in σ e distanti $\sqrt{2}$ da r

3) Determinare la distanza fra r e t con $t: \begin{cases} x+y+z=1 \\ 2x+y+z=1 \end{cases}$.

Svolgimento:

(2) un piano $\perp$ al vettore $v = \begin{pmatrix}1\\0\\1\end{pmatrix}$ deve appartenere al fascio

$\pi_K: x + z = K$ cerchiamo K tale che il punto $P = \begin{pmatrix}2\\1\\0\end{pmatrix}$, che appartiene ad r, appartenga anche a π_K



$\pi_K: x+z=K \quad \begin{pmatrix}2\\1\\0\end{pmatrix} \quad 2+0=K=2$

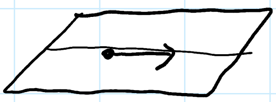
$\pi: x+z=2$ verifichiamo che $r \subseteq \pi$

$r: \underset{P}{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \left\langle \underset{v_r}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \right\rangle$

$P \in \pi \quad 2+0=2$ ok

$v_r \in V_\pi \quad \begin{pmatrix}0\\1\\0\end{pmatrix} \in V_\pi: x+z=0$

$0+0=0$ ok



$\Longrightarrow r \subseteq \pi$

$\text{rg}\begin{pmatrix}0 & a\\1 & b\\0 & -a\end{pmatrix} = 2 \iff a \neq 0$

2° modo

(1) $\pi: \begin{pmatrix}2\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix} \right\rangle \quad \text{rg}\begin{pmatrix}0 & a\\1 & b\\0 & c\end{pmatrix} = 2 \quad$ (1) $r \subseteq \pi$

$V_\pi \perp \begin{pmatrix}1\\0\\1\end{pmatrix}$

$\begin{cases} \begin{pmatrix}0\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 0 & 0=0 \text{ ok} \\ \begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 0 & a+c=0 \iff c=-a \end{cases}$

$$\pi: \begin{pmatrix}2\\1\\0\end{pmatrix} + \langle \underline{\overset{v_1}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}a\\b\\-a\end{pmatrix}}} \rangle = \begin{pmatrix}2\\1\\0\end{pmatrix} + \langle \overset{v_1}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \overset{v_2 - b v_1}{\begin{pmatrix}a\\0\\-a\end{pmatrix}} \rangle \quad \text{con } a \neq 0$$

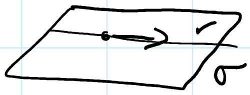
$$= \begin{pmatrix}2\\1\\0\end{pmatrix} + \langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix} \rangle$$

eq. cart. $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} + a\begin{pmatrix}0\\1\\0\end{pmatrix} + b\begin{pmatrix}1\\0\\-1\end{pmatrix} = \begin{pmatrix}2+b\\1+a\\-b\end{pmatrix}$ $\quad \begin{cases} x = 2+b \\ y = 1+a \\ z = -b \end{cases}$

$\begin{cases} a = y-1 \\ b = -z \end{cases}$ $\quad x = 2 - z \quad \pi: x + z = 2$

2) Verifichiamo che $r: \overset{P}{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \langle \overset{v_r}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \rangle$ è contenuna in

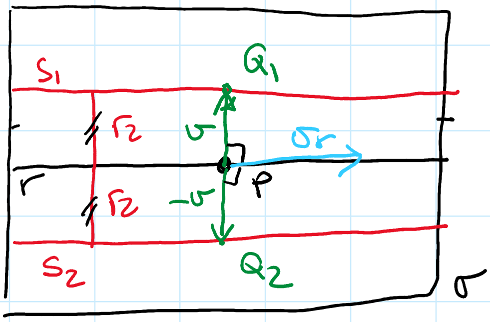
$\sigma: x - z = 2$



$\begin{cases} P \in \sigma \text{ ok} & 2-0=2 \text{ ok} \\ v_r \in V_\sigma \text{ ok} & v_r = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad V_\sigma: x - z = 0 \quad 0-0=0 \text{ ok} \end{cases} \Rightarrow r \subseteq \sigma.$

$\underset{P}{\begin{pmatrix}2\\1+a\\0\end{pmatrix}}$ $\quad \sigma: x - z = 2$

$\quad 2 - 0 = 2$



$r: \overset{P}{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \langle \overset{v_r}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \rangle$

$s_1: Q_1 + \langle \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle$

$s_2: Q_2 + \langle \begin{pmatrix}0\\1\\0\end{pmatrix} \rangle$

$Q_1 = P + v$

$Q_2 = P + v$

$$\begin{cases} v \in V_\sigma: \quad x - z = 0 & v = \begin{pmatrix}a\\b\\a\end{pmatrix} \\ v \cdot v_r = 0 & \begin{pmatrix}a\\b\\a\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = 0 \quad b = 0 \quad \begin{pmatrix}a\\0\\a\end{pmatrix} \\ \|v\| = \sqrt{2} & \left\| a\begin{pmatrix}1\\0\\1\end{pmatrix} \right\| = |a|\sqrt{2} = \sqrt{2} \quad |a| = 1 \end{cases}$$

$a = 1 \quad v = \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad a = -1 \quad -v = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$

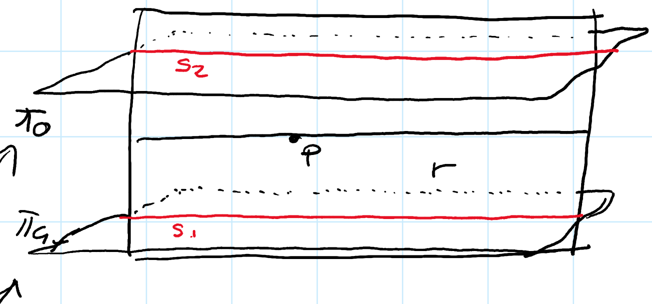
$s_1: \begin{pmatrix}3\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \qquad \begin{cases} x=3 \\ z=1 \end{cases} \qquad Q_1 = P+v = \begin{pmatrix}2\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$

$s_2: \begin{pmatrix}1\\1\\-1\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \qquad \begin{cases} x=1 \\ z=-1 \end{cases} \qquad Q_2 = P-v = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

2° modo: Trovo un vettore $v \in V_\sigma$ e $v \perp v_r$

$v = \begin{pmatrix}a\\b\\a\end{pmatrix}$ perché $V_\sigma: x-z=0$; $v \cdot v_r = 0 \Rightarrow b=0 \Rightarrow v = \begin{pmatrix}a\\0\\a\end{pmatrix}$

tutti multipli di $\begin{pmatrix}1\\0\\1\end{pmatrix}$ quindi cerco i piani $\perp$ a v e distanti $\sqrt{2}$ da r : $\pi_k: x+z=k$



$P = \begin{pmatrix}2\\1\\0\end{pmatrix}$

$$d(P,\pi_k) = \frac{|2+0-k|}{\sqrt{2}} = \sqrt{2}$$

$\Longleftrightarrow |2-k| = 2 \Rightarrow \begin{matrix} 2-k=2 & k=0 \\ 2-k=-2 & k=4 \end{matrix}$

$s_1: \begin{cases} x+z=4 \\ x-z=2 \end{cases}$ equiv. a $\begin{cases} x=3 \\ z=1 \end{cases}$

$s_2: \begin{cases} x+z=0 \\ x-z=2 \end{cases}$ equiv. a $\begin{cases} x=1 \\ z=-1 \end{cases}$

3) Distanza fra $r: \overset{R}{\begin{pmatrix}2\\1\\0\end{pmatrix}} + \left\langle \overset{v_r}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \right\rangle \qquad t: \begin{cases} x+y+z=1 \\ 2x+y+z=1 \end{cases}$

$V_r = \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle \qquad V_t: \begin{cases} x+y+z=0 \\ 2x+y+z=0 \end{cases} \qquad V_r \neq V_t \qquad 0+1+0 \neq 0$

$t: \begin{cases} x+y+z=1 \\ 2x+y+z=1 \end{cases} \quad \begin{cases} x=0 \\ y+z=1 \end{cases} \quad \begin{pmatrix}0\\1-z\\z\end{pmatrix} \qquad t: \overset{T}{\begin{pmatrix}0\\1\\0\end{pmatrix}} + \left\langle \overset{v_t}{\begin{pmatrix}0\\-1\\1\end{pmatrix}} \right\rangle \qquad \begin{pmatrix}0\\1-z\\z\end{pmatrix}$

$z=0 \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$$\dim \langle v_r, v_t, T-R \rangle = \dim \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}-2\\0\\0\end{pmatrix} \right\rangle = 3$$

$\Rightarrow$ r e t sono sghembe.

$d(r,t) = d(r,\pi) =$

$= d(R,\pi)$

$R = \begin{pmatrix}2\\1\\0\end{pmatrix}$ $\qquad \pi: x = 0$

$d(R,\pi) = \dfrac{|2|}{1} = 2$

π piano contenente t e // ad r

$\pi: \boxed{\begin{pmatrix}0\\1\\0\end{pmatrix} + \left\langle \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right.}\left.\right\rangle$